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# The Weighted Arithmetic Mean–Geometric Mean Inequality is Equivalent to the Hölder Inequality

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**Abstract:** In the current note, we investigate the mathematical relations among the weighted arithmetic mean–geometric mean (AM–GM) inequality, the Hölder inequality and the weighted power-mean inequality. Meanwhile, the proofs of mathematical equivalence among the weighted AM–GM inequality, the weighted power-mean inequality and the Hölder inequality are fully achieved. The new results are more generalized than those of previous studies.

**Keywords:** weighted AM–GM inequality; Hölder inequality; weighted power-mean inequality; L'Hospital's rule

## 1. Introduction

In the field of classical analysis, the weighted arithmetic mean–geometric mean (AM–GM) inequality (see e.g., [1], pp. 74–75) is often inferred from Jensen's inequality, which is a more generalized inequality compared to the AM–GM inequality; refer to, e.g., [1,2]. In addition, the Hölder inequality [2] found by *Leonard James Rogers* (1888) and discovered independently by *Otto Hölder* (1889) is a basic and indispensable inequality for studying integrals and  $L^p$  spaces, and is also an extension of the Cauchy–Bunyakovsky–Schwarz (CBS) inequality [3]. The Hölder inequality is used to prove the Minkowski inequality, which is the triangle inequality in the space  $L^p(\mu)$  [4,5]. The weighted power mean (also known as the generalized mean)  $M_r^m(a)$  for a sequence  $a = (a_1, a_2, \dots, a_n)$  is defined as  $M_r^m(a) = (m_1 a_1^r + m_2 a_2^r + \dots + m_n a_n^r)^{\frac{1}{r}}$ , which is a family of functions for aggregating sets of numbers, and plays a vital role in analytical inequalities; see [2,6] for instance.

In recent years, many researchers have been interested in studying the mathematical equivalence among some famous analytical inequalities, such as the Cauchy–Schwarz inequality, the Bernoulli inequality, the Wielandt inequality, and the Minkowski inequality; see [7–13] for details. Additionally, these studies note the relations among the weighted AM–GM inequality, the Hölder inequality, and the weighted power-mean inequality are still less clear, although one inequality is often helpful to prove another inequality [1,12]. Motivated by these aforementioned studies, in the present note, the mathematical equivalence among three such well-known inequalities is proved in detail; the result introduced in [14] is also extended.

The rest of the present note is organized as follows. In the next section, we will present the detailed proofs of mathematical equivalence among three celebrated mathematical inequalities. Finally, the paper ends with several concluding remarks in Section 3.

## 2. Main Results

The weighted AM–GM inequality, the Hölder inequality, and the weighted power-mean inequality [1] (pp. 111–112, Theorem 10.5) are first reviewed and they are often related to each other. Then the results of mathematical equivalence among three such inequalities will be shown.

**Weighted AM–GM Inequality.** If  $0 \leq c_i \in \mathbb{R}$  ( $i = 1, \dots, n$ ) and  $0 \leq \lambda_i \in \mathbb{R}$  ( $i = 1, \dots, n$ ) such that  $\sum_{i=1}^n \lambda_i = 1$ , then

$$\prod_{k=1}^n c_k^{\lambda_k} \leq \sum_{k=1}^n \lambda_k c_k. \quad (1)$$

**Hölder Inequality.** If  $0 \leq a_i, b_i \in \mathbb{R}$  ( $i = 1, \dots, n$ ) and  $p, q \in \mathbb{R}^+$  such that  $p^{-1} + q^{-1} = 1$ , then

$$\sum_{k=1}^n a_k b_k \leq \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{q}}. \quad (2)$$

**Weighted Power-Mean Inequality.** If  $0 \leq c_i, \lambda_i \in \mathbb{R}$  ( $i = 1, \dots, n$ ) such that  $\sum_{i=1}^n \lambda_i = 1$ , and  $r, s \in \mathbb{R}^+$  such that  $r \leq s$ , then

$$\left( \sum_{k=1}^n \lambda_k c_k^r \right)^{\frac{1}{r}} \leq \left( \sum_{k=1}^n \lambda_k c_k^s \right)^{\frac{1}{s}}. \quad (3)$$

The word “equivalence” between two statements **A** and **B**, by convention, is understood as follows: **A** implies **B** and **B** implies **A**. Two equivalent sentences have the same truth value. Thus, this note reveals a connection (in the sense of art) between these two well-known facts.

**Theorem 1.** *The Hölder inequality is equivalent to the weighted AM–GM inequality.*

**Proof.** To show that (2) implies (1), let  $a_k = (\lambda_k c_k)^{\frac{1}{p}}, b_k = (\lambda_k)^{\frac{1}{q}}$  in (2) for all  $k$ ; then

$$\sum_{k=1}^n \left[ (\lambda_k c_k)^{\frac{1}{p}} (\lambda_k)^{\frac{1}{q}} \right] \leq \left( \sum_{k=1}^n \lambda_k c_k \right)^{\frac{1}{p}} \left( \sum_{k=1}^n \lambda_k \right)^{\frac{1}{q}}. \quad (4)$$

Since  $p^{-1} + q^{-1} = 1$  and  $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$ , the inequality (4) can be rewritten as:

$$\sum_{k=1}^n \lambda_k c_k \geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{p}} \right)^p. \quad (5)$$

Now using the inequality (5) successively, it follows that

$$\begin{aligned} \sum_{k=1}^n \lambda_k c_k &\geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{p}} \right)^p \geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{p^2}} \right)^{p^2} \\ &\geq \dots \geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{p^m}} \right)^{p^m} \geq \dots \end{aligned} \quad (6)$$

By L'Hospital's rule, it is easy to see that

$$\lim_{x \rightarrow 0^+} \ln \left( \sum_{k=1}^n \lambda_k c_k^x \right) / x = \sum_{k=1}^n \lambda_k \ln c_k.$$

Thus,

$$\lim_{x \rightarrow 0^+} \left( \sum_{k=1}^n \lambda_k c_k^x \right)^{\frac{1}{x}} = \prod_{k=1}^n c_k^{\lambda_k}.$$

Thus, in (6), we can pass to the limit by  $m \rightarrow +\infty$ , giving  $\sum_{k=1}^n \lambda_k c_k \geq \prod_{k=1}^n c_k^{\lambda_k}$ ; hence, (2) implies (1). To show the converse, all that is needed is a special case of (1),

$$\lambda_1 c_1 + \lambda_2 c_2 \geq c_1^{\lambda_1} c_2^{\lambda_2}. \tag{7}$$

Since  $p^{-1} + q^{-1} = 1$ , and by (7), thus

$$\frac{1}{p} a_k^p \sum_{k=1}^n b_k^q + \frac{1}{q} b_k^q \sum_{k=1}^n a_k^p \geq a_k b_k \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{p}} \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{q}}.$$

Summing over  $k = 1, 2, \dots, n$ , it follows that

$$\sum_{k=1}^n a_k^p \sum_{k=1}^n b_k^q \geq \sum_{k=1}^n a_k b_k \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{p}} \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{q}}.$$

Thus, (1) implies (2).  $\square$

**Remark 1.** The CBS inequality (the AM–GM inequality) is the special case of the Hölder inequality (the weighted AM–GM inequality); therefore, Theorem 1 is a generalization of the result established in [14].

**Theorem 2.** The Hölder inequality is equivalent to the weighted power-mean inequality.

**Proof.** For  $r \leq s$ , let  $p = \frac{s}{s-r} \geq 1$  and  $c_k = d_k^s$  in (5) for  $k = 1, \dots, n$ , then

$$\sum_{k=1}^n \lambda_k d_k^s \geq \left( \sum_{k=1}^n \lambda_k d_k^r \right)^{\frac{s}{r}}. \tag{8}$$

Thus, rewriting (8) as  $\left( \sum_{k=1}^n \lambda_k d_k^s \right)^{\frac{1}{s}} \geq \left( \sum_{k=1}^n \lambda_k d_k^r \right)^{\frac{1}{r}}$ . Now the task is to prove that (3) implies (2);

let  $\lambda_k = b_k^q \left( \sum_{k=1}^n b_k^q \right)^{-1}$ ,  $d_k = a_k / b_k^{q-1}$ ,  $r = 1, s = p$  in (3), then

$$\begin{aligned} \sum_{k=1}^n a_k b_k &= \left( \sum_{k=1}^n b_k^q \right) \left( \sum_{k=1}^n \frac{b_k^q}{\sum_{k=1}^n b_k^q} \cdot \frac{a_k}{b_k^{q-1}} \right) \\ &\leq \left( \sum_{k=1}^n b_k^q \right) \left( \sum_{k=1}^n \frac{b_k^q}{\sum_{k=1}^n b_k^q} \cdot \left( \frac{a_k}{b_k^{q-1}} \right)^p \right)^{\frac{1}{p}} \\ &= \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{q}}. \quad \square \end{aligned}$$

**Remark 2.** Here, we can give another proof that (3) implies (2). Repeatedly using inequality (3), it follows that

$$\sum_{k=1}^n \lambda_k c_k \geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{2}} \right)^2 \geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{3}} \right)^3 \geq \dots \geq \left( \sum_{k=1}^n \lambda_k c_k^{\frac{1}{n}} \right)^n \geq \dots$$

By L'Hospital's rule, it is easy to see that

$$\sum_{k=1}^n \lambda_k c_k \geq \prod_{k=1}^n c_k^{\lambda_k}.$$

By using analogous methods from Theorem 1, it can be proved that

$$\sum_{k=1}^n a_k^p \sum_{k=1}^n b_k^q \geq \sum_{k=1}^n a_k b_k \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{p}} \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{q}}.$$

**Theorem 3.** The weighted power-mean inequality is equivalent to the weighted AM–GM inequality.

**Proof.** To show that (1) implies (3), we merely exploit a special case of (1),

$$a_1^{\lambda_1} a_2^{\lambda_2} \leq \lambda_1 a_1 + \lambda_2 a_2. \tag{9}$$

Here, we define  $U_n(a) = \lambda_1 a_1^s + \lambda_2 a_2^s + \dots + \lambda_n a_n^s$ ; let  $a_1 = \lambda_k a_k^s (U_n(a))^{-1}$ ,  $a_2 = \lambda_k$  and  $\lambda_1 = \frac{r}{s}$ ,  $\lambda_2 = 1 - \frac{r}{s}$  in (9), then

$$\lambda_k a_k^r (U_n(a))^{-\frac{r}{s}} \leq \frac{r}{s} \cdot \lambda_k a_k^s (U_n(a))^{-1} + \left(1 - \frac{r}{s}\right) \cdot \lambda_k.$$

Summing over  $k = 1, 2, \dots, n$ , then

$$\sum_{k=1}^n \lambda_k a_k^r (U_n(a))^{-\frac{r}{s}} \leq \sum_{k=1}^n \left[ \frac{r}{s} \cdot \lambda_k a_k^s (U_n(a))^{-1} + \left(1 - \frac{r}{s}\right) \cdot \lambda_k \right] = 1.$$

Therefore,

$$\left( \sum_{k=1}^n \lambda_k c_k^r \right)^{\frac{1}{r}} \leq \left( \sum_{k=1}^n \lambda_k c_k^s \right)^{\frac{1}{s}}.$$

The converse is trivial from Remark 2. □

### 3. Concluding Remarks

In this note, the mathematical equivalence among the weighted AM–GM inequality, the Hölder inequality, and the weighted power-mean inequality is investigated in detail. Moreover, the interesting conclusions of Lin's paper [14] are also extended. At the end of the present study, for convenience, the results on the equivalence of some well-known analytical inequalities can be summarized as follows:

- Equivalence of the Hölder's inequality and the Minkowski inequality; see [9].
- Equivalence of the Cauchy–Schwarz inequality and the Hölder's inequality; see [8].
- Equivalence of the Cauchy–Schwarz inequality and the Covariance–Variance inequality; see [7].
- Equivalence of the Kantorovich inequality and the Wielandt inequality; see e.g., [11]
- Equivalence of the AM–GM inequality and the Bernoulli inequality; see e.g., [10]
- Equivalence of the Hölder inequality and Artin's theorem; see e.g., [12] (pp. 657–663) for details.
- Equivalence of the Hölder inequality and the weighted AM–GM inequality; refer to Theorem 1.
- Equivalence of the Hölder inequality and the weighted power-mean inequality; see Theorem 2.
- Equivalence of the weighted power-mean inequality and the weighted AM–GM inequality; refer to Theorem 3.

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## Abbreviations

The following abbreviations are used in this manuscript:

AM–GM Arithmetic Mean–Geometric Mean  
CBS Cauchy–Bunyakovsky–Schwarz

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