Mathematical Excalibur

Volume 1, Number 1 January - February, 1995

Olympiad Corner

The 35th International Mathematical Olympiad was held in Hong Kong last summer. The following are the six problems given to the contestants. How many can you solve? (The country names inside the parentheses are the problem proposers.) - *Editors*

Problem 1. (France)
Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}$$

Problem 2. (Armenia/Australia)
ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

(*continued on page 4*)

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Acknowledgment: Thanks to Martha A. Dahlen, Technical Writer, HKUST, for her comments.

The editors welcome contributions from all students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is January 31, 1995. Send all correspondence to:

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Pigeonhole Principle

Kin-Yin Li

What in the world is the pigeonhole principle? Well, this famous principle states that if $n+1$ objects (pigeons) are taken from n boxes (pigeonholes), then at least two of the objects will be from the same box. This is clear enough that it does not require much explanation. A problem solver who takes advantage of this principle can tackle certain combinatorial problems in a manner that is more elegant and systematic than case-by-case. To show how to apply this principle, we give a few examples below.

Example 1. Suppose 51 numbers are chosen from 1, 2, 3, ..., 99, 100. Show that there are two which do not have any common prime divisor.

Solution. Let us consider the 50 pairs of consecutive numbers (1,2), (3,4), ..., (99,100). Since 51 numbers are chosen, the pigeonhole principle tells us that there will be a pair $(k, k+1)$ among them. Now if a prime number p divides $k+1$ and k, then p will divide $(k+1) - k = 1$, which is a contradiction. So, k and $k+1$ have no common prime divisor.

Example 2. Suppose 51 numbers are chosen from 1, 2, 3, ..., 99, 100. Show that there are two such that one divides the other.

Solution. Consider the 50 odd numbers 1, 3, 5, ..., 99. For each one, form a box containing the number and all powers of 2 times the number. So the first box contains 1, 2, 4, 8, 16, ... and the next box contains 3, 6, 12, 24, 48, ... and so on. Then among the 51 numbers chosen, the pigeonhole principle tells us that there are two that are contained in the same box. They must be of the form $2^m k$ and $2^n k$ with the same odd number k. So one will divide the other.

Note that the two examples look alike, however the boxes formed are quite different. By now, the readers must have observed that forming the right boxes is the key to success. Often a certain amount of experience as well as clever thinking are required to solve such problems. The additional examples below will help beginners become familiar with this useful principle.

Example 3. Show that among any nine distinct real numbers, there are two, say a and b, such that

$$0 < (a-b)/(1+ab) < \sqrt{2} - 1.$$

Solution. The middle expression $(a-b)/(1+ab)$ reminds us of the formula for $\tan(x-y)$. So we proceed as follow. Divide the interval $(-\pi/2, \pi/2]$ into 8 intervals $(-\pi/2, -3\pi/8]$, $(-3\pi/8, -\pi/4]$, ..., $(\pi/4, 3\pi/8]$, $(3\pi/8, \pi/2]$. Let the numbers be $a_1, a_2, \ldots, a_9$ and let $x_i = \arctan a_i$, $i = 1, 2, \ldots, 9$. By the pigeonhole principle, two of the x_i's, say x_j and x_k with $x_j > x_k$, must be in one of the 8 subintervals. Then we have $0 < x_j - x_k < \pi/8$, so $0 < \tan(x_j - x_k) = (a_j - a_k)/(1 + a_j a_k) < \tan(\pi/8) = \sqrt{2} - 1$.

Example 4. Suppose a triangle can be placed inside a square of unit area in such a way that the center of the square is not inside the triangle. Show that one side of the triangle has length less than 1. (This example came from the *XLI Mathematical Olympiad in Poland.*)

Solution. Through the center C of the square, draw a line L_1 parallel to the closest side of the triangle and a second line L_2 perpendicular to L_1 at C. The lines L_1 and L_2 divide the square into four congruent quadrilaterals. Since C is not

(*continued on page 2*)

Pigeonhole Principle

(*continued from page 1*)

inside the triangle, the triangle can lie in at most two (adjacent) quadrilaterals. By the pigeonhole principle, two of the vertices of the triangle must belong to the same quadrilateral. Now the furthest distance between two points in the quadrilateral is the distance between two of its opposite vertices, which is at most 1. So the side of the triangle with two vertices lying in the same quadrilateral must have length less than 1.

Below we provide some exercises for the active readers.

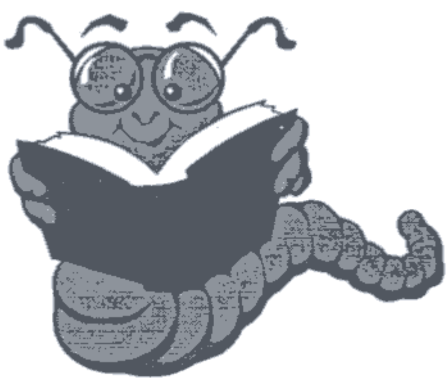
1. Eleven numbers are chosen from 1, 2, 3, ..., 99, 100. Show that there are two nonempty disjoint subsets of these eleven numbers whose elements have the same sum.

2. Suppose nine points with integer coordinates in the three dimensional space are chosen. Show that one of the segments with endpoints selected from the nine points must contain a third point with integer coordinates.

3. Show that among any six people, either there are three who know each other or there are three, no pair of which knows each other.

4. In every 16-digit number, show that there is a string of one or more consecutive digits such that the product of these digits is a perfect square. [*Hint:* The exponents of a factorization of a perfect square into prime numbers are even.] (This problem is from the *1991 Japan Mathematical Olympiad.*)

(*Answers can be found on page 3.*)



The Game of "Life"

Tsz-Mei Ko

The game of "Life" was first introduced by John Conway, a mathematician and a game hobbyist currently working at Princeton University. The game is played on an infinite chessboard, where each cell has eight neighboring cells. Initially, an arrangement of stones is placed on the board (the live cells) as the first generation. Each new generation is determined by two simple generic rules:

The Death Rule: Consider a live cell (occupied by a stone). If it has 0 or 1 live neighbors (among the eight neighboring cells), then it dies from isolation. If it has 4 or more live neighbors, then it dies from overcrowding. If it has 2 or 3 live neighbors, then it survives to the next generation.

The Birth Rule: Consider a dead (unoccupied) cell. If it has exactly 3 live neighbors, then it becomes a live cell (with a stone placed on it) in the next generation.

Here is an example. The six circles in Figure 1 indicate the live cells in the first generation. Those marked i and c will die due to isolation and overcrowding respectively (Death Rule). The empty cells marked b will become live cells in the next generation (Birth Rule). The second generation is shown in Figure 2.

What will happen in the third, fourth, and *n*th generation? Is there an initial generation that will grow infinitely?

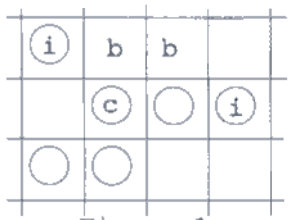


Figure 1

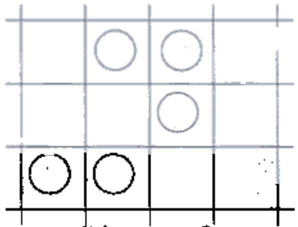
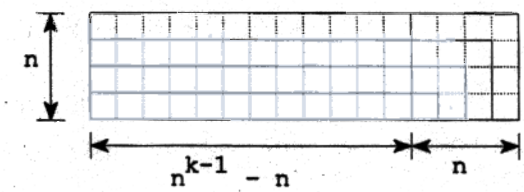


Figure 2

Proof Without Words

kth Power of a Natural Number n as the Sum of n Consecutive Odd Numbers (k = 2,3,...)



$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \cdots + (n^{k-1} - n + 2n - 1)$$

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *January 31st, 1995.*

Problem 1. The sum of two positive integers is 2310. Show that their product is not divisible by 2310.

Problem 2. Given N objects and $B(\geq 2)$ boxes, find an inequality involving N and B such that if the inequality is satisfied, then at least two of the boxes have the same number of objects.

Problem 3. Show that for every positive integer n, there are polynomials $P(x)$ of degree n and $Q(x)$ of degree n-1 such that $(P(x))^2 - 1 = (x^2-1)(Q(x))^2$.

Problem 4. If the diagonals of a quadrilateral in the plane are perpendicular, show that the midpoints of its sides and the feet of the perpendiculars dropped from the midpoints to the opposite sides lie on a circle.

Problem 5. (*1979 British Mathematical Olympiad*) Let $a_1, a_2, ..., a_n$ be n distinct positive odd integers. Suppose all the differences $|a_i - a_j|$ are distinct, $1 \leq i < j \leq n$. Prove that $a_1 + a_2 + \cdots + a_n \geq n(n^2+2)/3$.

Answers to Exercises in "Pigeonhole Principle"

1. The set of eleven numbers have 2^{11}-2 = 2046 nonempty subsets with less than eleven elements, and the maximal sum of the elements in any of these subsets is 91 + 92 + ··· + 99 + 100 = 955. So, by the pigeonhole principle, there are two nonempty subsets with the same sum. If they have common elements, then remove them from both subsets and we will get two nonempty disjoint subsets with the same sum.

2. For the nine points, each of the three coordinates is either even or odd. So, there are 2^3=8 parity patterns for the coordinates. By the pigeonhole principle, two of the nine points must have the same parity coordinate patterns. Then their midpoint must have integer coordinates.

3. Let the six people correspond to the six vertices of a regular hexagon. If two people know each other, then color the segment with the associated vertices red, otherwise blue. Solving the problem is equivalent to showing that a red triangle or a blue triangle exists.

Take any vertex. By the pigeonhole principle, of the five segments issuing from this vertex, three have the same color c. Consider the three vertices at the other ends of these segments and the triangle T with these vertices. If T has an edge colored c, then there is a triangle with color c. Otherwise, all edges of T are colored opposite to c. In both cases, there is a triangle with all edges the same color.

4. Let $d_1, d_2, ..., d_{16}$ be the digits of a 16-digit number. If one of the digits of the sixteen digits is either 0 or 1 or 4 or 9, then the problem is solved. So, we may assume each of the digits is 2, 3, 5, 6=2x3, 7 or $8=2^3$. Let $x_0 = 1$ and x_i be the product of $d_1, d_2, ..., d_i$ for $i = 1, 2, ..., 16$. Now each $x_i = 2^{p_i} \times 3^{q_i} \times 5^{r_i} \times 7^{s_i}$ for $i = 0, 1, 2, ..., 16$. Each of the p_i, q_i, r_i, s_i is either even or odd. So there are $2^4 = 16$ possible parity patterns. By the pigeonhole principle, the p_i, q_i, r_i, s_i for two of the seventeen x_i's, say x_j and x_k with $j < k$, must have the same parity pattern. Then $d_{j+1} \times \cdots \times d_k = x_k/x_j$ is a perfect square.

Mathematical Application: Pattern Design

Roger Ng

Mathematics is by far the most powerful tool that human race has created. We invite articles which can share with us different areas of applications in mathematics. We wish that this column will inspire students to study mathematics. - *Editors*

In this first issue, I would like to introduce an interesting application which exemplifies the power of mathematics to define an artistic work in a formalised manner.

Take a look at your school uniform. It is made up of patches of fabrics. Before the fabric is cut, the overall shape and measurement of each patch must be drawn. Each patch is known as a pattern piece.

When a pattern is drawn, it must match the surface of a human body. Therefore, the pattern design process is in fact a surface unfolding problem.

What makes the pattern design process an artistic activity is the drawing of curves in the pattern. Each person has his/her own preference. That is why some brand manufacturers can produce better looking garments.

To see how mathematics can be applied, let us consider a specific problem in curve drawing. Take a look at your pant or your skirt. Do you see any smooth overlapping at the center front where you button up the garment? If the pattern is not drawn correctly, you should see a scissor-like crossing at the opening along the waist.

You may imagine that to button up your pant is equivalent to let two curves meet at the same point x. In mathematics, we define two types of continuity conditions, namely, C^0 and C^1 (Figure 1). C^0 means that the two curves meet at the point x, i.e., $f(x^-) = f(x^+)$. C^1 means that the two curves have the same slope at x, i.e., $f'(x^-) = f'(x^+)$.

There will be a smooth overlapping when a pant is buttoned up if both continuity conditions C^0 and C^1 are met. Thus clever fashion designers use a ruler to keep track of the slope $f'(x)$ (Figure 1). This technique dramatically improves the quality of a garment.

In the above example, we see how the continuity concept in mathematics can help a fashion designer to improve the smoothness of a pattern and thus to design nice-looking garments. In fact, there are many other such areas where mathematics can be useful.

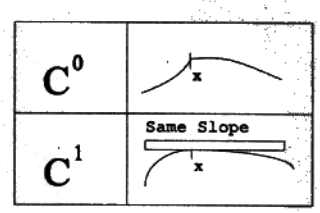


Figure 1

Olympiad Corner

(continued from page 1)

Problem 3. (Romania)
For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

(a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k)=m$.

(b) Determine all positive integers m for which there exists exactly one k with $f(k)=m$.

Problem 4. (Australia)
Determine all ordered pairs (m,n) of positive integers such that

$$\frac{n^3+1}{mn-1}$$

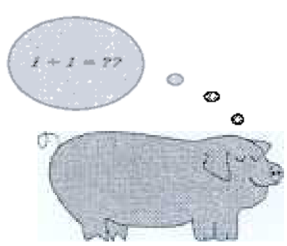
is an integer.

Problem 5. (United Kingdom)
Let S be the set of real numbers strictly greater than -1. Find all functions $f: S \to S$ satisfying the two conditions:

(i) $f(x+f(y)+xf(y)) = y+f(x)+yf(x)$ for all x and y in S;

(ii) $f(x)/x$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

Problem 6. (Finland)
Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$.



From Fermat Primes to Constructible Regular Polygons

Tsz-Mei Ko

Pierre de Fermat (1601-1665), an amateur mathematician, once guessed that all numbers in the form $2^{2^n}+1$ are prime numbers. If we try the first five n's ($n = 0, 1, 2, 3, 4$), they are in fact all primes:

n	$2^{2^n}+1$
0	3
1	5
2	17
3	257
4	65537

It was later discovered by Leonhard Euler (1707-1783) in 1732 that the next Fermat number ($n = 5$) can be factored as

$$2^{2^5}+1 = 641 \times 6700417$$

and thus not a prime. The story would have ended here if without an ingenious discovery by Carl Friedrich Gauss (1777-1855).

In 1794, at the age of seventeen, Gauss found that a regular "p-gon" (a polygon with p sides), where p is a prime, is constructible (i.e., using only ruler and compass) if and only if p is a "Fermat prime" (a prime number in the form $2^{2^n}+1$). He proved this by considering the solutions of certain algebraic equations. (The interested reader may refer to the book, "What Is Mathematics?" written by Courant and Robbins, Oxford University Press.) The young Gauss was so overwhelmed by his discovery that he then decided to devote his life to mathematics. After his death, a bronze statue in memory of him standing on a regular 17-gon pedestal was erected in Brauschweig – the hometown of Gauss.

Which regular polygons are constructible? From Gauss's result, we know that the regular triangle, pentagon, 17-gon, 257-gon and 65537-gon are constructible. (How?) We also know that regular polygons with 7, 11, 13, 19, ... sides are not constructible since they are primes but not Fermat primes. In addition, we know how to bisect an angle and thus regular polygons with 4, 8, 16, 32, ... or 6, 12, 24, 48, ... sides are also constructible. What about the others? Is a regular 15-gon constructible? The answer turns out to be yes since $1/15 = 2/5 - 1/3$ and thus we can divide a circle into 15 equal parts. What about a regular 9-gon? It can be proved that a regular 9-gon is not constructible. Can you find a general theorem on which regular polygons are constructible?

Are there any other constructible p-gons (where p is a prime) besides the five mentioned? This question is equivalent to asking whether there are any other Fermat primes. To date, no other Fermat number has been shown to be prime, and it is still not known whether there are more than five Fermat primes. Perhaps you can discover a new Fermat prime and make a note in the history of mathematics.

Right: A photo of the six members of the Hong Kong Team and one of the editors (far right) taken at the Shatin Town Hall after the closing ceremony of the 35th International Mathematical Olympiad.

From left to right are: Suen Yun-Leung, Chu Hoi-Pan, Tsui Ka-Hing, Wong Him-Ting, Ho Wing-Yip, Poon Wai-Hoi Bobby, and Li Kin-Yin.

Mathematical Excalibur

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Olympiad Corner

The following are the six problems from the two-day Final Selection Exam for the 1994 Hong Kong Mathematical Olympiad Team. Would you like to try these problems to see if you could have qualified to be a Hong Kong team member? *- Editors*

Instructions (the same instructions were given on both days): *Answer all three questions. Each question carries 35 points. Time allowed is 4½ hours.*

<u>First Day</u>

Question 1. In a triangle ΔABC, $\angle C = 2\angle B$. P is a point in the interior of ΔABC satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

Question 2. In a table-tennis tournament of 10 contestants, any two contestants meet only once. We say that there is a winning triangle if the following situation occurs: ith contestant defeated jth contestant, jth contestant defeated kth contestant, and kth contestant defeated ith contestant. Let W_i and L_i be respectively the number of games won and lost by the ith contestant. Suppose $L_i + W_j \geq 8$ whenever the ith contestant wins

(*continued on page 4*)

Acknowledgment: Thanks to Martha A. Dahlen, Technical Writer, HKUST, for her comments.

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Fractal Game of Escape

Roger Ng

Consider the following scenario. John, a secret agent, is being held captive in terrorists' headquarters. He has found an escape route, and knows it follows the quadratic equation $z_{n+1} = z_n^2 + c$ if the floor map is encoded as a complex z-plane (i.e., each point (x,y) is represented by a complex number $x+yi$). However, John does not know the value of the complex constant c. John only knows that he should start from the origin with $z_0 = 0 + 0i$. For which values of c, will John have not even a chance for a successful escape?

To help John to answer the above question, it is natural to first try $c = 0$ and see what will happen. The recursion becomes $z_{n+1} = z_n^2$ and thus $z_n = 0$ for all n. That is, John will be going nowhere but staying at the origin!

If we try other values of c, there are three possible outcomes: (1) the sequence z_n converges to a fixed point; (2) the sequence z_n repeats in a finite cycle of points and thus becomes a periodic sequence; or (3) the sequence z_n diverges from the origin, i.e., John may have a chance to escape successfully.

The above story is a dramatization for the definition of a fractal called the Mandelbrot set. (The word "fractal" was coined by Benoit Mandelbrot to describe sets with self-similarity, i.e., they look the same if you magnify a portion of them.) The Mandelbrot set can be defined as the set of complex numbers c for which the sequence $z_{n+1} = z_n^2 + c$ is bounded (i.e., does not diverge) when the starting point z_0 is the origin (0,0). Figure 1 shows the asymptotic behaviour of z_n for real c's that generate bounded sequences (i.e., outcomes 1 and 2). The number of points on a vertical line indicates the period of the asymptotic sequence. Figure 2 shows the values for c (the black area) that would keep z_n bounded, i.e., the Mandelbrot set.

Now if we modify our story slightly—assume that John knows the constant c but not the starting point z_0, this will lead us to the definition of Julia sets—named after the mathematician Gaston Julia (1893-1978). For any given complex number c, some initial points z_0 generate divergent sequences $z_{n+1} = z_n^2 + c$ while others generate nondivergent sequences. The Julia set is the boundary that separates the set of "diverging" starting points from the set of "nondiverging" starting points.

Here is a simple example. For $c = 0$, the equation is $z_{n+1} = z_n^2$. If the starting point lies within a distance of 1 from the origin, the subsequent points will get closer and closer to the origin. If the intial point is more than a distance of 1 from the origin, the subsequent points will get farther and farther away from the origin. The unit circle separates these two sets of starting points. This boundary is the Julia set corresponding to $c = 0$.

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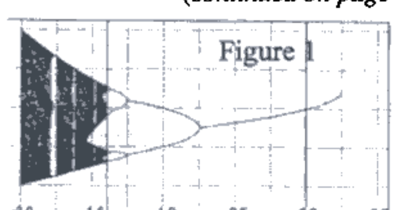


Figure 1

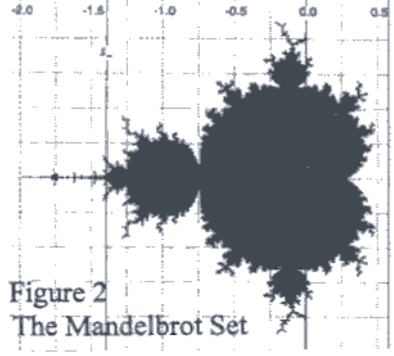


Figure 2
The Mandelbrot Set

Fractal Game of Escape

(continued from page 1)

By varying c, we will obtain an infinite number of different pictures of Julia sets. Some examples are shown in the figures on this page. However, no matter what c is, we observe that there are basically two major types of Julia sets. Either all the points z_0 are connected in one piece, or these points are broken into a number of pieces (in fact, an infinite number of pieces to form something called a Cantor set).

We may ask ourselves an interesting question. For which values of c, will the corresponding Julia set be connected? This seems to be a very hard problem. It seems that we need to look at all Julia sets to find out which one is connected, and it would take an eternity to compile this huge amount of data. But mathematicians John Hubbard and Adrien Douady found a quick way to carry out this task. They proved that a Julia set is connected if the sequence $z_{n+1} = z_n^2 + c$ is bounded when the starting point z_0 is the origin (0,0). That is, if c belongs to the Mandelbrot set, then its corresponding Julia set will be connected! Thus the Mandelbrot set is known as the table of contents for all Julia sets.

Besides this interesting relationship and the fascinating pictures, the Julia set and many other fractals provide us insight into many physical phenomenon. As an example, the Julia Set is directly related to the equipotential field lines of an electrostatic circular metal rod. The interested reader may refer to the book "Chaos and Fractals: New Frontiers of Science," written by H.O. Peitgen, H. Jürgens, and D. Saupe (Springer Verlag, 1992).

Due to the self-similarity of fractals, one usually needs only a few lines of computer programming to generate a fractal image. (Would you like to try?) There is also a free computer software FRACTINT (developed by the Stone Soup Group) that can generate many popular fractal images. If you would like to get a copy of this computer software, send a stamped self-addressed envelope and a PC-formatted high-density diskette to the author at the following address: *Roger Ng, Institute of Textile and Clothing, Hong Kong Polytechnic University, Hung Hom, Kowloon.* There are over a hundred fractal images for your investigation.

Pythagorean Triples

Kin-Yin Li

In geometry, we often encounter triangles whose sides are integers. Have you ever thought about how to produce many nonsimilar triangles of this kind *without guessing*? For this, we first define *Pythagorean triples* to be triples (a, b, c) of positive integers satisfying $a^2 + b^2 = c^2$. For example, (3, 4, 5) and (5, 12, 13) are Pythagorean triples. Clearly, if $a^2 + b^2 = c^2$, then $(ad)^2 + (bd)^2 = (cd)^2$ for any positive integer d. So, solutions of $a^2 + b^2 = c^2$ with a, b, c relatively prime (i.e., having no common prime divisors) are important. These are called *primitive solutions.* Below we will establish a famous theorem giving all primitive solutions.

Theorem. If u, v are relatively prime positive integers, $u > v$ and one is odd, the other even, then $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ give a primitive solution of $a^2 + b^2 = c^2$. Conversely, every primitive solution is of this form, with a possible permutation of a and b.

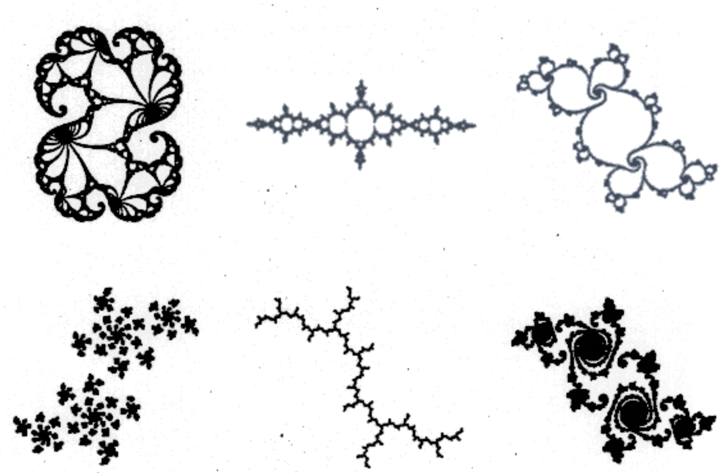
For example, $u = 2$, $v = 1$ corresponds to $a = 3$, $b = 4$, $c = 5$. Now let us try to see why the theorem is true. For the first statement, simple algebra shows $a^2 + b^2 = u^4 + 2u^2v^2 + v^4 = c^2$. If two of a, b, c have a common prime divisor p, then the equation will imply all three have p as a common divisor and $p \neq 2$. It will also follow that $(c - a)/2 = u^2$ and $(c + a)/2 = v^2$ are integers with p as a common divisor. This will contradict u, v *being* relatively prime. So a, b, c must be relatively prime.

For the second statement, we introduce modulo arithmetic. If r, s are integers having the same remainder upon division by a positive integer m, then we say *r is congruent to s modulo m* and let us denote this by $r \equiv s \pmod{m}$. For example, $r \equiv 0$ or $1 \pmod{2}$ depending on whether r is even or odd. From the definition, we see that congruence is an equivalence relation between r and s. Also, if $r \equiv s \pmod{m}$ and $r' \equiv s' \pmod{m}$, then $r + r' \equiv s + s' \pmod{m}$, $r - r' \equiv s - s' \pmod{m}$, $rr' \equiv ss' \pmod{m}$ and $r^k \equiv s^k \pmod{m}$ for any positive integer k.

In working with squares, modulo 4 is often considered. This comes from the observation that $r^2 \equiv 0$ or $1 \pmod{4}$ depending on r is even or odd. Now, if $a^2 + b^2 = c^2$, then $a^2 + b^2 \equiv 0$ or $1 \pmod{4}$.

(continued on page 4)

Julia Sets for Various Values of c



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* Solutions to the following problems should be submitted by *March 31, 1995.*

Problem 6. For quadratic polynomials $P(x) = ax^2 + bx + c$ with real coefficients satisfying $|P(x)| \le 1$ for $-1 \le x \le 1$, find the maximum possible values of b and give a polynomial attaining the maximal b coefficient.

Problem 7. If positive integers a, b, c satisfy $a^2 + b^2 = c^2$, show that there are at least three noncongruent right triangles with integer sides having hypotenuses all equal to c^3.

Problem 8. (1963 Moscow Mathematical Olympiad) Let $a_1 = a_2 = 1$ and $a_n = (a_{n-1}^2 + 2)/a_{n-2}$ for $n = 3, 4, \cdots$. Show that a_n is an integer for $n = 3, 4, \cdots$.

Problem 9. On sides AD and BC of a convex quadrilateral $ABCD$ with $AB < CD$, locate points F and E, respectively, such that $AF/FD = BE/EC = AB/CD$. Suppose EF when extended beyond F meets line BA at P and meets line CD at Q. Show that $\angle BPE = \angle CQE$.

Problem 10. Show that every integer $k > 1$ has a multiple which is less than k^4 and can be written in base 10 with at most four different digits. [*Hint:* First consider numbers with digits 0 and 1 only.] (This was a problem proposed by Poland in a past IMO.)

****

Solutions

Problem 1. The sum of two positive integers is 2310. Show that their product is not divisible by 2310.

Solution: W. H. FOK, Homantin Government Secondary School.

Let x, y be two positive integers such that $x + y = 2310$. Suppose xy is divisible by 2310, then $xy = 2310n$ for some positive integer n. We get $x + (2310n/x) = 2310$. So $x^2 - 2310x + 2310n = 0$. It follows the discriminant $\Delta = 2310^2 - 4(2310n) = 2^2 \times 3 \times 5 \times 7 \times 11 \times (1155 - 2n)$ must be a perfect square. Then for some positive integer k, $1155 - 2n = 3 \times 5 \times 7 \times 11 \times k^2 = 1155k^2 \ge 1155$, which is a contradiction. So xy is not divisible by 2310.

Comments: A similar problem appeared in the magazine *Quantum*, Sept./Oct. 1993, p. 54, published by Springer-Verlag.

Other commended solvers: **AU Kwok Nin** (Tsung Tsin College), **HO Wing Yip** (Clementi Secondary School), **POON Wai Hoi Bobby** (St. Paul's College) and **SZE Hoi Wing** (St. Paul's Co-ed College).

Problem 2. Given N objects and $B(\ge 2)$ boxes, find an inequality involving N and B such that if the inequality is satisfied, then at least two of the boxes have the same number of objects.

Solution: POON Wai Hoi Bobby, St. Paul's College.

Denote the number of objects in the kth box by N_k. Suppose no two boxes have the same number of objects. Then $N = N_1 + N_2 + \cdots + N_B \ge 0 + 1 + 2 + \cdots + (B-1) = B(B-1)/2$. So if $N < B(B-1)/2$, then at least two of the boxes have the same number of objects.

Other commended solvers: **CHAN Wing Sum** (HKUST), **W. H. FOK** (Homantin Government Secondary School), and **HO Wing Yip** (Clementi Secondary School).

Problem 3. Show that for every positive integer n, there are polynomials $P(x)$ of degree n and $Q(x)$ of degree n-1 such that $(P(x))^2 - 1 = (x^2-1)(Q(x))^2$.

Solution: POON Wai Hoi Bobby, St. Paul's College.

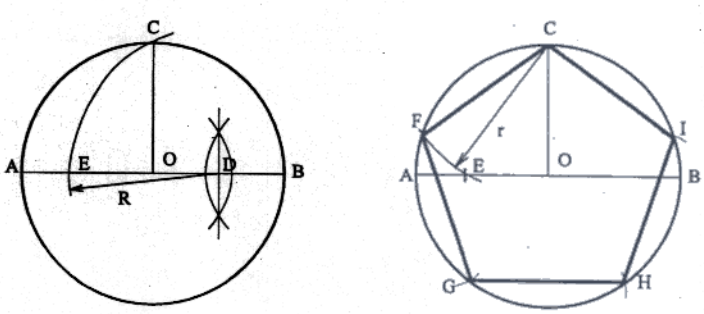
For $k = 1, 2, \cdots$, define $P_k(x)$, $Q_k(x)$ by $P_1(x) = x$, $Q_1(x) = 1$, $P_{k+1}(x) = xP_k(x) + (x^2-1)Q_k(x)$ and $Q_{k+1}(x) = P_k(x) + xQ_k(x)$. We can check that the degree of P_n is n and the degree of Q_n is n-1 by showing inductively that $P_n(x) = 2^{n-1}x^n + \cdots$ and $Q_n(x) = 2^{n-1}x^{n-1} + \cdots$. For the problem, when $n = 1$, $P_1(x)^2 - 1 = x^2 - 1 = (x^2-1)Q_1(x)^2$. Suppose the case $n = k$ holds. Then

$$\begin{aligned} P_{k+1}(x)^2 - 1 &= [xP_k(x) + (x^2-1)Q_k(x)]^2 - 1 \\ &= (x^2-1)[P_k(x)^2 + 2xP_k(x)Q_k(x) \\ &\quad + (x^2-1)Q_k(x)^2] + P_k(x)^2 - 1 \\ &= (x^2-1)[P_k(x)^2 + 2xP_k(x)Q_k(x) \\ &\quad + (x^2-1)Q_k(x)^2] + (x^2-1)Q_k(x)^2 \\ &= (x^2-1)Q_{k+1}(x)^2. \end{aligned}$$

Comments: The solvers mainly observed that if we substitute $x = \cos\theta$, then $P_k(\cos\theta) = \cos k\theta$ and $Q_k(\cos\theta) = \sin k\theta / \sin\theta$. The recurrence relations for P_{k+1} and Q_{k+1} are just the usual identities for $\cos(k\theta + \theta)$ and $\sin(k\theta + \theta)$. The polynomials P_k, Q_k are

(*continued on page 4*)

Construction Without Words: Inscribe a Regular Pentagon in a Unit Circle



Side length $= 2\sin 36° = \dfrac{\sqrt{10 - 2\sqrt{5}}}{2} = \sqrt{1 + \left(\dfrac{\sqrt{5}-1}{2}\right)^2}$.

How would you construct a regular 17-gon inscribed in a given circle?

Problem Corner

(*continued from page 3*)

called *Chebychev polynomials* and have many interesting properties.

We thank Professor Andy Liu (University of Alberta, Canada) for informing us that his colleague Professor Murray Klamkin located this problem in Goursat-Hedrick's "A Course in Mathematical Anaysis", vol. 1, p. 32, published by Ginn and Company in 1904. Professor Klamkin has a calculus solution, first showing Q divides P', then obtaining $Q = nP'$ and solving a differential equation in P to get $P(x) = \cos(n \arccos x)$. Professor Liu also forwarded an alternative recurrence approach by Byung-Kyu Chun, a Korean-Canadian secondary school student. He observed that $P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x)$ and $Q_n(x) = 2xQ_{n-1}(x) - Q_{n-2}(x)$ and showed by simultaneous induction that $P_n(x)P_{n-1}(x) - x = (x^2-1)Q_n(x)Q_{n-1}(x)$ and $P_n(x)^2 - 1 = (x^2-1)Q_n(x)^2$.

Other commended solver: **HO Wing Yip** (Clementi Secondary School).

Problem 4. If the diagonals of a quadrilateral in the plane are perpendicular, show that the midpoints of its sides and the feet of the perpendiculars dropped from the midpoints to the opposite sides lie on a circle.

Solution: Independent solution by **W. H. FOK** (Homantin Government Secondary School) and **POON Wai Hoi Bobby** (St. Paul's College).

Let $ABCD$ be a quadrilateral such that AC is perpendicular to BD. Let E, F, G, H be the midpoints of AB, BC, CD, DA, respectively. By the midpoint theorem, EH, BD, FG are parallel to each other and so are EF, AC, HG. Since AC and BD are perpendicular, $EFGH$ is a rectangle. Hence E, F, G, H are concyclic.

Let M be the foot of the perpendicular from E to CD, then $\angle EMG = \angle EFG = 90^0$. So E, F, M, G, H lie on a circle. Similarly, the other feet of perpendiculars are on the same circle.

Problem 5. (*1979 British Mathematical Olympiad*) Let $a_1, a_2, ..., a_n$ be n distinct positive odd integers. Suppose all the differences $|a_i\text{-}a_j|$ are distinct, $1 \le i < j \le n$. Prove that $a_1 + a_2 + \cdots + a_n \ge n(n^2+2)/3$.

Solution: Independent solution by **Julian CHAN Chun Sang** (Lok Sin Tong Wong Chung Ming Secondary School), **W. H. FOK** (Homantin Government Secondary School) and **HO Wing Yip** (Clementi Secondary School).

Without loss of generality, suppose $a_1 < a_2 < \cdots < a_n$. For $k = 2, 3, ..., n$, since the differences are distinct, $a_k = a_1 + (a_2 - a_1) + \cdots + (a_k - a_{k-1}) \ge 1 + (2 + 4 + \cdots + 2(k-1)) = 1 + k^2 - k$. Summing from $k = 1$ to n, we get $a_1 + a_2 + \ldots + a_n \ge n(n^2+2)/3$.

Comments: Ho Wing Yip proved the result by induction on n, which did not require the formula for summing k^2 in the last step.

Pythagorean Triples

(*continued from page 2*)

So, if a, b, c are also relatively prime, then one of a or b is odd and the other is even. Let us say a is odd and b is even. Then c is odd and it follows $m = (c-a)/2$ and $n = (c+a)/2$ are positive integers. Note a $(= m-n)$ and c $(= m+n)$ relatively prime implies m, n cannot have a common prime divisor. Now considering the prime factorization of $(b/2)^2$, which equals mn, it follows that both m and n are perfect squares with no common prime divisors. Let us say $m = u^2$ and $n = v^2$. Then $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$.

Example 1. Show that there are exactly three right triangles whose sides are integers while the area is twice the perimeter as numbers. (This was a problem on the 1965 Putnam Exam, a North American Collegiate Competition.)

Solution: For such a triangle, the sides are of the form $a = (u^2 - v^2)d$, $b = 2uvd$ and $c = (u^2 + v^2)d$, where u, v are relatively prime, $u > v$, one is odd, the other even and d is the greatest common divisor of the three sides. The condition $ab/2 = 2(a+b+c)$ expressed in terms of u, v, d can be simplified to $(u\text{-}v)vd = 4$. It follows that $u - v$ being odd must be 1. Then $v = 1, 2$ or 4; $u = 2, 3$ or 5; $d = 4, 2$ or 1 corresponding to the 12-16-20, 10-24-26 and 9-40-41 triangles.

Example 2. Show that there are infinitely many points on the unit circle such that the distance between any two of them is rational. (This was essentially a problem in the 1975 International Mathematical Olympiad).

Solution: Let $A = (-1, 0)$, $B = (1, 0)$ and O be the origin. Consider all points P such that $AP = 2(u^2\text{-}v^2)/(u^2+v^2)$ and $BP = 4uv/(u^2+v^2)$, where u, v are as in the theorem. Since $AP^2 + BP^2 = AB^2$, all such P's are on the unit circle. Using similar triangles, we find the coordinates of P is (x,y), where $x = (AP^2/2) - 1$ and $y = \pm AP{\cdot}BP/2$ are both rational. Let $\theta = \angle BOP = 2\angle BAP$. Then $\cos(\theta/2) = (1+x)/AP$ and $\sin(\theta/2) = |y|/AP$ are rational. Finally, for two such points P and P', $PP' = 2|\sin(\theta\text{-}\theta')/2| = 2|\sin(\theta/2)\cos(\theta'/2) - \cos(\theta/2)\sin(\theta'/2)|$ is rational.

Example 3. Find all positive integral solutions of $3^x + 4^y = 5^z$. (cf. W. Sierpinski, On the Equation $3^x + 4^y = 5^z$ (Polish), Wiadom. Mat.(1955/56), pp. 194-5.)

Solution. We will show there is exactly one solution set, namely $x = y = z = 2$. To simplify the equation, we consider modulo 3. We have $1 = 0 + 1^y \equiv 3^x + 4^y = 5^z \equiv (-1)^z \pmod 3$. It follows that z must be even, say $z = 2w$. Then $3^x = 5^z - 4^y = (5^w + 2^y)(5^w - 2^y)$. Now $5^w + 2^y$ and $5^w - 2^y$ are not both divisible by 3, since their sum is not divisible by 3. So, $5^w + 2^y = 3^x$ and $5^w - 2^y = 1$. Then, $(-1)^w + (-1)^y \equiv 0 \pmod 3$ and $(-1)^w - (-1)^y \equiv 1 \pmod 3$. Consequently, w is odd and y is even. If $y > 2$, then $5 \equiv 5^w + 2^y = 3^x \equiv 1$ or $3 \pmod 8$, a contradiction. So $y = 2$. Then $5^w - 2^y = 1$ implies $w = 1$ and $z = 2$. Finally, we get $x = 2$.

Olympiad Corner

(*continued from page 1*)

the jth contestant. Prove that there are exactly 40 winning triangles in this tournament.

Question 3. Find all the non-negative integers x, y, and z satisfying that $7^x + 1 = 3^y + 5^z$.

<u>Second Day</u>

Question 4. Suppose that $yz + zx + xy = 1$ and x, y, and $z \ge 0$. Prove that $x(1\text{-}y^2)(1\text{-}z^2) + y(1\text{-}z^2)(1\text{-}x^2) + z(1\text{-}x^2)(1\text{-}y^2) \le 4\sqrt{3}/9$.

Question 5. Given that a function $f(n)$ defined on natural numbers satisfies the conditions: $f(n) = n - 12$ if $n > 2000$, and $f(n) = f(f(n+16))$ if $n \le 2000$.

(a) Find $f(n)$.

(b) Find all solutions to $f(n) = n$.

Question 6. Let m and n be positive integers where m has d digits in base ten and $d \le n$. Find the sum of all the digits (in base ten) of the product $(10^n - 1)m$.

Mathematical Excalibur

Volume 1, Number 3 May - June, 1995

Olympiad Corner

The Seventh Asian Pacific Mathematics Olympiad was held on March 18, 1995. The five problems given in this contest are listed below for you to try. Time allowed was four hours. - *Editors*

Question 1. Determine all sequences of real numbers $a_1, a_2, \cdots, a_{1995}$ which satisfy:

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1)$$

for $n = 1, 2, \cdots, 1994$, and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

Question 2. Let $a_1, a_2, \cdots, a_n$ be a sequence of integers with values between 2 and 1995 such that:

i) any two of the a_i's are relatively prime.

ii) each a_i is either a prime or a product of different primes.

Determine the smallest possible value of n to make sure that the sequence will contain a prime number.

Question 3. Let $PQRS$ be a cyclic quadrilateral (i.e., P, Q, R, S all lie on a circle) such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set A of points of tangency of circles in these two sets.

(*continued on page 4*)

Editors: Cheung, Pak-Hong, Curr. Studies, HKU
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Leung, Tat-Wing, Appl. Math Dept, HKPU
Li, Kin-Yin, Math Dept, HKUST
Ng, Keng Po Roger, ITC, HKPU

Artist: Yeung, Sau-Ying Camille, Fine Arts Dept, CU

Acknowledgment: Thanks to Martha A. Dahlen, Technical Writer, HKUST, for her comments.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email address, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is June 10, 1995. Send all correspondence to:

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Similar Triangles via Complex Numbers

Kin-Yin Li

Similar triangles are familiar to students who studied geometry. Here we would like to look at an algebraic way of describing similar triangles by complex numbers. Recall that every point Z on the coordinate plane corresponds to a complex number $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$ are the polar coordinates of z. (From now on, we will use capital letters for points and small letters for the corresponding complex numbers.)

In general, there are two possible cases for similar triangles. Two triangles are said to be *directly similar* if one can be obtained by translating and rotating the other on the plane, then scaling up or down. (Note a triangle is not directly similar to its reflection unless it is isosceles or equilateral.) Suppose $\Delta Z_1Z_2Z_3$ is directly similar to $\Delta W_1W_2W_3$. Then $Z_2Z_1/Z_3Z_1 = W_2W_1/W_3W_1$ and $\angle Z_2Z_1Z_3 = \angle W_2W_1W_3$. These two equations are equivalent to $|z_2-z_1|/|z_3-z_1| = |w_2-w_1|/|w_3-w_1|$ and $\arg((z_2-z_1)/(z_3-z_1)) = \arg((w_2-w_1)/(w_3-w_1))$, which say exactly that

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

Reversing steps, we see that the equation implies the triangles are directly similar. For the case $\Delta Z_1Z_2Z_3$ directly similar to the reflection of $\Delta W_1W_2W_3$, the equation is

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{\overline{w_2} - \overline{w_1}}{\overline{w_3} - \overline{w_1}}$$

because $\overline{w_1}, \overline{w_2}, \overline{w_3}$ provide a reflection of w_1, w_2, w_3.

Let $\Delta W_1W_2W_3$ be the equilateral triangle with vertices at $1, \omega, \omega^2 (= \overline{\omega})$, where $\omega = (-1 \pm i\sqrt{3})/2$ is a cube root of unity. We observe that $w_1+\omega w_2+\omega^2 w_3 = 1+\omega^2+\omega^4 = 0$. One can show that this equation is satisfied by any equilateral triangle in general. A triangle $\Delta Z_1Z_2Z_3$ is equilateral if and only if $(z_3-z_1)/(z_2-z_1) = (w_3-w_1)/(w_2-w_1) = -\omega^2$. (Note that $-\omega^2 = \pm(\cos 60° + i\sin 60°)$.) This equation can be simplified to $z_1+\omega z_2+\omega^2 z_3 = 0$ by utilizing $1+\omega+\omega^2=0$. Therefore, a triangle $\Delta Z_1Z_2Z_3$ is equilateral if and only if $z_1+\omega z_2+\omega^2 z_3 = 0$. Here $\omega = (-1 + i\sqrt{3})/2$ when Z_1, Z_2, Z_3 are in counterclockwise direction and $\omega = (-1 - i\sqrt{3})/2$ when Z_1, Z_2, Z_3 are in clockwise direction.

Example 1. (Napolean Triangle Theorem) Given ΔABC. Draw equilateral triangles DBA, ECB, FAC on the opposite sides of AB, BC, CA as ΔABC, respectively. Let G, H, I be the centroids of ΔDBA, ΔECB, ΔFAC, respectively. Show that ΔGHI is equilateral.

Solution. Since $d + \omega b + \omega^2 a = 0$, $e + \omega c + \omega^2 b = 0$, $f + \omega a + \omega^2 c = 0$ and $\omega^3 = 1$, we have

$$\begin{aligned} &g + \omega h + \omega^2 i \\ &= (a+d+b)/3+\omega(b+e+c)/3+\omega^2(c+f+a)/3 \\ &= [(d+\omega b+\omega^2 a) + \omega(e+\omega c+\omega^2 b) \\ &\quad + \omega^2(f+\omega a+\omega^2 c)]/3 = 0. \end{aligned}$$

Example 2. Given an acute triangle $A_1A_2A_3$, let H_1, H_2, H_3 be the feet of the altitudes dropped from A_1, A_2, A_3, respectively. Show that each of the triangles $A_1H_2H_3$, $A_2H_3H_1$, $A_3H_1H_2$ is similar to $\Delta A_1A_2A_3$.

Solution. Set up coordinates so that $A_1 = (0,0)$, $A_2 = (t,0)$ and $A_3 = (x,y)$, i.e., $a_1 = 0$, $a_2 = t$, $a_3 = x+iy$. Observe that $A_1H_2 = A_1A_2 \cos \angle A_1 = tx/\sqrt{x^2+y^2}$. Thus $h_2 = (tx/\sqrt{x^2+y^2})(a_3/|a_3|) = tx(x+iy)/(x^2+y^2)$. Also, $h_3 = x$. Now

$$\frac{h_2 - a_1}{h_3 - a_1} = \frac{t(x+iy)}{x^2+y^2} = \frac{t}{x-iy} = \frac{\overline{a_2} - \overline{a_1}}{\overline{a_3} - \overline{a_1}}.$$

So, in fact, $\Delta A_1H_2H_3$ is similar to (the reflection of) $\Delta A_1A_2A_3$. By changing indices, we also get similarity for the other two triangles.

(*continued on page 4*)

From The Editors' Desk:

This is the last issue for the 94-95 academic year. Thanks for all the supports, comments, suggestions, and especially the elegant solutions for the Problem Corner. We will give out a few book prizes to show our appreciation. We are also planning a Best Paper Award for articles to be submitted in the next academic year. Details will be given in the September issue. Meanwhile, we encourage our readers to spend some spare time writing intriguing articles for the Mathematical Excalibur?

****

For the 95-96 academic year, we plan to have five issues to be delivered on Sept, Nov, Jan, Mar and May. If you would like to receive your personal copy directly, send five stamped self-addressed envelopes to Dr. Tsz-Mei Ko, Hong Kong University of Science and Technology, Department of Electrical and Electronic Engineering, Clear Water Bay, Kowloon. Please write "Math Excalibur 95-96" at the lower left corner on all five envelopes.

****

We have sent out the computer program FRACTINT to all interested readers. If you have requested but not yet received the software, contact Roger Ng.

****

Are you interested in math or in winning a math olympiad gold medal? The Preliminary Selection Exam for the 1996 Hong Kong Math Olympiad Team will be held in Hong Kong Polytechnic University on May 27, 1995. You may ask your math teacher for further information if you are interested in participating in this exam. The 1996 IMO will be held in India.

Cryptarithms and Alphametics

Tsz-Mei Ko

A cryptarithm or alphametic is a puzzle to find the original digits in an encrypted equation which is made by substituting distinct letters for distinct digits in a simple arithmetic problem. Here is an example. Consider the alphametic

```
   AT
+   A
  TEE
```

in which each letter represents a distinct digit. The puzzle is to find the original digits each letter represents so that the result is arithmetically correct.

To solve this puzzle, we may reason as follows. Since T is the "carry" from the "tens" column, T must be equal to 1 and thus we get

```
   A1
+   A
  1EE
```

Now, on the tens column, since $A \neq E$, there must be a carry from the units column, i.e., A+1 = 10+E. Thus A=9 and E=0. Therefore, the solution should be

```
   91
+   9
  100
```

We may check our solution that it is arithmetically correct and each letter indeed represents a distinct digit (with A=9, E=0 and T=1). Also, from our reasoning, we see that the solution for this puzzle is unique.

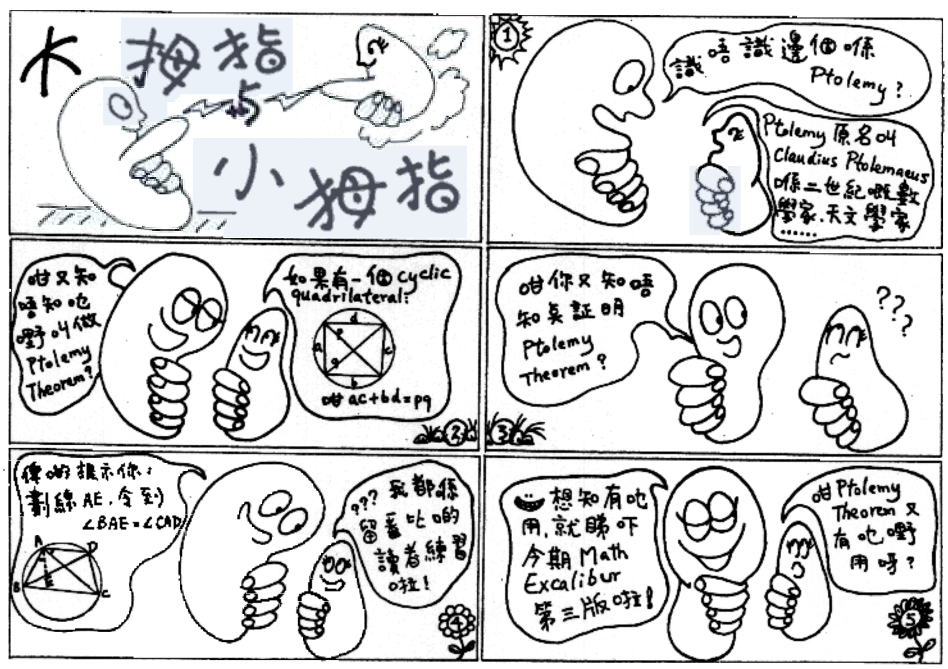
There are many amusing alphametics that make sense in English or some language. Here is one with a unique solution. Do you think you can solve it?

```
  FORTY
    TEN
+   TEN
  SIXTY
```

How about this cryptarithm in which the phrase "Qui Trouve Ceci" means "Who can solve this?" Each letter represents a distinct digit and each # represents any digit (not necessary to be distinct).

```
         CECI
QUI ) TROUVE
      ###
       ###
       ###
        ###
        ##E
         ###
         ###
           V
```

Mathematical Thumbnotes:



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* Solutions to the following problems should be submitted by *June 10, 1995.*

Problem 11. Simplify

$$\sum_{n=1}^{1995} \tan(n)\tan(n+1).$$

(There is an answer with two terms involving tan 1, tan 1996 and integers.)

Problem 12. Show that for any integer $n > 12$, there is a right triangle whose sides are integers and whose area is between n and $2n$. (*Source:* 1993 Korean Mathematical Olympiad.)

Problem 13. Suppose x_k, y_k ($k = 1, 2, \cdots, 1995$) are positive and $x_1 + x_2 + \cdots + x_{1995} = y_1 + y_2 + \cdots + y_{1995} = 1$. Prove that

$$\sum_{k=1}^{1995} \frac{x_k y_k}{x_k + y_k} \le \frac{1}{2}.$$

Problem 14. Suppose ΔABC, $\Delta A'B'C'$ are (directly) similar to each other and $\Delta AA'A''$, $\Delta BB'B''$, $\Delta CC'C''$ are also (directly) similar to each other. Show that $\Delta A''B''C''$ is (directly) similar to ΔABC.

Problem 15. Is there an infinite sequence $a_0, a_1, a_2, \cdots$ of nonzero real numbers such that for $n = 1, 2, 3, \cdots$, the polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has exactly n distinct real roots? (*Source:* 1990 Putnam Exam.)

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Solutions

Problem 6. For quadratic polynomials $P(x) = ax^2 + bx + c$ with real coefficients satisfying $|P(x)| \le 1$ for $-1 \le x \le 1$, find the maximum possible values of b and give a polynomial attaining the maximal b coefficient.

Solution: Independent solution by **KWOK Wing Yin** (St. Clare's Girls' School), **Bobby POON Wai Hoi** (St. Paul's College), **SZE Hoi WING** (St. Paul's Co-ed College) and **WONG Chun Keung** (St. Paul's Co-ed College).

Since $b = (P(1) - P(-1))/2 \le 2/2 = 1$, the maximum possible values of b is at most 1. Now the polynomial $P(x) = x^2/2 + x - 1/2 = (x+1)^2/2 - 1$ satisfy the condition $|P(x)| \le 1$ for $-1 \le x \le 1$ because $0 \le x+1 \le 2$. So the maximum of b is 1.

Comments: With $-1 \le x \le 1$ replaced by $0 \le x \le 1$, the problem appeared in the 1968 Putnam Exam.

Other commended solvers: **CHAN Wing Sum** (HKUST), **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School), **W. H. FOK** (Homantin Government Secondary School), **Michael LAM Wing Young** (St. Paul's College), **LIN Kwong Shing** (University of Illinois) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 7. If positive integers a, b, c satisfy $a^2 + b^2 = c^2$, show that there are at least three noncongruent right triangles with integer sides having hypotenuses all equal to c^3.

Solution: Independent solution by **LIN Kwong Shing** (University of Illinois) and **LIU Wai Kwong** (Pui Tak Canossian College).

Without loss of generality, assume $a \ge b$. The first triangle comes from $(c^3)^2 = (a^2+b^2)c^4 = (ac^2)^2 + (bc^2)^2$. The second triangle comes from $(c^3)^2 = (a^2 + b^2)^2c^2 = (a^4 - 2a^2b^2 + b^4 + 4a^2b^2)c^2 = [(a^2-b^2)c]^2 + [2abc]^2$. The third triangle comes from $(c^3)^2 = (a^2+b^2)^3 = (a^6 - 6a^4b^2 + 9a^2b^4) + (9a^4b^2 - 6a^2b^4 + b^6) = [a|a^2-3b^2|]^2 + [b(3a^2-b^2)]^2$.

For the first and second triangles, $2abc = ac^2$ or bc^2 implies $c = 2b$ or $2a$. Substitute $c = 2b$ or $2a$ into $a^2 + b^2 = c^2$ will lead to the contradiction $\sqrt{3} = a/b$ or b/a. So these two triangles cannot be congruent.

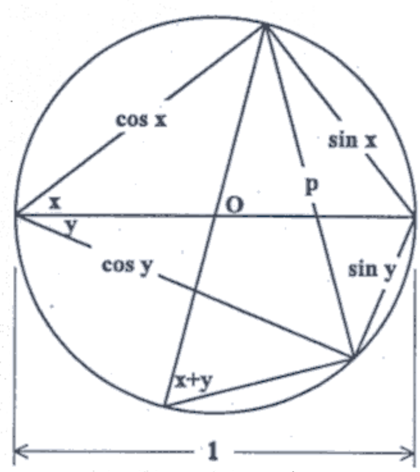
Similarly, for the first and third triangles, since $b(3a^2-b^2) = ac^2$ or bc^2 will lead to $\sqrt{2} = (a+b)/a$ or c/a by simple algebra, these two triangles cannot be congruent.

Finally, for the second and third triangles, $b(3a^2-b^2) = (a^2-b^2)c$ or $2abc$ will lead to $\sqrt{5} = (c-b)/b$ or $(c+a)/a$ (again by simple algebra). So these two triangles cannot be congruent.

Comments: Au Kwok Nin obtained the same triangles *systematically* by writing $c^6 = (c^3\cos n\theta)^2 + (c^3\sin n\theta)^2$ for $n = 1, 2, 3$ and expressed $\cos n\theta$, $\sin n\theta$ in terms of $\cos\theta = a/c$, $\sin\theta = b/c$. Cheung Kwok

(continued on page 4)

Proof Without Words



sin(x+y) = p = sin x cos y + cos x sin y

Problem Corner

(*continued from page 3*)

Koon observed that the greatest common divisors of the sides of the triangles were divisible by different powers of c, hence the triangles could not be congruent.

Other commended solvers: **AU Kwok Nin** (Tsung Tsin College), **CHAN Wing Sum** (HKUST), **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School) and **FUNG Tak Kwan & POON Wing Chi** (La Salle College).

Problem 8. (1963 Moscow Mathematical Olympiad) Let $a_1 = a_2 = 1$ and $a_n = (a_{n-1}^2 + 2)/a_{n-2}$ for $n = 3, 4, \cdots$. Show that a_n is an integer for $n = 3, 4, \cdots$.

Solution: Independent solution by **CHAN Chi Kin** (Pak Kau English School), **Michael LAM Wing Young** and **Bobby POON Wai Hoi** (St. Paul's College).

Since $a_1 = a_2 = 1$ and $a_n a_{n-2} = a_{n-1}^2 + 2$ for all integer $n \geq 3$, we have $a_n \neq 0$ and $a_n a_{n-2} - a_{n-1}^2 = 2 = a_{n+1}a_{n-1} - a_n^2$ for $n \geq 3$. We obtain $(a_{n+1}+a_{n-1})/a_n = (a_n+a_{n-2})/a_{n-1}$ by rearranging terms. Hence, the value of $(a_n+a_{n-2})/a_{n-1}$ is constant for $n \geq 3$. Since $(a_3+a_1)/a_2 = 4$, we have $(a_n+a_{n-2})/a_{n-1} = 4$, i.e., $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 3$. This shows that a_n is in fact an odd integer for all $n \geq 1$.

Comments: Most solvers observed that a_n depends on a_{n-1} and a_{n-2}, and thus guessed that a_n can be expressed as $ra_{n-1} + sa_{n-2}$ for some r, s. They went on to find $r = 4$ and $s = -1$ by setting $n = 3, 4$, then confirmed the guess by mathematical induction.

Other commended solvers: **CHAN Wing Sum** (HKUST), **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School), **HUI Yue Hon Bernard** (HKUST), **LIN Kwong Shing** (University of Illinois), **LIU Wai Kwong** (Pui Tak Canossian College) and **Alex MOK Chi Chiu** (Homantin Government Secondary School).

Problem 9. On sides AD and BC of a convex quadrilateral $ABCD$ with $AB < CD$, locate points F and E, respectively, such that $AF/FD = BE/EC = AB/CD$. Suppose EF when extended beyond F meets line BA at P and meets line CD at Q. Show that $\angle BPE = \angle CQE$.

Solution: Bobby POON Wai Hoi, St. Paul's College.

First construct parallelograms $ABGF$ and $CDFH$. Since BG, AD, CH are parallel, $\angle GBE = \angle HCE$. Also, $BG/CH = AF/DF = AB/CD = BE/CE$. So, ΔBGE is similar to ΔCHE. Then G, E, H must be collinear and $GE/HE = AB/CD = GF/HF$. Therefore, $\angle GFE = \angle HFE$ or $\angle BPE = \angle CQE$.

Other commended solvers: **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School), **W. H. FOK** (Homantin Government Secondary School), **Michael LAM Wing Young** (St. Paul's College) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 10. Show that every integer $k > 1$ has a multiple which is less than k^4 and can be written in base 10 with at most four different digits. [*Hint:* First consider numbers with digits 0 and 1 only.] (This was a problem proposed by Poland in a past IMO.)

Solution: Official IMO solution.

Choose n such that $2^{n-1} \leq k < 2^n$. Let S be the set of nonnegative integers less than 10^n that can be written with digits 0 or 1 only. Then S has 2^n elements and the largest number m in S is composed of n ones. Since $2^n > k$, by the pigeonhole principle, there are two numbers x, y in S which have the same remainder upon division by k, i.e., $x \equiv y \pmod k$. Then $|x-y|$ is a multiple of k and

$$|x-y| \leq m < 10^{n-1} \times 1.2 < 16^{n-1} \leq k^4.$$

Finally, considering the cases of subtracting a 0,1 digit by another 0,1 digit with possible carries, we see that $|x - y|$ can be written with digits 0, 1, 8, 9 only.

Similar Triangles ...

(*continued from page 1*)

Example 3. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. For $s \geq 4$, define $A_s = A_{s-3}$. For $k \geq 0$, define P_{k+1} to be the image of P_k under rotation with center at A_{k+1} through angle 120° clockwise. Prove that if $P_{1986} = P_0$, then $\Delta A_1A_2A_3$ is equilateral. (This was a problem on the 1986 IMO.)

Solution. We have $p_{k+1}-a_{k+1} = \omega(p_k-a_{k+1})$, where $\omega = \cos 120° - i\sin 120° = (-1-i\sqrt{3})/2$. Adding proper multiples of these equations (so as to cancel all p_k's), we consider

$$\begin{aligned}&(p_{1986}-a_{1986})+\omega(p_{1985}-a_{1985})+\omega^2(p_{1984}-a_{1984}) + \cdots + \omega^{1985}(p_1-a_1)\\ &= \omega(p_{1985} - a_{1986}) + \omega^2(p_{1984} - a_{1985}) + \omega^3(p_{1983} - a_{1984}) + \cdots + \omega^{1986}(p_0 - a_1).\end{aligned}$$

Cancelling common terms on both sides, noting $\omega^{1986}p_0 = p_0 = p_{1986}$, then transposing all terms on the left side to the right, we get

$$\begin{aligned}0 &= (1-\omega)(a_{1986}+\omega a_{1985}+\omega^2 a_{1984}+\cdots+\omega^{1985}a_1)\\ &= 662(1-\omega)(a_3+\omega a_2+\omega^2 a_1)\end{aligned}$$

by the definition of a_k and the fact $\omega^3 = 1$. Since $\omega \neq 1$, $\Delta A_1A_2A_3$ is equilateral.

Olympiad Corner

(*continued from page 1*)

Question 4. Let C be a circle with radius R and center O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles $SAMB$, $SBN'A'$, $SA'M'B'$, and $SB'NA$. Find the set of all points M, N', M', and N when A moves around the whole circle.

Question 5. Find the minimum positive integer k such that there exists a function f from the set Z of all integers to $\{1, 2, \cdots, k\}$ with the property that $f(x) \neq f(y)$ whenever $|x-y| \in \{5, 7, 12\}$.

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Olympiad News:

Congratulations to CHEUNG Kwok Koon (F. 7, SKH Bishop Mok Sau Tseng Secondary School), HO Wing Yip (F. 6, Clementi Secondary School), MOK Tze Tao (F. 5, Queen's College), POON Wai Hoi Bobby (F. 6, St. Paul's College), WONG Him Ting (F. 7, Salesian English School) and YU Chun Ling (F. 6, Ying Wa College) for being selected as the 1995 Hong Kong Mathematical Olympiad Team Members. The selection was based on their outstanding performances in the Hong Kong Math Olympiad Training Program. They will represent Hong Kong to participate in the 36th International Mathematical Olympiad (IMO) to be held in Toronto, Canada this summer. Hong Kong was ranked 16 among 69 participating teams in 1994.

Mathematical Excalibur

Volume 1, Number 4 September-October, 1995

Olympiad Corner

The 36th International Mathematical Olympiad wad held in Toronto, Canada on July, 1995. The following six problems were given to the contestants. (The country inside the parantheses are the problem proposers.) -Editors

First Day

Question 1. (Bulgaria)
Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

Question 2. (Russia)
Let a, b and c be positve real numbers such that abc=1. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

(continued on page 4)

Editors: Cheung, Pak-Hong, Curr. Studies, HKU
Ko, Tsz-Mei, EEE Dept, HKUST
Leung, Tat-Wing, Appl. Math Dept, HKPU
Li, Kin-Yin, Math Dept, HKUST
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Artist: Yeung, Sau-Ying Camille, MFA, CU

Acknowledgment: Thanks to Debbie Leung for her help in typesetting.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is October 15, 1995. Send all correspondence to:

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Descartes' Rule of Signs

Andy Liu
University of Alberta, Canada

Let $P(x)$ be a polynomial of degree n with complex coefficients. The Fundamental Theorem of Algebra tells us that it has exactly n complex roots. We are interested in the number of real roots in the case where the coefficients are real. We may assume that the leading coefficient is 1 and the constant term is non-zero.

As an example, consider

$$p(x) = x^6-6x^5+10x^4-2x^3-3x^2+4x-12.$$

As it turns out, it has four real roots -1, 2 (with multiplicity 2) and 3, and two non-real roots i and $-i$.

In general, we may not be able to find the roots of $P(x)$. However, we can obtain some information about the number of positive roots from the number of sign-switches of $P(x)$. If we consider the sequence of the signs of the non-zero coefficients of $P(x)$ in order, a sign-switch is said to occur if a + is followed immediately by a – or vice versa.

For $p(x)$ above, the sequence is + – + – – + –. Hence the number of sign switches is 5.

The first part of Descartes' Rule of Signs states that the number of positive roots of $P(x)$ has the same parity as the number of sign-switches of $P(x)$. Clearly, the latter is even if and only if the constant term of $P(x)$ is positive (because the sign sequence begins and ends with +). What we have to prove is that the same goes for the number of positive roots of $P(x)$.

From the Fundamental Theorem of Algebra, $P(x)$ is a product of linear factors and irreducible quadratic factors. Now the constant term of a quadratic factor with a negative discriminant must be positive. The constant term of a linear factor is positive if and only if it corresponds to a negative root. It follows that the sign of the constant term of $P(x)$ is positive if and only if the number of positive roots of $P(x)$ is even.

Since the number of sign-switches of $p(x)$ is 5, we can tell that it has an odd number of positive roots without trying to find them.

The second part of Descartes' Rule of Signs states that the number of positive roots of $P(x)$ is less than or equal to the number of sign-switches of $P(x)$. We shall build up $P(x)$ as follows. Start with the product of all irreducible quadratic factors and all linear factors corresponding to negative roots. What we have to prove is that the number of sign-switches increases every time we introduce a linear factor corresponding to a positive root.

For any polynomial $Q(x)$ with real coefficients, leading coefficient 1 and a non-zero constant term, we group consecutive terms of the same signs together to express $Q(x)$ as an alternating sum of polynomials of positive coefficients. Then the sign-switches occur precisely between summands. We claim that when we multiply $Q(x)$ by $x-t$ for some positive number t, the original sign-switches are preserved, while at least one additional sign-switch occurs.

Consider each summand in turn. The leading coefficient is positive. This does not change after multiplication by x. However, we may have to combine it with $-t$ times the last term of the preceding summand. Since there is a sign-switch

(continued on page 2)

Descartes' Rule of Signs

(continued from page 1)

between the two summands, the term with which it is to be combined is also positive. This justifies the first claim. The second claim follows since the constant terms of $Q(x)$ and $(x-t)Q(x)$ have opposite signs. This completes the proof of Descartes' Rule of Signs.

Let us illustrate the proof of the second part with

$$p(x) = (x^2+1)(x+1)(x-2)^2(x-3).$$

We first let

$$q(x) = (x^2+1)(x+1) = x^3+x^2+x+1.$$

Since the number of sign-switches is 0, there is only one summand. We have

$$\begin{aligned} q_1(x) &= (x-2)\,q(x) \\ &= (x-2)\,(x^3+x^2+x+1) \\ &= x^4-x^3-x^2-x-2 \\ &= x^4-(x^3+x^2+x+2). \end{aligned}$$

$$\begin{aligned} q_2(x) &= (x-2)\,q_1(x) \\ &= (x-2)x^4 - (x-2)(x^3+x^2+x+2) \\ &= (x^5-2x^4) - (x^4-x^3-x^2-4) \\ &= x^5 - 3x^4 + x^3 + x^2 + 4. \end{aligned}$$

Note that we have combined the terms $-x^4$ and $-2x^4$ which have the same sign. Finally,

$$\begin{aligned} p(x) &= (x-3)\,q_2(x) \\ &= (x-3)x^5 - (x-3)(3x^4) + (x-3)(x^3+x^2+4) \\ &= (x^6-3x^5)-(3x^5-9x^4)+(x^4-2x^3-3x^2+4x-12) \\ &= x^6 - 6x^5 + 10x^4 - 2x^3 - 3x^2 + 4x - 12. \end{aligned}$$

We point out that using the same argument, we can prove that the number of negative roots of $P(x)$ is not greater than the number of sign-switches in $P(-x)$, and differs from it by an even number. For example, the number of sign-switches in $p(-x) = x^6+6x^5+10x^4+2x^3-3x^2-4x-12$ is 1, and we can conclude that $p(x)$ has exactly one negative root.

希臘幾何學的發展

林達威, 鄧智傑, 王俊威
Form 5, St. Paul's Co-ed College

「幾何」一詞，拉丁文是 geometria，其中"geo-"代表「地」（與 geography, geology 中"geo-"的意思一致），而"metria"則與今天英文的"metric"相關，代表「量度」。兩部份合起來，就是「量地的學問」-----原來古埃及的尼羅河每年都泛濫一次，摧毀河畔的農地，洪水過後，政府爲重新劃定的農地量度面積，以決定每戶所須繳付的賦稅；幾何學就在這情況下應運而生。後來，這些知識輾轉傳入希臘，逐漸發展成一套完整的學說。

希臘數學史可分爲三個時期：第一段從愛奧尼亞學派到柏拉圖學派爲止；第二段是亞歷山大前期，從歐幾里得到羅馬攻陷希臘爲止；最後則是亞歷山大後期。

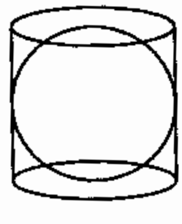
第一段時期的希臘共有六個主要學派，其中以奉行素食的畢達哥拉斯(Pythagoras)學派最負盛名，他們的「畢氏定理」(Pythagorean Theorem)是理科必備的工具。這時期，很多數學家都有重要的發現，雖然沒有建立一套完整的學說，但爲日後阿基米得等人的數學理論建立了一個良好的基礎。

第二段時期中最偉大的數學家可算是歐幾里得(Euclid)。歐幾里得深諳柏拉圖幾何的精髓，經過嚴謹的演繹和推論，寫成了<<原本>>(Elements)一書。<<原本>>爲幾何建立了一套完整的理論，在1637年笛卡爾(Descartes)引入「坐標幾何」前，它佔幾何學的領導地位。它也是用公理法建立起演繹數學體系的最早典範-----所謂「公理法」，就是從一些大家都公認可以接受，毋須加以証明的「公理」出發，通過合乎邏輯的推論而得出被驗証的結果。這可說是人類從直觀事物邁向抽象思維的重要一步！

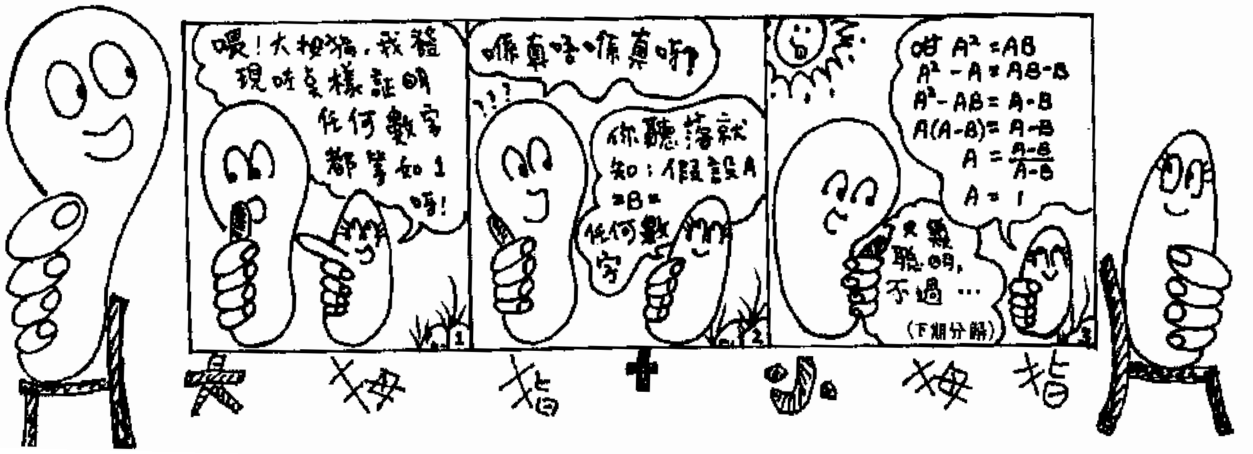
亞歷山大前期的另一位偉大的數學家是阿基米德。在他所著的<<圓的量度>>中，阿基米德利用外切與內接九十六邊形求得圓周率π的兩個近似值：

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}\text{。}$$

假設圓的半徑爲1，則圓周剛好爲2π，此數值必須大於內接六邊形的周界，而六邊形的周界爲6。故此求得π的下限爲3。同理，利用圓形的外切六邊形，可求得π的上限爲3.4641。假若我們把六邊形換作十二邊形、廿四邊形，…，則（內接或外切）多邊形的周界會越來越接近圓，而相應的上限及下限也會趨近π的眞實數值。阿基米得利用圓形的外切與內接九十六邊形，求得$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$。阿基米得的另一項建樹是體積的計算，例如圓球體積是它的外接圓柱體積的三份二。阿基米德還發現圓球的表面積恰巧也是外接圓柱表面積的三份二，他非常欣賞這定理，吩咐親人把這個圖形刻在他的墓碑上（見圖）。



最後一段是亞歷山大後期。這個時期的數學家，以歐幾里得的<<原本>>爲根據，作出了不少增潤修補的工作，而我們現在所學的平面幾何，亦在這時候逐漸形成。讀到這裏，同學們該明白到這門學問是眾多數學家集體智慧的結晶及長期辛勞的成果。



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address and school affiliation. Please send submissions to *Dr. Tsz-Mei Ko, Dept of EEE, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is October 15, 1995.

Problem 16. Let a, b, c, p be real numbers, with a, b, c not all equal, such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}=p$. Determine all possible values of p and prove that $abc+p=0$. (Source: 1983 Dutch Mathematical Olympiad.)

Problem 17. Find all sets of positive integers x, y and z such that $x \le y \le z$ and $x^y+y^z=z^x$.

Problem 18. For real numbers a, b, c, define

$f(a,b,c)=a+b-|a-b|-|a+b+|a-b|-2c|$.

Show that $f(a,b,c)>0$ if and only if $f(b,c,a)>0$ if and only if $f(c,a,b)>0$.

Problem 19. Suppose A is a point inside a given circle and is different from the center. Consider all chords (excluding the diameter) passing through A. What is the locus of the intersection of the tangent lines at the endpoints of these chords?

Problem 20. For $n>1$, let $2n$ chess pieces be placed on any $2n$ squares of an $n \times n$ chessboard. Show that there are 4 pieces among them that formed the vertices of a parallelogram. (Note that if $2n-1$ pieces are placed on the squares of the first column and the first row, then there is no parallelogram. So $2n$ is the best possible.)

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Solutions

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Problem 11. Simplify

$$\sum_{n=1}^{1995} \tan n \tan(n+1).$$

(There is an answer with two terms involving tan1, tan1996 and integers.)

Solution: Independent solutions by **Iris CHAN Chau Ping** (St. Catherine's School for Girls, Kwun Tong), **CHAN Chi Kin** (Pak Kau English School), **CHAN Sze Tai, Angie** (Ming Kei College), **CHAN Wing Sum** (HKUST), **CHOW Chak On** (HKUST), **CHUI Yuk Man** (Queen Elizabeth School), **LEUNG Ka Fai** (Ju Ching Chu Sceondary School (Yuen Long)), **LIU Wai Kwong** (Pui Tak Canossian College), **Alex MOK Chi Chiu** (Homantin Government Secondary School), **TAM Tak Wing** (Delia Memorial School (Yuet Wah)) and **WOO Chin Yeung** (St. Peter's Secondary School).

From $\tan 1 = \tan[(n+1)-n] = (\tan(n+1)-\tan n)/(1+\tan n\tan(n+1))$, we get

$$\sum_{n=1}^{1995}\tan(n)\tan(n+1)=\sum_{n=1}^{1995}\left(\frac{\tan(n+1)-\tan(n)}{\tan 1}-1\right)$$
$$=\frac{\tan 1996-\tan 1}{\tan 1}-1995=\frac{\tan 1996}{\tan 1}-1996.$$

Comments: This problem illustrates the *telescoping method* of summing a series, i.e., by some means, write a_n as $b_{n+1}-b_n$, then summing a_n will result in many cancellations yielding a simple answer.

Problem 12. Show that for any integer $n>12$, there is a right triangle whose sides are integers and whose area is between n and $2n$. (Source: 1993 Korean Mathematical Olympiad.)

Solution: **WONG Chun Keung**, St. Paul's Co-ed College.

Consider triangle A with sides $3d$, $4d$, $5d$, which has area $6d^2$. So for n in the interval $(3d^2+1, 6d^2-1)$, triangle A has an area between n and $2n$. For $d \ge 3$, $6d^2-1-[3(d+1)^2+1]=3(d-1)^2-8>0$. So the intervals $(3d^2+1, 6d^2-1)$ with $d=3, 4, 5, \ldots$ cover all positive integers n greater than or equal to 28. For $d=2$, triangle A has area 24, which takes care of the cases $n=13, 14, \ldots, 23$. Finally, the cases $n=24, 25, 26, 27$ are taken care of by the triangle with sides 5, 12, 13, which has area 30.

Other commended solvers: **CHAN Wing Sum** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 13. Suppose x_k, y_k $(k=1, 2, \ldots, 1995)$ are positive and $x_1+x_2+\cdots+x_{1995}=y_1+y_2+\cdots+y_{1995}=1$. Prove that

$$\sum_{k=1}^{1995}\frac{x_k y_k}{x_k+y_k}\le\frac{1}{2}.$$

Solution: Independent solution by **CHAN Chi Kin** (Pak Kau English School), **CHAN Wing Sum** (HKUST), **KWOK Wing Yin** (St. Clare's Girls' School) and **LEUNG Ka Fai** (Ju Ching Chu Secondary School (Yuen Long)).

Since $x_k y_k/(x_k+y_k) \le (x_k+y_k)/4$ (is equivalent to $(x_k-y_k)^2 \ge 0$ by simple algebra), we get

$$\sum_{k=1}^{1995}\frac{x_k y_k}{x_k+y_k}\le\sum_{k=1}^{1995}\frac{x_k+y_k}{4}=\frac{1}{2}.$$

Other commended solvers: **Iris CHAN Chau Ping** (St. Catherine's School for Girls, Kwun Tong), **CHEUNG Lap Kin** (Hon Wah Middle School), **CHOW Chak On** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College), **Alex MOK Chi Chiu** (Homantin Government Secondary School), **TAM Tak Wing** (Delia Memorial School (Yuet Wah)), **WONG Chun Keung** (St. Paul's Co-ed College) and **WOO Chin Yeung** (St. Peter's Secondary School).

Problem 14. If ΔABC, $\Delta A'B'C'$ are (directly) similar to each other and $\Delta AA'A''$, $\Delta BB'B''$, $\Delta CC'C''$ are also (directly) similar to each other, then show that $\Delta A''B''C''$, ΔABC are (directly) similar to each other.

Solution: Independent solution by **CHAN Wing Sum** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

We will use capital letters for points and small letters for the corresponding complex numbers. Since $\Delta AA'A''$, $\Delta BB'B''$, $\Delta CC'C''$ are (directly) similar to each other,

$$\frac{a''-a}{a'-a}=\frac{b''-b}{b'-b}=\frac{c''-c}{c'-c}=r.$$

Then $a''=ra'+(1-r)a$, $b''=rb'+(1-r)b$, $c''=rc'+(1-r)c$. Since ΔABC, $\Delta A'B'C'$ are (directly) similar to each other,

$$\frac{b-a}{c-a}=\frac{b'-a'}{c'-a'}.$$

Then

$$\frac{b''-a''}{c''-a''}=\frac{r(b'-a')+(1-r)(b-a)}{r(c'-a')+(1-r)(c-a)}=\frac{b-a}{c-a},$$

(continued on page 4)

Problem Corner

(continued from page 3)

which is equilvalent to $\Delta A''B''C''$ (directly) simlar to ΔABC.

Problem 15. Is there an infinite sequence $a_0, a_1, a_2, \cdots$ of non-zero real numbers such that for $n = 1, 2, 3, \cdots$, the polynomial

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

has exactly n distinct real roots? (Source: 1990 Putnam Exam.)

Solution: Yes. Take $a_0 = 1$, $a_1 = -1$ and proceed by induction. Suppose $a_0, \cdots, a_n$ have been chosen so that $P_n(x)$ has n distinct real roots and $P_n(x) \to \infty$ or $-\infty$ as $x \to \infty$ *depending upon whether* a_n *is positive or negative*. Suppose the roots of $P_n(x)$ is in the interval $(-T, T)$. Let $a_{n+1} = (-1)^{n+1}/M$, where M is chosen to be very large so that T^{n+1}/M is very small. Then $P_{n+1}(x) = P_n(x) + (-x)^{n+1}/M$ is very close to $P_n(x)$ on $[-T, T]$ because $|P_{n+1}(x) - P_n(x)| \le T^{n+1}/M$ for every x on $[-T, T]$. So, $P_{n+1}(x)$ has a sign change very close to every root of $P_n(x)$ and has the same sign as $P_n(x)$ at T. Since $P_n(x)$ and $P_{n+1}(x)$ take on different sign when $x \to \infty$, there must be another sign change beyond T. So $P_{n+1}(x)$ must have n+1 real roots.

Comments: **Liu Wai Kwong** sent in a more detail solution showing that the numbers can even be chosen to have the same sign.

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College).

Olympiad Corner

(continued from page 1)

Question 3. (Czech Republic)
Determine all integers $n > 3$ for which there exist n points $A_1, A_2, \cdots, A_n$ in the plane, and real numbers $r_1, r_2, \cdots, r_n$ satisfying the following two conditions:

(i) no three of the points $A_1, A_2, \cdots, A_n$ lie on a line;

(ii) for each triple i, j, k $(1 \le i < j < k \le n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Second Day

Question 4. (Poland)
Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \cdots, x_{1995}$ satisfying the two conditions:

(i) $x_0 = x_{1995}$;

(ii) $x_{i+1} + \dfrac{2}{x_{i-1}} = 2x_i + \dfrac{1}{x_i}$ for each i = 1, 2, $\cdots$, 1995.

Question 5. (New Zealand)
Let $ABCDEF$ be a convex hexagon with

$$AB = BC = CD,$$
$$DE = EF = FA,$$

and

$$\angle BCD = \angle EFA = 60^\circ.$$

Let G and H be two points in the interior of the hexagon such that:

$$\angle AGB = \angle DHE = 120^\circ.$$

Prove that

$$AG + GB + GH + DH + HE \ge CF.$$

Question 6. (Poland)
Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \cdots, 2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

IMO-95, Toronto, Canada

Kin Y. Li

On July 16, the Hong Kong team started their journey to Toronto, Canada for the thirty-sixth International Mathematical Olympiad. The flight took about 18 hours with one stop at Anchorage, Alaska. Shortly after arrival, the team was interviewed by local Chinese media. The Canadian host certainly publicized the event very well. During the entire period, the team stayed at the beautiful York University campus. The quarters provided were very comfortable; each person had his own room!

Opening ceremony came two days later and the examination followed. Team leaders and deputy leaders began markings and coordination soon afterward, while the students were given tours to Toronto's top attractions, such as Skydome, Ontario Science Center, Downtown Toronto, CN (Canadian National) Tower, Canada's Wonderland and of course, Niagara Falls. Meanwhile the scores were quickly decided. This year the team brought home two silvers, three bronzes and one honorable mention. (One silver was actually one mark short of a gold!) In the closing ceremony, the winners received their medals. Also, for entertainment, there were impressive performances, which included an awesome laser show. Throughout the events, there were many opportunities for students from different countries to get to know each other. Enjoying every moment of the whole trip, the team finally came home reluctantly on the evening of July 25. Everybody had fond memories and developed new friendships.

Photo at left: The 1995 Hong Kong Math Olympiad Team taken at the Kai Tak Airport before departure. From left to right are: LI Kin-Yin (leader), MOK Tze Tao, HO Wing Yip, POON Wai Hoi, CHEUNG Kwok Koon, YU Chun Ling, WONG Him Ting, and KWOK, Ka Keung (deputy leader).

Mathematical Excalibur

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Olympiad Corner

The following are five problems from the 24th USA Mathematical Olympiad held in April 27, 1995. The time limit for this competition was three and a half hours. -Editors

Problem 1. Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0, a_1 = 1, \ldots, a_{p-2} = p-2$ and, for all $n \geq p-1$, a_n is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n, a_n is the number obtained by writing n in base $p-1$ and reading the result in base p.

Problem 2. A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

(*continued on page 4*)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is December 30, 1995.

For individual subscription for the remaining three issues for the 95-96 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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談談質數

王元教授

自然數是指1, 2, 3, … 之一。整數則是指 …, −2, −1, 0, 1, 2, … 之一。自然數即正整數。二整數間可以定義和、差、乘運算，其結果仍爲整數，即“整數集合對加、減、乘運算是自封的”。

定理 1 (歐氏除法)：任二整數a及b (>0)，必有整數q及r滿足

$$a = bq + r,\ 0 \leq r < b.$$

若在上式中 $r = 0$，即 $a = bq$，則稱a爲b之倍數，或b爲a之因數，記爲$b|a$。否則記爲$b \nmid a$。

自然數可以分成三類：

1：只有自然數1爲其因數；

p：恰有1與p爲其因數，這種數稱之爲質數。

n：除1與n之外，還有其他因數，這種數稱爲複合數。

凡能被2整除的整數稱爲偶數，否則稱爲奇數。

定理 2 (算術的基本定理 (Fundamental Theorem of Arithmetic))：非1之自然數皆可以唯一地表示爲質數之積。

由定理 2 可見在自然數中質數是基本的。最初之若干質數是由Eratosthenes 篩法得到的。例如要找出不超過50之質數，則先找出不超過 $\sqrt{50}$ 的質數，即 2, 3, 5, 7，再將 1, 2, ⋯, 50 排列如下：

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,
11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
21, 22, 23, 24, 25, 26, 27, 28, 29, 30,
31, 32, 33, 34, 35, 36, 37, 38, 39, 40,
41, 42, 43, 44, 45, 46, 47, 48, 49, 50,

去掉1；去掉2的倍數4, 6, ⋯, 50；在剩下的數中去掉3的倍數9, ⋯, 45及在剩下的數中去掉5與7的倍數(保留5與7)，最後剩下的都是質數：2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

質數表都是根據這一方法加以改進而造出來的。如Kulik曾編出不超過10^8的質數表。自從有了電腦之後，質數表就更大了。六十年代初，美國就在電腦中儲存了前5×10^8個質數。但不管怎樣，我們至今仍然只知道有限多個質數，雖然有下面的著名定理：

定理 3 (歐幾里德 (Euclid))：質數有無窮多。

證：我們用反證法。若質數個數有限，則可以依次排列爲 $p_1, p_2, \ldots, p_s$。令$N = p_1p_2 \ldots p_s + 1$，若$N$爲質數，則$N > p_s$，矛盾。若$N$爲複合數，則$p_1, p_2, \ldots, p_s$ 皆非N之因數，否則就能整除1，此不可能，所以N有異於$p_1, p_2, \ldots, p_s$的質因數，矛盾。因此，質數有無窮多。定理證完。

目前所知道的大質數都是一些特殊形式的質數。形如

$$M_p = 2^p - 1,\ \ p\text{爲質數}$$

的質數叫梅森林質數(Mersenne Prime)。當 p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 132049, 216091時，M_p爲質數，最大者

$$2^{216091} - 1$$

共65050位，這是目前所知道的最大質數。但是否有無窮多Mersenne Prime？仍有待澄清。

還有費馬數 (Fermat Number)

$$F_n = 2^{2^n} + 1.$$

當 $n = 0, 1, 2, 3, 4$ 時，F_n都是質數。Fermat 曾猜想，當 $n = 0, 1, 2, \cdots$ 時，F_n都是質數。歐拉 (Euler) 證明了：

$$F_5 = 2^{2^5} + 1 = 641 \times 6700417.$$

從而否定了 Fermat 猜想。

****

下面講兩個質數論方面的中心問題：

一、質數分佈問題。命$\pi(x)$表示不超過x的質數個數。例如 $\pi(50) = 15$, $\pi(100) = 25$ 等等。質數分佈論的中心問題之一是研究

(*continued on page 2*)

談談質數：

(*continued from page 1*)

$\pi(x)$的性質：我們如何來估計$\pi(x)$？定理3可以記為：

$$\pi(x) \to \infty.$$

定理 4 (車比雪夫 (Chebychev))：當$n \geq 2$時

$$\frac{n}{8\log n} \leq \pi(n) \leq \frac{12n}{\log n},$$

其中$\log n$表示n的自然對數 (natural logarithm)。

這一定理比定理3精密多了。在此就不證明了。高斯 (Gauss) 與蘭讓德 (Legendre) 曾猜想：

$$\pi(x) \sim \frac{x}{\log x}$$

即當$x \to \infty$時，$\pi(x)$與$\frac{x}{\log x}$之比趨於1。注意：Gauss猜想的形式與上式稍有不同。因此當x較大時，用$\frac{x}{\log x}$來估計$\pi(x)$時，應該是很精密的。例如：

x	$\pi(x)$	$\frac{x}{\log x}$
1,000	168	145
10,000	1229	1086
100,000	9592	8686
1,000,000	78498	72382
10,000,000	664579	620417

Gauss 與Legendre猜想是Hadamard與 de la Vallee Poussin 獨立證明的。人們稱它為質數定理。即

定理 5 (質數定理 (Prime Number Theorem))：$\pi(x) \sim \frac{x}{\log x}$.

質數的另一重要問題是兩個相鄰質數的間隔長度估計問題。有所謂的Bertrand假設：當$x \geq 1$時，在x與$2x$之間必有一個質數。這一假設也是Chebychev證明的，即

定理 6 (Chebychev)：當$x \geq 1$時，在x與$2x$之間必有一個質數。

定理6還可以有較大改進。作為定理3的進一步，我們可以考慮算術序列 (Arithmetic Progression)中的質數問題：若l, q為正整數，且適合$(l,q)=1$ (即它們的最大公因數(l,q)等於1) 問序列

$$l,\ l+q,\ l+2q,\ \cdots$$

中是否有無窮多個質數？當$l=q=1$時，

(*continued on page 4*)

Systems of Distinct Representatives

Kin-Yin Li

Suppose in a school, there are some clubs. In the science club, the members are Bob and Cathy. In the dance club, the members are Bob, Mary, Joe and Emma. In the bridge club, the members are Joe, Emma, Paul and Cathy. In the debate club, the members are Bob and Cathy. Suppose a representative is to be elected from each club and no two clubs are allowed to have the same representative. Is this possible?

In the example, one possibility is to have Bob for science, Mary for dance, Joe for bridge and Cathy for debate. We say the collection Bob, Mary, Joe and Cathy is a *system of distinct representatives* (SDR) for the four clubs because each represents a different club.

If a new drama club is formed with only Bob and Cathy as members, then there is not any SDR for these five clubs because the science, debate and drama clubs together have only two members. So far, to decide whether there is a SDR for clubs or not is simple because there are not too many clubs. If the number of clubs increases, then the problem will become difficult. Naturally we would like to know if there is a method for knowing whether there exists any SDR for clubs or not. Also, we would like to know, when a SDR exists, how to find such a SDR.

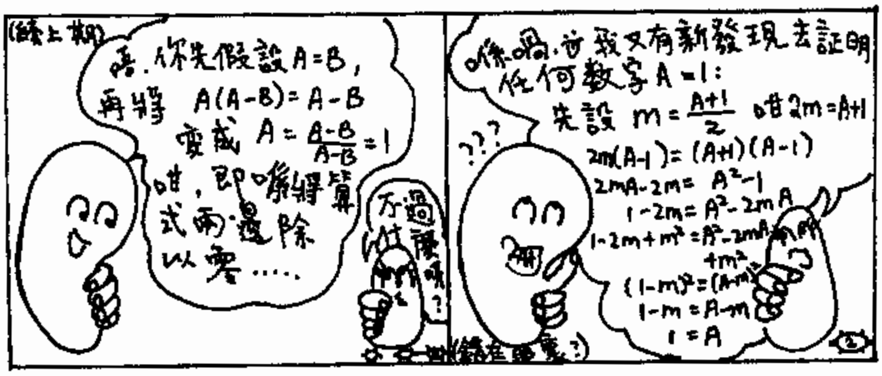
Suppose there are n clubs. From the drama club situation above, we learned that if these n clubs have a SDR, then every set of m ($\leq n$) clubs together must have at least m members. This gives us a necessary condition to check. In fact, there is a famous theorem, due to Philip Hall, that asserts the condition is also sufficient.

Hall's Theorem. *There exists a SDR for n clubs if and only if every set of m ($\leq n$) clubs together has at least m members.*

Briefly, here is how to get a SDR inductively when the condition is met. If we are lucky that every set of k ($< n$) clubs together has *more than* k members, then pick a member as representative for a club and remove this member from the other $n - 1$ clubs. The condition for the $n - 1$ clubs will still be met. Inductively, we can find a SDR for these $n - 1$ clubs.

If we are unlucky that there are k ($< n$) clubs together having exactly k members. Since $k < n$, inductively we can find a SDR for these k clubs. Now remove these k members from the other $n - k$ clubs. After removal, we can check that the condition for the remaining $n - k$ clubs will still be met. (This is because any j of these remaining clubs together will contain the members of the $j + k$ clubs together, minus the k removed members. That is, every set of j ($\leq n - k$) remaining clubs has at least $(j + k) - k = j$ members.) So inductively we can find a SDR for the remaining $n - k$ clubs.

For another application of SDR, consider the situation of n boys and n girls in a party. Each boy knows some of the girls and vice versa. When is it possible to match each boy with a unique girl that he knows? This is simple if you understand Hall's theorem. For each boy, form a fan club consists of all the girls he knows. There is a matching if and only if there is a SDR for the n fan clubs, i.e., every set of m ($\leq n$) boys together must know at least m girls.



Problem Corner

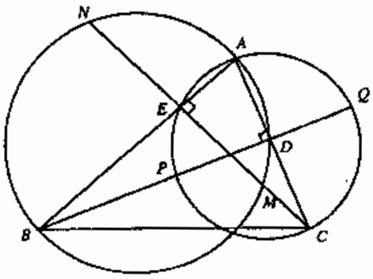
We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is December 30, 1995.

Problem 21. Show that if a polynomial $P(x)$ satisfies

$$P(2x^2-1)=\frac{P(x)^2}{2}-1,$$

it must be constant.

Problem 22. An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CE and its extension at points M and N, and the circle with diameter AC intersects altitude BD and its extension at P and Q. Prove that the points M, N, P, Q lie on a common circle. (*Source*: 1990 USA Mathematical Olympiad).



Problem 23. Determine all sequences $\{a_1, a_2, \ldots\}$ such that $a_1 = 1$ and $|a_n - a_m| \le 2mn/(m^2+n^2)$ for all positive integers m and n. (*Source*: Past IMO problem proposed by Finland).

Problem 24. In a party, n boys and n girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the k-th tallest boy and the k-th tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.

Problem 25. Are there any positive integers n such that the first four digits from the left side of $n!$ (in base 10 representation) is 1995?

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Solutions

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Problem 16. Let a, b, c, p be real numbers, with a, b, c not all equal, such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}=p$. Determine all possible values of p and prove that $abc + p = 0$. (Source: 1983 Dutch Mathematical Olympiad.)

Solution: Official Solution.

Since $ca + 1 = ap$ and $bc + 1 = cp$, we get $ap^2 = cap + p = a(bc+1) + p = abc + a + p$. Hence $a(p^2-1) = abc + p$. Similarly, $b(p^2-1) = abc + p$ and $c(p^2-1) = abc + p$. Since a, b, c are not all equal, $p = \pm 1$ and then $abc + p = 0$. Both values of p are possible by considering $(a,b,c) = (2,-1,1/2)$ and $(-2,1,-1/2)$.

Comments: Most solvers use repeated substitution to obtain the equation $(p^2 - 1)(a^2 - ap + 1) = 0$ (and similar equations for b and c) and then show that $p = \pm 1$. (Otherwise, $a^2 - ap + 1 = 0$ and the other two similar equations will lead to the contradiction $a = b = c$.) Solvers then use different approaches to find abc for the two possible values of p to prove $abc + p = 0$.

Other commended solvers: **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **Wallis LEUNG Ka-Wo** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 17. Find all sets of positive integers x, y and z such that $x \le y \le z$ and $x^y + y^z = z^x$.

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College).

Since $3^{1/3} > 4^{1/4} > 5^{1/5} > \ldots$, we have $y^z \ge z^y$ if $y \ge 3$. Hence the equation has no solution if $y \ge 3$. Since $1 \le x \le y$, the only possible values for (x,y) are $(1,1)$, $(1,2)$ and $(2,2)$. These lead to the equations $1 + 1 = z$, $1 + 2^z = z$ and $4 + 2^z = z^2$. The third equation has no solution since $2^z \ge z^2$ for $z \ge 4$ and $(2,2,3)$ is not a solution to $x^y + y^z = z^x$. The second equation has no solution either since $2^z > z$. The first equation leads to the unique solution $(1,1,2)$.

Other commended solvers: **HO Wing Yip** (Clementi Secondary School), **LIU Wai Kwong** (Pui Tak Canossian College) and **WONG Him Ting** (Salesian English School).

Problem 18. For real numbers a, b, c, define

$f(a,b,c) = a + b - |a-b| - |a+b+|a-b|-2c|$.

Show that $f(a,b,c) > 0$ if and only if $f(b,c,a) > 0$ if and only if $f(c,a,b) > 0$.

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College).

We have $f(a,b,c) > 0$ if and only if $|a + b + |a-b| - 2c| < a + b - |a-b|$. Applying the fact that $|x| < y$ if and only if $x < y$ and $-x < y$ to the last inequality and simplifying, we see that $f(a,b,c) > 0$ if and only if $|a-b| < c$ and $c < a + b$. Applying the fact again to $|a-b| < c$ and transposing terms, we see that $f(a,b,c) > 0$ if and only if $a < b + c$ and $b < c + a$ and $c < a + b$. The assertion follows.

Comments: **LIU Wai Kwong** considers the six possible orderings $a \ge b \ge c$, $a \ge c \ge b$, etc. to show that $f(a,b,c) = f(b,c,a) = f(c,a,b) = 2(a + b + c - 2\max\{a,b,c\})$ and thus the assertion follows.

Other commended solvers: **Wallis LEUNG Ka-Wo** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 19. Suppose A is a point inside a given circle and is different from the center. Consider all chords (excluding the diameter) passing through A. What is the locus of the intersection of the tangent lines at the endpoints of these chords?

Solution: WONG Him Ting (Salesian English School).

Let O be the center and r be the radius. Let A' be the point on OA extended beyond A such that $OA \times OA' = r^2$. suppose BC is one such chord passing through A and the tangents at B and C intersect at D'. By symmetry, D' is on the line OD, where D is the midpoint of BC. Since $\angle OBD' = 90°$, $OD \times OD' = OB^2$ ($= OA \times OA'$.) So ΔOAD is similar to $\Delta OD'A'$. Since $\angle ODA = 90°$, D' is on the line L perpendicular to OA at A'.

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Conversely, for D' on L, let the chord through A perpendicular to OD' intersect the circle at B and C. Let D be the intersection of the chord with OD'. Now ΔOAD and $\Delta OD'A'$are similar right triangles. So $OD \times OD' = OA \times OA' = OB^2 = OC^2$, which implies $\angle OBD' = \angle OCD' = 90°$. Therefore, D' is on the locus. This shows the locus is the line L.

Other commended solvers: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **Wallis LEUNG Ka-Wo** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College) and **Bobby POON Wai Hoi** (St Paul's College).

Problem 20. For $n > 1$, let $2n$ chess pieces be placed on any $2n$ squares of an $n \times n$ chessboard. Show that there are 4 pieces among them that formed the vertices of a parallelogram. (Note that if $2n - 1$ pieces are placed on the squares of the first column and the first row, then there is no parallelogram. So $2n$ is the best possible.)

Solution: Edmond MOK Tze Tao (Queen's College).

Let m be the number of rows that have at least 2 pieces. (Then each of the remaining $n - m$ rows contains at most 1 piece.) For each of these m rows, locate the leftmost square that contains a piece. Record the distances (i.e., number of squares) between this piece and the other pieces on the same row. The distances can only be 1, 2, …, n-1 because there are n columns.

Since the number of pieces in these m rows altogether is at least $2n - (n - m) = n + m$, there are at least $(n + m) - m = n$ distances recorded altogether for these m rows. By the pigeonhole principle, at least two of these distances are the same. This implies there are at least two rows each containing 2 pieces that are of the same distance apart. These four pieces yield a parallelogram.

Other commended solvers: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **HO Wing Yip** (Clementi Secondary School) and **WONG Him Ting** (Salesian English School).

談談質數：

(*continued from page 2*)

即得自然數集合，由定理3可知其中有無窮多質數。算術序列中包有無窮多質數這個問題是狄里希勒 (Dirichlet) 解決的，即

定理 7 (Dirichlet)：算術序列中有無窮多個質數。

我們還可以問：在算術序列中最小質數$P(l,q)$的上界估計？林尼克(Linnik)首先證明了：

定理 8 (Linnik)： $P(l,q) \le c_1 q^{c_2}$ ，其中c_1, c_2是兩個常數。

中國數學家潘承洞首先給出估計$c_2 = 5448$。現在最佳估計$c_2 = 5.5$是希斯-布朗(Heath-Brown)得到的。

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二、哥德巴赫猜想 (Goldbach Conjecture)。在Goldbach與Euler的通信中提出了這樣的猜想：
(i) 每一個偶數≥ 6 都是兩個奇質數之和。
(ii)每一個奇數≥ 9 都是三個奇質數之和。

猜想(ii)是猜想(i)的推論。事實上，如果(i)成立及n爲一個奇數≥ 9 ，則n−3爲偶數≥ 6，從而由(i)可知它是兩個質數p_1與p_2之和，即$n-3 = p_1 + p_2$，所以$n = 3 + p_1 + p_2$ ，即(ii)成立。維諾格拉朵夫(Vinogradov)首先基本上證明了猜想(ii)，即

定理 9 (Vinogradov)：每個充分大的奇數都是三個奇質數之和。

關於猜想(i)，迄今仍然只有一些數值驗算，說明它可能是對的。例如有人在電腦上驗證過猜想(i)對於不超過3×10^8的偶數都成立。首先是布倫(Brun)將Eratosthenes篩法加以改進，並證明了下面結果：

定理 10 (Brun)：每個大偶數都是兩個質因數不超過9的整數之和，簡記爲(9,9)。

Brun的結果與方法被很多數學家加以發展與改進。目前最好的結果是中國數學家陳景潤證明的，即

定理 11 (陳景潤)：每個大偶數都是一個質數及一個質因數不超過2的整數之和，簡記爲(1,2)。

類似地，還有所謂學生質數猜想：3, 5；5, 7；11, 13；…；10016957, 10016959；…；$10^9 + 7$, $10^9 + 9$；… 皆爲相差爲2的質數對，我們稱這樣一對質數爲學生質數對 (twin primes)。巳知小於100,000者有1,224對，小於1,000,000者有8,164對，現在所知道的最大學生質數對爲

$$260497545 \times 2^{6625} - 1,\ \ 260497545 \times 2^{6625} + 1.$$

有一個著名猜想爲：學生質數對有無窮多？這是定理3的深化，迄今仍未能證明。這一猜想與Goldbach猜想(i)有深刻的內在聯繫。考慮不定方程式

$$ax + by = c.$$

其中a,b,c爲給予整數，試求這一方程式的質數解x,y問題？當$a = b = 1$，c爲偶數≥ 6時，方程的可解性即Goldbach猜想(i)。當$a = 1$, $b = -1$, $c = 2$，方程式有無窮多解即相當於學生質數對猜想。用陳景潤證明(1,2)的方法可以證明：

定理 12 (陳景潤)：存在無窮多個質數p使p+2爲不超過2個質數之乘積。

Olympiad Corner

(*continued from page 1*)

Problem 3. Given a nonisosceles, nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that ΔOAA_1 is similar to ΔOA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent, i.e., these three lines intersect at a point.

Problem 4. Suppose q_0, q_1, q_2, ... is an infinite sequence of integers satisfying the following two conditions:

(i) $m - n$ divides $q_m - q_n$ for $m > n \ge 0$,
(ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

Problem 5. Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1 - 4q/n^2)$ or fewer amicable pairs.

Mathematical Excalibur

Volume 2, Number 1 January-February, 1996

Olympiad Corner

Fourth Mathematical Olympiad of Taiwan:

First Day
Taipei, April 13, 1995

Problem 1. Let $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with complex coefficients. Suppose the roots of $P(x)$ are $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $|\alpha_1| > 1$, $|\alpha_2| > 1, \cdots, |\alpha_j| > 1$, and $|\alpha_{j+1}| \le 1, \cdots, |\alpha_n| \le 1$. Prove:

$$\prod_{i=1}^{j}|\alpha_i| \le \frac{\sqrt{|a_0|^2+|a_1|^2+\cdots+|a_n|^2}}{|a_n|}.$$

Problem 2. Given a sequence of integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$. One constructs a second sequence $|x_2-x_1|$, $|x_3-x_2|$, $|x_4-x_3|$, $|x_5-x_4|$, $|x_6-x_5|$, $|x_7-x_6|$, $|x_8-x_7|$, $|x_1-x_8|$. Such a process is called a single operation. Find all the 8-terms integral sequences having the following property: after finitely many single operations it becomes an integral sequence with all terms equal.

(continued on page 4)

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is February 28, 1996.

For individual subscription for the remaining two issues for the 95-96 academic year, send us two stamped self-addressed envelopes. Send all correspondence to:

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Solution by Linear Combination

Kin-Yin Li

In mathematics, often we are interested in finding a solution to equations. Consider the following two problems:

Problem 1. Given real numbers $m_1, m_2, \ldots, m_n$ (all distinct) and $a_1, a_2, \ldots, a_n$, find a polynomial $v(x)$ such that $v(m_1) = a_1$, $v(m_2) = a_2, \ldots, v(m_n) = a_n$.

Problem 2. Given positive integers $m_1, m_2, \ldots, m_n$ (pairwise relatively prime) and integers $a_1, a_2, \ldots, a_n$, find an integer v such that $v \equiv a_1 \pmod{m_1}$, $v \equiv a_2 \pmod{m_2}, \ldots, v \equiv a_n \pmod{m_n}$.

Problem 1 comes up first in algebra and analysis (later in engineering and statistics). It is an *interpolation* problem, where we try to fit the values a_i at m_i (i.e., to find a polynomial whose graph passes through the points (m_1,a_1), $(m_2,a_2), \ldots, (m_n,a_n)$). Problem 2 comes up in number theory. It is a *congruence* problem, where we try to count objects by inspecting the remainders (i.e., to find a number which has the same remainder as a_i upon division by m_i).

There is a technique that can be applied to both problems. The idea is to solve first the special cases, where exactly one of the a_i's is 1 and all others 0. For problem 1, this is easily solved by defining (for $i = 1, 2, \ldots, n$) the polynomial $P_i(x)$ to be $(x-m_1)(x-m_2)\cdots(x-m_n)$ with the factor $(x-m_i)$ *omitted*, i.e.,

$$P_i(x) = \prod_{\substack{j=1 \\ j\ne i}}^{n}(x-m_j),$$

and $v_i(x) = P_i(x)/P_i(m_i)$. Then $v_i(m_i) = 1$ and $v_i(m_k) = 0$ for $k \ne i$ because $P_i(m_k) = 0$ (for $k \ne i$).

For problem 2, this is solved similarly by first defining (for $i = 1, 2, \ldots, n$) the integer P_i to be $m_1m_2\cdots m_n$ with the factor m_i *omitted*. Consider $P_i, 2P_i, \ldots, m_iP_i$. Upon division by m_i, no two of these will have the same remainder because the difference of any two of them is not divisible by m_i. So one of these, say c_iP_i, has remainder 1. Let $v_i = c_iP_i$, then $v_i \equiv 1 \pmod{m_i}$ and $v_i \equiv 0 \pmod{m_k}$ for $k \ne i$ because $P_i \equiv 0 \pmod{m_k}$.

Finally to solve problem 1 or 2 in general, we use the special case solutions $v_1, v_2, \ldots, v_n$ to form $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. It is now easy to check that the expression v solves both problems 1 and 2.

For problem 1,

$$v(x) = a_1\frac{P_1(x)}{P_1(m_1)} + \cdots + a_n\frac{P_n(x)}{P_n(m_n)}$$

is called *Lagrange's interpolation formula*. For problem 2, although the c_i's may be tedious to find, we know a solution $v = a_1c_1P_1 + \cdots + a_nc_nP_n$ exists. This is the assertion of the *Chinese remainder theorem*. Note also that if we add to v any multiple of $(x-m_1)(x-m_2)\cdots(x-m_n)$ in problem 1 or any multiple of $m_1m_2\cdots m_n$ in problem 2, we get other solutions.

The expression of v, involving a sum of multiples of $v_1, v_2, \ldots, v_n$, is so common in similar problems that it is now come to be called a *linear combination* of $v_1, v_2, \ldots, v_n$. In passing, note that the a_i's are numbers. However, the v_i's are polynomials in problem 1 and numbers in problem 2. Like vectors expressed in coordinates, the v_i's are objects that may take on different values at different positions. So functions corresponding to solutions of equations are often viewed as vectors (with infinitely many coordinates). Concepts like these are the foundation of *Linear*

(continued on page 2)

Solution by Linear Combination:
(*continued from page 1*)

Algebra, which studies the properties of solutions of these kind of problems in an abstract manner.

Example 1. If $f(x)$ is a polynomial of degree at most n and $f(k) = (n+1-k)/(k+1)$ for $k = 0, 1, \ldots, n$, find $f(n+1)$.

Solution 1. Applying Lagrange's interpolation formula, we define $P_k(x) = x(x-1)\cdots(x-n)$ with the factor $(x-k)$ *omitted.* Then $P_k(n+1) = (n+1)!/(n+1-k)$, $P_k(k) = (-1)^{n-k}k!(n-k)!$ and

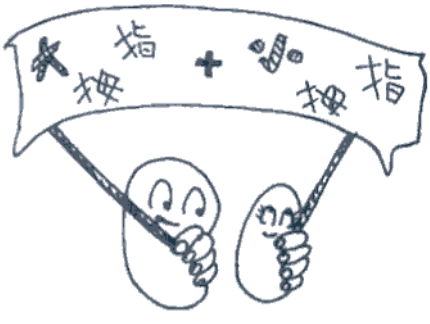
$$f(n+1) = (-1)^{n+1}\sum_{k=0}^{n}(-1)^{k+1}\frac{(n+1)!}{(k+1)!(n-k)!} = (-1)^n$$

where we used the binomial expansion of $(1-1)^{n+1}$ in the last step.

Solution 2. The polynomial $g(x) = (x+1)f(x) - (n+1-x)$ has degree at most $n+1$. We are given that $g(0) = g(1) = \cdots = g(n) = 0$. So $g(x) = Cx(x-1)\cdots(x-n)$. To find C, we set $x = -1$ and get $g(-1) = -(n+2) = C(-1)^{n+1}(n+1)!$. Therefore, $C = (-1)^n(n+2)/(n+1)!$ and $g(n+1) = (n+2)f(n+1) = (-1)^n(n+2)$, which implies $f(n+1) = (-1)^n$.

Example 2. Prove that for each positive integer n there exist n consecutive positive integers, none of which is an integral power of a prime number. (*Source:* 1989 IMO.)

Solution. Let $p_1, p_2, \ldots, p_{2n}$ be $2n$ distinct prime numbers and consider the congruence problem $v \equiv -1 \pmod{p_1p_2}$, $v \equiv -2 \pmod{p_3p_4}$, $\ldots$, $v \equiv -n \pmod{p_{2n-1}p_{2n}}$. Since $p_1p_2, p_3p_4, \ldots, p_{2n-1}p_{2n}$ are pairwise relatively prime, by the Chinese remainder theorem, there is a positive integer solution v. Then each of the n consecutive numbers $v+1, v+2, \ldots, v+n$ is divisible by more than one prime number. So each is not a power of a prime number.



中國剩餘定理

郭宇權
香港科技大學數學系

在中國南北朝時代著成的數學經典《孫子算經》中，有一道千古名題，名爲【物不知數】問題是

問： 今有物不知其數，三三數之賸二，五五數之賸三，七七數之賸二，問物幾何？

答曰： 二十三

這問題頗有猜謎的趣味，在中國民間頗廣流傳，在西方數學史上被稱爲【中國剩餘定理】。如果用現代數學符號來表示【孫子問題】，我們由已知條件

$N \equiv 2 \pmod 3$；
$N \equiv 3 \pmod 5$；
$N \equiv 2 \pmod 7$

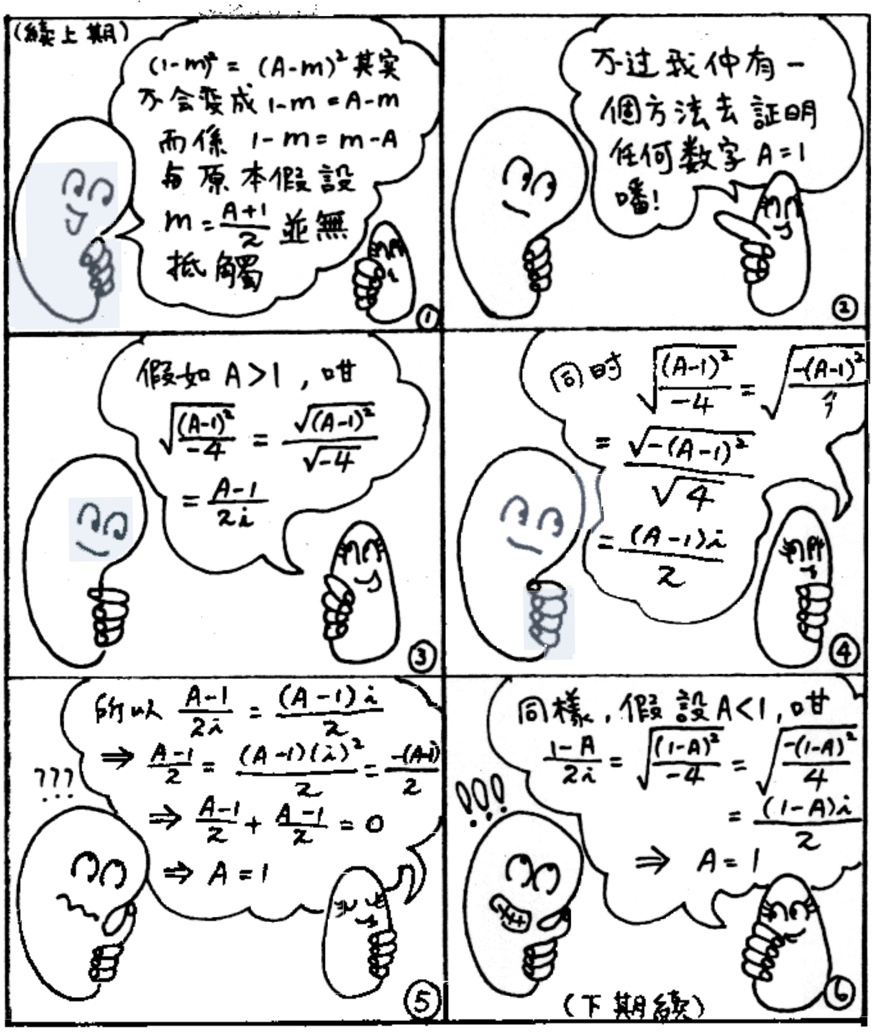
求最小的正整數N。這是一個一次同餘式組的問題，問題的求解法被編成【孫子歌】，是一首五絕詩

三人同行七十稀
五樹梅花廿一枝
七子團圓正半月
減百零五便得知

用現代算式表示是

$N = 70\times2+21\times3+15\times2-105\times2=23$

求解的手段是先找7與5的公倍數，用它除3餘1，這數是70；因此，70×2除3餘2。類似地，21是7與3的公倍數，用它除5餘1，所以21×3除5餘3；接著15×2除盡3與5，除7餘2。這三數的總和，滿足了所需條件；最後，減去3與5與7的公倍數，使N變爲滿足所有條件的最小正整數。這就得出所需答案。

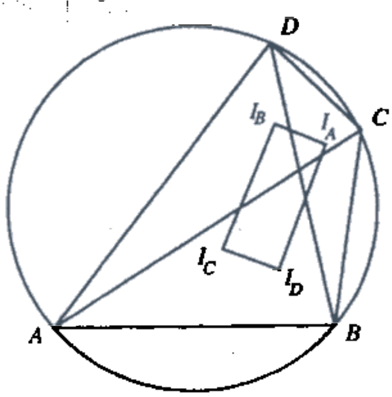


Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is February 28, 1996.

Problem 26. Show that the solutions of the equation $\cos\pi x = \frac{1}{3}$ are all irrational numbers. (*Source:* 1974 Putnam Exam.)

Problem 27. Let *ABCD* be a cyclic quadrilateral and let I_A, I_B, I_C, I_D be the incenters of ΔBCD, ΔACD, ΔABD, ΔABC, respectively. Show that $I_AI_BI_CI_D$ is a rectangle.



Problem 28. The positive integers are separated into two subsets with no common elements. Show that one of these two subsets must contain a three term arithmetic progression.

Problem 29. Suppose $P(x)$ is a nonconstant polynomial with integer coefficients and all coefficients are greater than or equal to -1. If $P(2) = 0$, show that $P(1) \neq 0$.

Problem 30. For positive integer $n > 1$, define $f(n)$ to be 1 plus the sum of all prime numbers dividing n multiplied by their exponents, *e.g.*, $f(40) = f(2^3 \times 5^1) = 1 + (2\times3 + 5\times1) = 12$. Show that if $n > 6$, the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$ must eventually be repeating 8, 7, 8, 7, 8, 7, ….

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Solutions

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Problem 21. Show that if a polynomial $P(x)$ satisfies

$$P(2x^2-1) = \frac{P(x)^2}{2}.$$

it must be constant.

Solution 1: Independent solution by **LIU Wai Kwong** (Pui Tak Canossian College) and **YUNG Fai** (CUHK).

Construct a sequence $u_1 = 1$, $u_2 = -1$ and $u_n = \sqrt{\frac{u_{n-1}+1}{2}}$ for $n \geq 3$. We have $u_n < u_{n+1} < 1$ for $n \geq 2$ and $P(u_n) = (P(u_{n+1})^2/2) - 1$ for $n \geq 1$. Note that $P(u_n) \neq 0$ for $n \geq 1$ (otherwise $P(u_n) = 0$ would imply $P(u_{n-1})$, $P(u_{n-2})$, $\cdots$, $P(u_1)$ are rational, but $P(1) = 1 \pm \sqrt{3}$.) Differentiating the functional equation for P, we get $4xP'(2x^2-1) = P(x)P'(x)$. Since $P(1) \neq 4$, we get $P'(u_1) = P'(1) = 0$. This implies $0 = P'(u_2) = P'(u_3) = \cdots$. Therefore, $P'(x)$ is the zero polynomial and so $P(x)$ is constant.

Comments: This problem was from the 1991 USSR Math Winter Camp. Below we will provide a solution without calculus.

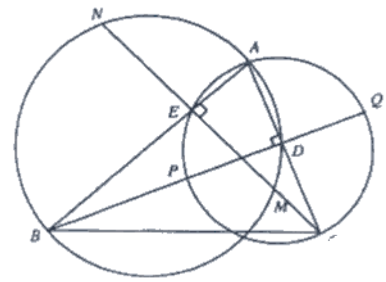
Solution 2: Suppose $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is such a polynomial with degree $n \geq 1$. Then

$$a_0(2x^2-1)^n + a_1(2x^2-1)^{n-1} + \cdots + a_n = \frac{(a_0x^n + a_1x^{n-1} + \cdots + a_n)^2}{2} - 1.$$

Comparing the coefficients of x^{2n}, we find $a_02^n = a_0^2/2$, so $a_0 = 2^{n+1}$. Suppose a_0, a_1, …, a_k are known to be rational. Comparing the coefficients of x^{2n-k-1}, the left side yields a rational number involving a_0, …, a_k, but the right side yields a number of the form a_0a_{k+1} plus a rational number involving a_0, …, a_k. So a_{k+1} is also rational. Hence a_0, a_1, …, a_n are all rational. Then $P(1) = a_0 + a_1 + \cdots + a_n$ is rational. However, $P(1) = (P(1)^2/2) - 1$ forces $P(1) = 1 \pm \sqrt{3}$, a contradiction. Therefore $P(x)$ must be constant.

Other commended solver: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College).

Problem 22. An acute-angled triangle *ABC* is given in the plane. The circle with diameter *AB* intersects altitude *CE* and its extension at points *M* and *N*, and the circle with diameter *AC* intersects altitude *BD* and its extension at *P* and *Q*. Prove that the points *M, N, P, Q* lie on a common circle. (*Source*: 1990 USA Mathematical Olympiad).



Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College).

If *M, N, P, Q* are concyclic, then *A* must be the center because it is the intersection of the perpendicular bisectors of *PQ* and *MN*. So it suffices to show *AP* = *AM*.

Considering the similar triangles *ADP* and *APC*, we get *AD*/*AP* = *AP*/*AC*, i.e., $AP^2 = AD\times AC$. Similarly, $AM^2 = AE\times AB$. Since $\angle BEC = \angle BDC$, points *B, C, D, E* are concyclic. Therefore, $AD\times AC = AE\times AB$ and so *AP* = *AM*.

Other commended solvers: **HO Wing Yip** (Clementi Secondary School), **LIU Wai Kwong** (Pui Tak Canossian College), **Edmond MOK Tze Tao** (Queen's College), **WONG Him Ting** (HKU) and **YU Chun Ling** (Ying Wa College).

Problem 23. Determine all sequences $\{a_1, a_2, \ldots\}$ such that $a_1 = 1$ and $|a_n - a_m| \leq 2mn/(m^2+n^2)$ for all positive integers *m* and *n*. (*Source*: Past IMO problem proposed by Finland).

Solution: Independent solution by **CHAN Wing Sum** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College) and **YUNG Fai** (CUHK).

For fixed *m*,

$$\lim_{n\to\infty}|a_n - a_m| \leq \lim_{n\to\infty}\frac{2mn}{m^2+n^2} = 0$$

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

So for all m,

$$a_m = \lim_{n\to\infty} a_n,$$

It follows that all terms are equal (to $a_1 = 1$.)

Problem 24. In a party, n boys and n girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the k-th tallest boy and the k-th tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.

Solution: Independent solution by **HO Wing Yip** (Clementi Secondary School) **and YU Chun Ling** (Ying Wa College).

Let $b_1 \ge b_2 \ge \cdots \ge b_n$ be the heights of the boys and $g_1 \ge g_2 \ge \cdots \ge g_n$ be those of the girls. Suppose for some k, $|b_k - g_k| \ge 10$. In the case $b_k - g_k \ge 10$, we have $b_i - g_j \ge 10$ for $1 \le i \le k$ and $k \le j \le n$. Consider the boys of height b_i $(1 \le i \le k)$ and the girls of height g_j $(k \le j \le n)$. By the pigeonhole principle, two of these n+1 must be paired originally. However, $b_i - g_j \ge 10$ contradicts the hypothesis. (The case $g_k - b_k \ge 10$ is handled similarly.) So $|b_k - g_k| < 10$ for all k.

Comments: This was a problem from the 1984 Tournament of the Towns, a competition started in 1980 at Moscow and Kiev and is now participated by students in dozens of cities in different continents.

Other commended solvers: **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, **KU Yuk Lun** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College) and **WONG Him Ting** (HKU).

Problem 25. Are there any positive integers n such that the first four digits from the left side of $n!$ (in base 10 representation) is 1995?

Solution 1: LIU Wai Kwong (Pui Tak Canossian College).

Let $[x]$ be the greatest integer not exceeding x and $\{x\} = x - [x]$. Also, let $a_j = 1 + j\times 10^{-8}$, $b_0 = \log 10^8!$ and $b_j = \log 10^8! + (\log a_1 + \cdots + \log a_j)$ for $j > 0$. (For this solution, log means $\log_{10}$.) Observe that

(i) $0 < \log a_k \le \log a_{30000} < \log\dfrac{1996}{1995}$ for $k = 1, 2, \ldots, 30000$;

(ii) $\sum_{j=1}^{30000} \log a_j > 15000(\log a_1 + \log a_{30000}) > 1.$

Note the distance between $\{\log 1995\}$ and $\{\log 1996\}$ is $\log(1996/1995)$. Now $b_0, b_1, \ldots, b_{30000}$ is increasing and

$$b_{30000} - b_0 > 1 \quad \text{(by (ii))},$$

but

$$0 < b_{j+1} - b_j < \log\frac{1996}{1995} \quad \text{(by (i))}.$$

So there is a $k \le 30000$ such that

$$\{\log 1995\} < \{b_k\} < \{\log 1996\}$$

Now

$$\log 10^8! + \sum_{j=1}^{k} \log a_j = \log(10^8 + k)! - 8k$$

implies

$$\{\log 1995\} < \{\log(10^8+k)!\} < \{\log 1996\}.$$

Adding $[\log 1995] = [\log 1996] = 3$, we have

$$\log 1995 < \log(10^8+k)! - m < \log 1996$$

for $m = [\log(10^8+k)!] - 3$. Therefore,

$$1995\times 10^m < (10^8+k)! < 1996\times 10^m.$$

Consequently, the number $(10^8+k)!$ begins with 1995.

Comments: With 1995 replaced by 1993, this problem appeared in the 1993 German Mathematical Olympiad. Below we will provide the (modified) official solution.

Solution 2: Let $m = 1000100000$. If $k < 99999$ and $(m+k)! = abcd\cdots$ (in base 10 representation), then $(m+k+1)! = abcd\cdots \times 10001\cdots = efgh\cdots$, where $efgh$ equals $abcd$ or the first four digits of $abcd$+1. So, the first four digits of each of $(m+1)!, (m+2)!, \ldots, (m+99999)!$ must be the same as or increase by 1 compared with the previous factorial. Also, because the fifth digit of $m+k$ ($k < 99999$) is 1, the fifth digit of $(m+k)!$ will be added to the first digit of $(m+k)!$ in computing $(m+k+1)!$. So, in any ten consecutive factorials among $(m+1)!, (m+2)!, \ldots, (m+99999)!$, there must be an increase by 1 in the first four digits. So the first four digits of $(m+1)!, (m+2)!, \ldots, (m+99999)!$ must take on all 9000 possible choices. In particular, one of these is 1995.

Olympiad Corner

(*continued from page 1*)

Problem 3. Suppose n persons meet in a meeting, every one among them is familiar with exactly 8 other participants of that meeting. Furthermore suppose that each pair of two participants who are familiar with each other have 4 acquaintances in common in that meeting, and each pair of two participants who are not familiar with each other have only 2 acquaintances in common. What are the possible values of n?

****

Second Day
Taipei, April 15, 1995

Problem 4. Given n (where $n \ge 2$) distinct integers $m_1, m_2, \cdots, m_n$. Prove that there exist a polynomial $f(x)$ of degree n and with integral coefficients which satisfies the following conditions:

(i) $f(m_i) = -1$, for all $1 \le i \le n$.

(ii) $f(x)$ cannot be factorized into a product of two nonconstant polynomials with integral coefficients.

Problem 5. Let P be a point on the circumscribed circle of $\Delta A_1A_2A_3$. Let H be the orthocenter of $\Delta A_1A_2A_3$. Let B_1 (B_2, B_3 respectively) be the point of intersection of the perpendicular from P to A_2A_3 (A_3A_1, A_1A_2 respectively). It is known that the three points B_1, B_2, B_3 are colinear. Prove that the line $B_1B_2B_3$ passes through the midpoint of the line segment $\overline{PH}$.

Problem 6. Let a, b, c, d be integers such that $ad - bc = k > 0$, $(a,b) = 1$, and $(c,d) = 1$. Prove that there are exactly k ordered pairs of real numbers (x_1, x_2) satisfying $0 \le x_1, x_2 < 1$ and both $ax_1 + bx_2$ and $cx_1 + dx_2$ are integers.

Erratum: In the article 談談質數 in the last issue, 定理 4 (Chebychev Theorem) should be corrected as

$$\frac{n}{8\log n} \le \pi(n) \le \frac{12n}{\log n} \quad (\text{when } n \ge 2).$$

Mathematical Excalibur

Volume 2, Number 2 March-April, 1996

Olympiad Corner

Eighth Asian Pacific Mathematical Olympiad, March 19, 1996:

Time Allowed: Four hours.

Problem 1. Let $ABCD$ be a quadrilateral with $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of the hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Problem 2. Let m and n be positive integers such that $n \le m$. Prove that

$$2^n n! \le \frac{(m+n)!}{(m-n)!} \le (m^2+m)^n.$$

Problem 3. Let P_1, P_2, P_3, P_4 be four points on a circle. and let I_1 be the incenter of the triangle $P_2P_3P_4$, I_2 be the incenter of the triangle $P_1P_3P_4$, I_3 be the

(*continued on page 2*)

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秦九韶與「三斜求積術」

香港道教聯合會青松中學
梁子傑

周教六藝，數實成之，…大則可以通神明，順性命，小則可以經世務，類萬物，詎容以淺近窺哉？
——《數書九章序》秦九韶

前言

眾所周知，三角形面積公式是：面積＝底×高÷2。這數式雖然簡單，但是實際地使用起來，又似乎不大「方便」。試想想：如果我們在一個球場上繪畫了一個很大的三角形，要在地面上準確地定出某條底邊的高並不容易，而且過程亦很繁複。不過，明顯得很，這個三角形三條邊的邊長就不難求得，祇要用一把夠長的尺去量度就可以了。於是我們自然會問：知道三角形三條斜邊的長度，可以求到該三角形的面積嗎？

上述問題的答案是肯定的，而且該面積公式亦已被發現了好幾百年。現在就讓我爲大家介紹中國南宋時期的數學家秦九韶，與及他的「三斜求積術」。

秦九韶

秦九韶生於1202年，卒於1261年，正是我國戰亂頻生的南宋時期，雖然秦九韶的父親是一名太守，但仍然逃不過需要四處遷徙逃避戰禍的命運。正因此，秦九韶自小就跟父親到過很多地方；此外，他自細就思想活躍，對天文、音律、算術、建築等學問，都有濃厚的興趣。在1247年，他從他以往曾研究過的數學問題中，精選了81道題目，將它們編寫成一本名叫《數書九章》的書。由於這本書的內容豐富，題目生動有趣，所以深受後世數學家的重視和喜愛，因此該書亦被認爲是我國數學史上的巨著之一。

在《數書九章》的第三章中，秦九韶就提出了以下的問題：

問沙田一段，其小斜一十三里，中斜一十四里，大斜一十五里。…卻知爲田幾何？

意思就是叫讀者求邊長分別爲13里、14里和15里的一個三角形的面積。秦九韶稱這道題目爲「三斜求積」，而後世人就稱書中求面積的方法爲「三斜求積術」了。

三斜求積術

在《數書九章》中，秦九韶對求面積的方法有以下的闡釋：

以小斜冪，並大斜冪，減中斜冪，餘半之，自乘於上；以小斜冪乘大斜冪，減上，餘四約之爲實，…開平方得積。

轉爲現代的數學符號，就即是話：如果三角形三條斜邊的長度爲a、b和c，則

$$面積 = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2} 。$$

依照此公式，代入$a = 13$, $b = 14$, $c = 15$，就得到三角形面積爲84平方單位了。

此公式看來很複雜，而且正如中國其他古代數學書一樣，秦九韶並沒有在書中給出這公式的證明，但是祇要大家懂得「勾股定理」和代數式自乘的法則，就不難獲得此式，方法如下：

(*continued on page 4*)

Olympiad Corner:

(*continued from page 1*)

incenter of the triangle $P_1P_2P_4$, I_4 be the incenter of the triangle $P_2P_3P_1$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Problem 4. The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

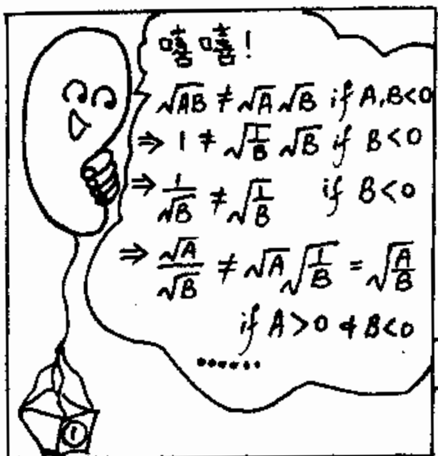
1) All members of a group must be the same sex, i.e., they are either all male or all female.
2) The difference in the size of any two groups is either 0 or 1.
3) All groups have at least one member.
4) Each person must belong to one and only one group.

Find all values of n, $n \le 1996$, for which this is possible. Justify your answer.

Problem 5. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$$

and determine when equality occurs.



Stirling's Inequality

Andy Liu
University of Alberta, Canada

It is useful to have a good approximation for $n!$, the factorial of a positive integer n. This is given by Stirling's Inequality which states that for $n \ge 2$,

$$\frac{n^{n+\frac{1}{2}}}{e^n} < n! < \frac{n^{n+\frac{1}{2}}}{e^{n-1}}.$$

This can be proved using elementary calculus.

We first deal with the upper bound. Consider the area under the curve $\ln x$ over the interval $[1, n]$. We divide it into $n-1$ subintervals of width 1. For $1 \le k \le n-1$, we approximate the area of the k-th strip over $[k, k+1]$ by replacing the curve with the chord joining the left endpoint $(k, \ln k)$ to the right endpoint $(k+1, \ln(k+1))$. The area of this trapezoid is $\frac{1}{2}(\ln k + \ln(k+1))$. Since $\ln x$ is concave down, it is less than the area of the strip. It follows that

$$\int_1^n \ln x dx > \frac{1}{2}(\ln 1 + 2\ln 2 + \cdots + 2\ln(n-1) + \ln n).$$

Using integration by parts, we have

$$n\ln n - n + 1 > \ln(n!) - \frac{1}{2}\ln n$$

or $$\ln\left(\frac{n^n}{e^{n-1}}\right) > \ln\left(\frac{n!}{n^{\frac{1}{2}}}\right).$$

The desired upper bound follows from the fact that $\ln x$ is increasing.

We now turn our attention to the lower bound. Consider the area under the curve $\ln x$ over the interval $[\frac{3}{2}, n]$. We divide it into $n-2$ subintervals of width 1 and a final interval $[n-\frac{1}{2}, n]$. For $1 \le k \le n-2$, we approximate the area of the k-th strip over $[k+\frac{1}{2}, k+\frac{3}{2}]$ by replacing the curve with its tangent at the midpoint $(k+1, \ln(k+1))$. The area of this trapezoid is $\ln(k+1)$. Since $\ln x$ is concave down, it is greater than the area of the strip. For the last strip, we replace the curve with a horizontal line through the right endpoint $(n, \ln n)$. The area of this rectangle is $\frac{1}{2}\ln n$. Since $\ln x$ is increasing, it is greater than the area of the strip. It follows that

$$\int_{\frac{3}{2}}^n \ln x dx < \ln 2 + \ln 3 + \cdots + \ln(n-1) + \frac{1}{2}\ln n.$$

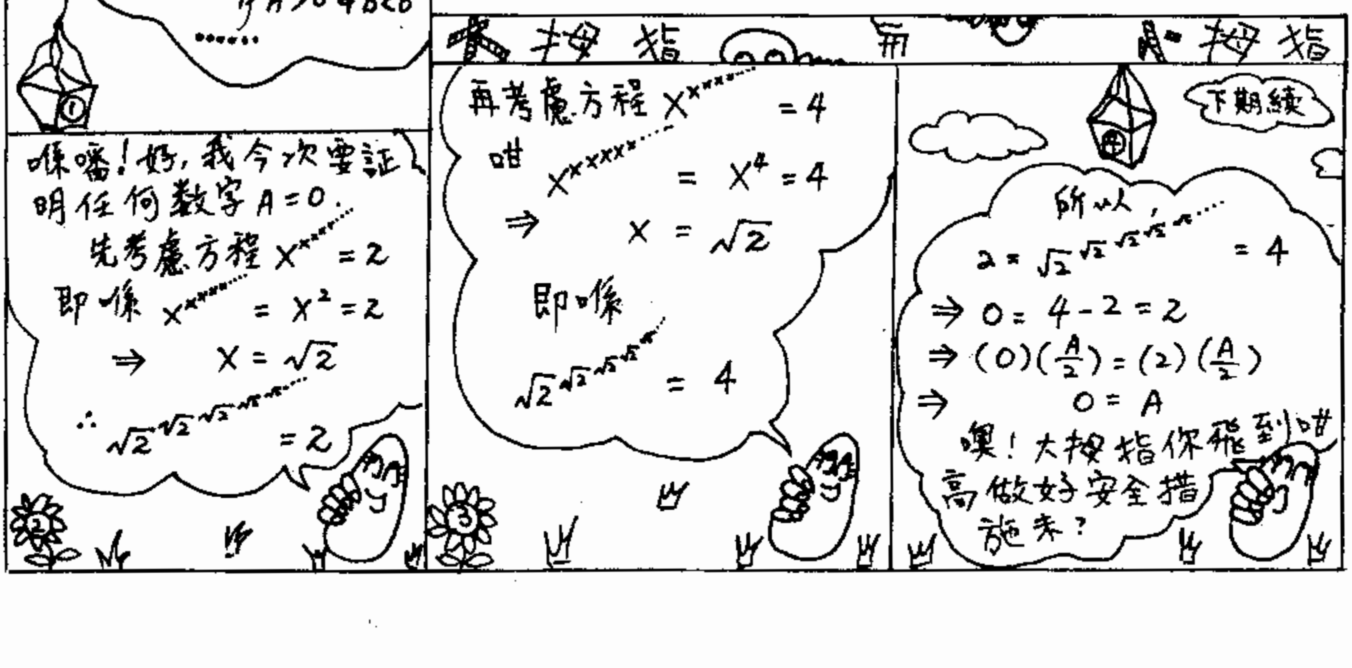
Using integration by parts, we have

$$n\ln n - n + \frac{3}{2}(1-\ln\frac{3}{2}) < \ln(n!) - \frac{1}{2}\ln n.$$

We can drop the term $\frac{3}{2}(1-\ln\frac{3}{2})$ since $1 > \ln\frac{3}{2}$. Hence

$$\ln\left(\frac{n^n}{e^n}\right) < \ln\left(\frac{n!}{n^{\frac{1}{2}}}\right).$$

The desired lower bound follows from the fact that $\ln x$ is increasing.



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is April 30, 1996.

Problem 31. Show that for any three given odd integers, there is an odd integer such that the sum of the squares of these four integers is also a square.

Problem 32. Let $a_0 = 1996$ and $a_{n+1} = a_n^2/(a_n + 1)$ for $n = 0, 1, 2, \ldots$. Prove that $[a_n] = 1996 - n$ for $n = 0, 1, 2, \ldots, 999$, where $[x]$ is the greatest integer less than or equal to x.

Problem 33. Let A, B, C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$. (A problem proposed by Germany in the last IMO.)

Problem 34. Let $n > 2$ be an integer, c be a nonzero real number and z be a nonreal root of $X^n + cX + 1$. Show that

$$|z| \geq \frac{1}{\sqrt[n]{n-1}}.$$

Problem 35. On a blackboard, nine 0's and one 1 are written. If any two of the numbers on the board may both be replaced by their average in one operation, what is the least *positive* number that can appear on the board after a finite number of such operations?

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Solutions

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Problem 26. Show that the solutions of the equation $\cos \pi x = \frac{1}{3}$ are all irrational numbers. (*Source:* 1974 Putnam Exam.)

Solution: Official Solution.

Assume $x = m/n$ (where m, n are nonzero integers and n positive) is a solution of $\cos \pi x = 1/3$. Consider $a_k = \cos k\pi x = \cos km\pi/n$ for positive integer k. Since cosine is 2π-periodic,

$$a_{k+2n} = \cos (km\pi/n + 2m\pi) = a_k,$$

so there are at most $2n$ different possible values of a_k. Using $\cos 2\theta = 2\cos^2\theta - 1$, we have

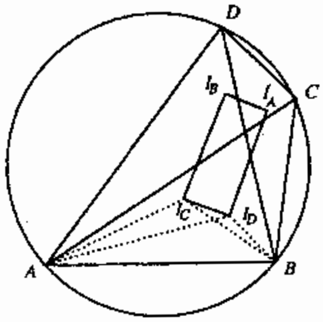
$$a_2 = -\tfrac{7}{9},\ a_4 = \tfrac{17}{81},\ \cdots,\ a_{2^p} = \frac{c_p}{3^{2^p}},\ \cdots$$

where the numerators

$$c_1 = -7,\ \ldots,\ c_p = 2c_{p-1}^2 - 3^{2^p},\ \ldots$$

are integers not divisible by 3 via mathematical induction. So the numbers a_2, a_4, a_8, a_{16}, … are all different, a contradiction.

Problem 27. Let $ABCD$ be a cyclic quadrilateral and let I_A, I_B, I_C, I_D be the incenters of ΔBCD, ΔACD, ΔABD, ΔABC, respectively. Show that $I_AI_BI_CI_D$ is a rectangle.



Solution: Independent solution by **CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 4) and **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5).

Draw segments AI_C, AI_D, BI_C, BI_D. Since $\angle ADB = \angle ACB$, we get

$$\angle DAB + \angle DBA = \angle CAB + \angle CBA.$$

Then

$$\begin{aligned}\angle I_CAI_D &= \angle I_CAB - \angle I_DAB\\ &= \tfrac{1}{2}\angle DAB - \tfrac{1}{2}\angle CAB\\ &= \tfrac{1}{2}\angle CBA - \tfrac{1}{2}\angle DBA\\ &= \angle I_DBA - \angle I_CBA\\ &= \angle I_CBI_D.\end{aligned}$$

So A, B, I_D, I_C are concyclic. Similarly, A, D, I_B, I_C are concyclic. Now

$$\begin{aligned}\angle I_BI_CI_D &= 360° - (\angle I_DI_CA + \angle I_BI_CA)\\ &= \angle I_DBA + \angle I_BDA\\ &= \tfrac{1}{2}\angle CBA + \tfrac{1}{2}\angle ADC\\ &= 90°.\end{aligned}$$

Similarly, the other three angles of $I_AI_BI_CI_D$ are right angles.

Comments: Surprisingly, this problem is the same as Problem 3 of the recently held APMO (c.f. Olympiad Corner on page 1).

Other commended solver: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College).

Problem 28. The positive integers are separated into two subsets with no common elements. Show that one of these two subsets must contain a three term arithmetic progression.

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College).

Let x be an integer greater than 6. If $x + 2$, $x + 4$, $x + 6$ are in the same subset, then we found a three term arithmetic progression there. Otherwise, x and (at least) one of $x + 2$, $x + 4$, $x + 6$ (call it $x + 2y$) are in the same subset. If this subset also contains one of $x - 2y$, $x + y$, $x + 4y$, then again there is a three term arithmetic progression. If not, then $x - 2y$, $x + y$, $x + 4y$ are in the other subset and they form a three term arithmetic progression there.

Comments: This problem is a special case of Van der Waerden's Theorem, which asserts that for every $m > 1$ and $n > 2$, there is a least integer $w(m, n)$ such that no matter how the numbers 1, 2, 3, …, $w(m, n)$ are separated into m subsets with no pairs having any common element, there will be at least one subset having an n term arithmetic progression. Two solvers, Chan Wing Sum and Alan Leung Wing Lun, independently pointed out that $w(2, 3) = 9$.

Other commended solvers: **CHAN Wing Sum** (HKUST), **KU Yuk Lun** (HKUST), **LEUNG Wing Lun** (S.T.F.A. Leung Kau Kui College, Form 4) and **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **POON Wing Chi** (La Salle College).

Problem 29. Suppose $P(x)$ is a nonconstant polynomial with integer coefficients and all coefficients are

(continued on page 4)

Problem Corner

(*continued from page 3*)

greater than or equal to -1. If $P(2) = 0$, show that $P(1) \neq 0$.

Solution: Independent solution by **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **Bobby POON Wai Hoi** (St Paul's College) and **WONG Him Ting** (HKU).

Since $P(2) = 0$, $P(x) = (x-2)(a_n x^n + \cdots + a_0)$, where $n \geq 0$, $a_n \neq 0$, $a_n, \ldots, a_0$ are integers. We may assume $n > 0$ as the case $n = 0$ is easy. Since the coefficients of $P(x)$ are at least -1, we have $-2a_0 \geq -1$, $a_{i-1} - 2a_i \geq -1$ for $i = 1, \ldots, n$ and $a_n \geq -1$. So $a_0 (\leq 1/2)$ is 0 or a negative integer. Inductively, if $a_{i-1} \leq 0$, then $a_i \leq (a_{i-1} + 1)/2 \leq 1/2$ will also be 0 or negative. Hence, $a_0, \ldots, a_n \leq 0$. Then $a_n = -1$ and $P(1) = -(a_n + \ldots + a_0) \geq -a_n = 1$.

Comments: This is a variation of a problem on the 1988 Tournament of the Towns.

Problem 30. For positive integer $n > 1$, define $f(n)$ to be 1 plus the sum of all prime numbers dividing n multiplied by their exponents, *e.g.*, $f(40) = f(2^3 \times 5^1) = 1 + (2\times3 + 5\times1) = 12$. Show that if $n > 6$, the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$ must eventually be repeating 8, 7, 8, 7, 8, 7, ….

Solution: Independent solution by **Bobby POON Wai Hoi** (St Paul's College) and **WONG Him Ting** (HKU).

Considering the factorizations of n, we see that $f(n) \leq 6$ if and only if $n \leq 6$. Clearly, $f(7) = 8$, $f(8) = 7$. For $n \geq 9$ and not prime, we will first show $f(n) \leq n - 2$ by induction.

We have $f(9) = 7$. Suppose it is true for 9 to $n - 1$. For $n > 9$ and not prime, there are positive integers $r, s > 1$ such that $n = rs$ and $(r-1)(s-1) \geq 4$. (This is because $(r-1)(s-1) \leq 3$ implies $rs \leq 2 \times 4 = 8$.) If $2 \leq r \leq 8$ or r prime, then $f(r) \leq r + 1$. Otherwise, $9 \leq r < n$ and r is not prime, which imply by the induction step that $f(r) \leq r - 2 < r + 1$. Similarly, $f(s) \leq s + 1$. From the definition of f, we get

$$\begin{aligned} f(n) &= f(r) + f(s) - 1 \\ &\leq (r+1) + (s+1) - 1 \\ &= n + 2 - (r-1)(s-1) \\ &\leq n - 2, \end{aligned}$$

which completes the induction.

Now suppose the problem is true for $n = 7, 8, \ldots, m-1$, i.e., the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$ eventually repeats 8, 7, 8, 7, 8, 7, …. For the case $n = m$, if m is not prime, then $7 \leq f(m) \leq m - 2$. By the induction step, the case $f(m)$ is true, so the case m will also be true. If m is prime, then $f(m) = 1 + m$ is not prime and so $7 \leq f(f(m)) \leq f(m) - 2 = m - 1$. By the induction step, the case $f(f(m))$ is true, so the case m will also be true.

秦九韶與「三斜求積術」：

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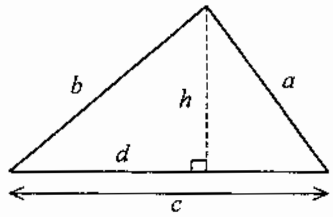
如下圖，



$$d = c - \sqrt{a^2 - h^2}$$

$$\therefore \quad b^2 = \left(c - \sqrt{a^2 - h^2}\right)^2 + h^2$$

$$= a^2 + c^2 - 2c\sqrt{a^2 - h^2}$$

$$\therefore \quad c\sqrt{a^2 - h^2} = ½(a^2 + c^2 - b^2)$$

兩端平方得

$$a^2c^2 - c^2h^2 = \left(\frac{a^2 + c^2 - b^2}{2}\right)^2$$

但，

$$面積 = ½ch$$

$$= \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2 + c^2 - b^2}{2}\right)^2}$$

（證畢）

秦氏三角

後世人對《數書九章》都有不少的批評，其中一項就是指它「脫離現實」。好似上面討論過的例子，我們跟本不可能會有一塊田，它的一邊會長「一十五里」。這個數字實在大了一點。不過，批評秦九韶的人似乎忘記了，秦九韶所引用的例子，其實有一個非常特別的特性：這個三角形的邊長是三個連續整數，而且面積亦剛好又是一個整數！祇要大家細心想想就會瞭解，這是一個很難得的「巧合」。秦九韶一定是經過細心選擇，才引用這個例子的。

因此有數學家就稱具有上述特性的三角形爲「秦氏三角」。除了(13, 14, 15)可以組成「秦氏三角」之外，還有(3, 4, 5)、(51, 52, 53)、(193, 194, 195)、(723, 724, 725)等等。現在更有數學家發現：如果k是一個正整數，祇要$\sqrt{3(k^2-1)}$亦是一個整數，則$(2k-1, 2k, 2k+1)$就可組成一個「秦氏三角」了。

希羅公式 (Heron's formula)

最後一提的是，秦九韶並不是歷史上第一位懂得「三斜求積」方法的人。大約在公元100年左右，希臘數學家希羅(Heron)早已提出了一個計算面積而且更簡單的公式。不過，他當時提出的證明就非常複雜，如果從秦九韶的公式出發，我們會更容易獲得那「希羅公式」：

$$面積 = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2}$$

$$= \sqrt{\frac{1}{16}\left(4a^2c^2 - (a^2+c^2-b^2)^2\right)}$$

$$= \sqrt{\frac{1}{16}(2ac+a^2+c^2-b^2)(2ac-a^2-c^2+b^2)}$$

$$= \sqrt{\frac{1}{16}\left((a+c)^2 - b^2\right)\left(b^2 - (a-c)^2\right)}$$

$$= \sqrt{\frac{1}{16}(a+c+b)(a+c-b)(b+a-c)(b-a+c)}$$

$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)}$$

最後設 $s = ½(a + b + c)$，則

$$面積 = \sqrt{s(s-a)(s-b)(s-c)}\ ,$$

這就是「希羅公式」了。

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Mathematical Excalibur

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Olympiad Corner

1996 Canadian Mathematical Olympiad:

Problem 1. If α, β and γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}.$$

Problem 2. Find all real solutions to the following system of equations:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y, \\ \dfrac{4y^2}{1+4y^2} = z, \\ \dfrac{4z^2}{1+4z^2} = x. \end{cases}$$

Carefully justify your answer.

Problem 3. We denote an arbitrary permutation of the integers $1, 2, \cdots, n$ by $a_1, a_2, \cdots, a_n$. Let $f(n)$ be the number of these permutations such that

(i) $a_1 = 1$;

(*continued on page 4*)

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Fermat's Little Theorem and Other Stories

T.W. Leung
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Pierre de Fermat (1601-1665), a councilor of the provincial High Court of Judicature in Toulouse, south of France, practised mathematics during his spare time. He discussed his findings with his friends via letters. As it turned out, his works significantly influenced the development of modern mathematics. During Fermat's time, the following "Chinese hypothesis" was around:

p is a prime if and only if $2^p \equiv 2 \pmod p$.

One direction of the hypothesis is not true. In fact $2^{341} - 2$ is divisible by 341, yet $341 = 11 \times 31$ is composite (not prime). However the other direction is indeed valid. From the manuscripts and letters of Fermat, we conclude that Fermat knew (and most likely could prove) the following facts:

(1) If n is not a prime, then $2^n - 1$ is not a prime.
(2) If n is a prime, then $2^n - 2$ is a multiple of $2n$.
(3) If n is a prime, and p is a prime divisor of $2^n - 1$, then $p - 1$ is a multiple of n.

The first statement can be proved directly by factoring $2^n - 1$. If $n = pq$ (with $p > 1$ and $q > 1$), then

$$\begin{aligned} 2^n - 1 &= 2^{pq} - 1 \\ &= (2^p - 1)(2^{p(q-1)} + 2^{p(q-2)} + \cdots + 1). \end{aligned}$$

The other two statements are variations of the more general statement, indicated in his other letter:

> Given any prime p, and any geometric progression $1, a, a^2, \cdots$, the number p must divide some number $a^n - 1$, for which n divides $p-1$; if then N is any multiple of the smallest number n for which this is so, p divides also $a^N - 1$.

With modern mathematical notation, we may rewrite Fermat's statement as the following which will be referred to as Fermat's Little Theorem:

> If p is a prime number and a is any integer, then $a^p \equiv a \pmod p$. In particular, if p does not divide a, then $a^{p-1} \equiv 1 \pmod p$.

Now we see how Fermat made use of his little theorem. He was challenged to determine if there is any even perfect number lying between 10^{20} and 10^{22}. (A positive integer n is called a perfect number if the sum of all proper factors (i.e., excluding n) of n is equal to n. For example, $6 = 1+2+3$ and $28 = 1+2+4+7+14$ are perfect numbers.) This problem can be reduced (how?) to check if $2^{37} - 1$ is prime. Suppose the number is not prime, and p is an odd prime divisor of that number, then from the third statement, $p - 1$ is a multiple of 37, or $p = 37k + 1$, observe that p is odd, so k is even, or p is of the form $74k' + 1$. The first few candidates are 149, 223, $\cdots$. One then check that

$$\begin{aligned} 2^{37} - 1 &= 137438953471 \\ &= 223 \times 616318177. \end{aligned}$$

It is more difficult to check that the second factor is a prime, however Fermat succeeded in showing that $2^{37} - 1$ is not prime.

Another side story comes from the fact that if $2^m + 1$ is prime, then m must be of the form 2^n. Fermat conjectured that all these numbers are prime. Now $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$ and $2^{2^4} + 1 = 65537$ are indeed prime numbers. However,

$$2^{2^5} + 1 = 4294967297$$

is not a prime. In fact, if p is a prime factor of $2^{2^n} + 1$, then 2^{n+1} is the smallest

(*continued on page 2*)

Fermat's Little Theorem …

(*continued from page 1*)

m satisfying $2^m \equiv 1 \pmod p$, thus 2^{n+1} divides $p-1$, or p is of the form $k2^{n+1}+1$, hence to look for prime factors of $2^{2^5}+1 = 2^{32}+1$, we should consider primes of the form $64k+1$. The possible candidates are 193, 257, 449, 577, 641, $\cdots$. Unfortunately, neither Fermat nor his contemporaries had enough patience to check that 641 indeed divides $2^{32}+1$. (For readers who are familiar with the law of quadratic reciprocity, one can prove that a prime divisor of $2^{2^n}+1$ is actually of the form $k2^{n+2}+1$.)

Fermat did not explicitly give any proof of the Fermat's little theorem, and it was Euler who first proved by induction the following fact: if p is a prime then $a^p \equiv a \pmod p$. Clearly the statement is true if $a = 1$. Now

$$(a+1)^p \equiv a^p + \binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} + \cdots + 1 \equiv a+1 \pmod p,$$

where $\binom{p}{i} = \dfrac{p!}{i!(p-i)!} \equiv 0 \pmod p$ for $1 \le i \le p-1$.

There is also another version of the theorem, namely, if p is a prime and a is relatively prime to p, then $a^{p-1} \equiv 1 \pmod p$. Euler also gave the first proof by noting that the terms of the series $1, a, a^2, \cdots \pmod p$ must repeat. So for some $r \ge 0$, and some $s \ge 0$, we must have $a^{r+s} \equiv a^r \pmod p$, i.e., $a^s \equiv 1 \pmod p$. Let s be the smallest positive integer such that $a^s \equiv 1 \pmod p$, then one can arrange the $p-1$ non-zero congruence classes modulo p into sets $\{b, ba, \cdots, ba^{s-1}\}$, where each set consists of s elements and the sets are disjoint. Thus s must divide $p-1$. For example, with $p = 7$ and $a = 2$, one obtains $s = 3$ and the numbers 1 to 6 can be grouped into two disjoint sets $\{1, 2, 4\}$ and $\{3, 6, 5\}$. We also observe that $p-1 = 6$ is divisible by $s = 3$. Euler generalized this argument to prove the famous Euler's theorem:

If a is relatively prime to n, then $a^{\phi(n)} \equiv 1 \pmod n$,

where $\phi(n)$ is the Euler totient function that counts the number of integers between 1 and n that are relatively prime to n. For example, $\phi(12) = 4$ since only 1, 5, 7, 11 (among the numbers 1-12) are relatively prime to 12.

A formal proof of Euler's theorem goes as follows: Let a be an integer relatively prime to n and let $\{a_1, a_2, \ldots, a_{\phi(n)}\}$ be the set of reduced residues modulo n (i.e., the $\phi(n)$ positive integers less than n that are relatively prime to n). Then the set $\{aa_1, aa_2, \ldots, aa_{\phi(n)}\}$ is also a set of reduced residues modulo n. Hence,

$$a_1a_2\cdots a_{\phi(n)} \equiv a^{\phi(n)}a_1a_2\cdots a_{\phi(n)} \pmod n$$

or $\quad a^{\phi(n)} \equiv 1 \pmod n$.

There is however another colouring argument for Fermat's little theorem. Arrange p boxes in a circle and colour them with a colours. There are a^p possible colouring patterns. Among all these possible colourings, a of them are such that every box has the same colour. The remaining $a^p - a$ colouring patterns can be grouped into sets of p patterns that are rotations of each other. The p rotations of any one of these colourings are all distinct and thus p divides $a^p - a$. (Where did we use "p is prime"?) Hence, in essence, the Fermat's little theorem can be proved using the pigeonhole principle.

The following are some applications of Fermat's little theorem and Euler's theorem.

Example 1: If n is an integer > 1, then n does not divide $2^n - 1$.

Solution: If n is even, then the statement is certainly true since $2^n - 1$ is an odd integer. For n odd, denote by p the smallest prime divisor of n. Suppose n (and thus also p) divides $2^n - 1$. By the Fermat's little theorem, p divides $2^{p-1} - 1$ too. Consequently, p divides $2^d - 1$, where d is the greatest common divisor of $p-1$ and n, Since p is the smallest prime divisor of n, $d = 1$ which leads to the contradiction p divides 1.

Example 2: Let n be an odd number not divisible by 5, then n divides a number of the form $99\cdots9$.

Solution: If n is odd and not divisible by 5, then n is relatively prime to 10. By the Euler's theorem, $10^{\phi(n)} \equiv 1 \pmod n$, i.e., n divides $10^{\phi(n)} - 1$, which is a number of the form $99\cdots9$.

Example 3: Let p be an odd prime number. Then for any set of $2p-1$ integers, there exists a set of p integers whose sum is divisible by p.

Sketch of Solution: There are $n = \binom{2p-1}{p}$ distinct sets that each contains p elements. Denote their sums by $s_1, s_2, \ldots, s_n$. Suppose none of them is divisible by p. Then, by the Fermat's little theorem, $\sum_{i=1}^{n} s_i^{\,p-1} = \sum_{i=1}^{n} 1 = n$, which is nonzero modulo p. On the other hand, one may use the multinomial expansion to show that $\sum_{i=1}^{n} s_i^{\,p-1}$ is, in fact, divisible by p, and thus lead to a contradiction.

It is interesting to observe that we use a number theoretic approach to solve a combinatorial problem while using a counting argument to prove Fermat's little theorem.

We have mentioned that the converse of Fermat's little theorem is not true. That is, there exists composite numbers n such that n divides $a^{n-1} - 1$. For example, as stated at the beginning of this article, the composite number 341 divides $2^{340} - 1$. Composite numbers n (which must be odd) that divides $2^{n-1} - 1$ are called pseudoprimes (in base 2). One may show that there exist infinitely many such pseudoprimes. In fact, if n is a pseudoprime, then $m = 2^n - 1$ will be composite (since n is composite). Also, $m - 1 = 2^n - 2 = nk$ and thus $2^{m-1} - 1 = 2^{nk} - 1$ is divisible by $2^n - 1 = m$. That is, m is another pseudoprime (in base 2).

We may of course try another base. For our example, we find that 341 is no longer a pseudoprime (in base 3), i.e., 341 does not divide $3^{340} - 1$. Well, we may then ask: is it possible to find a composite number n such that for every a relatively prime to n, $a^{n-1} \equiv 1 \pmod n$. Such a number is called a Carmichael number. Surprisingly, not only that they exist (with 561 being the smallest), there are infinitely many Carmichael numbers, which, in fact, was proved recently!

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is July 10, 1996.

The following problems are selected from the International Mathematics Tournament of the Towns, held in April 7, 1996.

Problem 36. Let a, b and c be positive numbers such that $a^2 + b^2 - ab = c^2$. Prove that $(a-c)(b-c) \le 0$.

Problem 37. Two non-intersecting circles λ_1 and λ_2 have centres O_1 and O_2 respectively. A_1 and A_2 are points on λ_1 and λ_2 respectively, such that A_1A_2 is an external common tangent of the circles. The segment O_1O_2 intersects λ_1 and λ_2 at B_1 and B_2 respectively. The lines A_1B_1 and A_2B_2 intersect at C, and the line through C perpendicular to B_1B_2 intersects A_1A_2 at D. Prove that D is the midpoint of A_1A_2.

Problem 38. Prove that from any sequence of 1996 real numbers, one can choose a block of consecutive terms whose sum differs from an integer by at most 0.001.

Problem 39. Eight students took part in a contest with eight problems.

(a) Each problem was solved by 5 students. Prove that there were two students who between them solved all eight problems.

(b) Prove that this is not necessarily the case if 5 is replaced by 4. (A counterexample is enough.)

Problem 40. ABC is an equilateral triangle. For a positive integer $n \ge 2$, D is the point on AB such that $AD = \frac{1}{n}AB$. $P_1, P_2, \cdots, P_{n-1}$ are points on BC which divide it into n equal segments. Prove that $\angle AP_1D + \angle AP_2D + \cdots + \angle AP_{n-1}D = 30°$.

[*Hint*: Consider Q_i such that ADP_iQ_i is a parallelogram.]

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Solutions

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Problem 31. Show that for any three given odd integers, there is an odd integer such that the sum of the squares of these four integers is also a square.

Solution: Independent solution by **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 5), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **PAI Hung Ming Tedward** (S.K.H. Tang Shiu Kin Secondary School, Form 6).

Let $x = 2a + 1, y = 2b + 1, z = 2c + 1$ be three given odd integers, then $x^2 + y^2 + z^2 = 2w + 1$, where $w = 2(a^2 + a + b^2 + b + c^2 + c) + 1$ is odd. So $x^2 + y^2 + z^2 + w^2 = (w + 1)^2$.

Other commended solver: **CHAN Wing Chiu** (La Salle College, Form 3), **CHENG Wing Kin** (S.K.H. Lam Woo Secondary School, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **W. H. FOK** (Homantin Government Secondary School), **Alan LEUNG Wing Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College) and **YAU Kwan Kiu** (Queen's College, Form 7).

Problem 32. Let $a_0 = 1996$ and $a_{n+1} = a_n^2/(a_n + 1)$ for $n = 0, 1, 2, \ldots$. Prove that $[a_n] = 1996 - n$ for $n = 0, 1, 2, \ldots, 999$, where $[x]$ is the greatest integer less than or equal to x.

Solution: Independent solution by **CHAN Wing Sum** (HKUST), **W. H. FOK** (Homantin Government Secondary School) and **KU Yuk Lun** (HKUST).

Note that $a_n > 0$ implies $a_{n+1} > 0$ and

$$a_n - a_{n+1} = 1 - \frac{1}{a_n + 1} > 0.$$

Hence $a_0 > a_1 > a_2 > \cdots$. Now

$$\begin{aligned} a_n &= a_0 + (a_1 - a_0) + \cdots + (a_n - a_{n-1}) \\ &= 1996 - n + \frac{1}{a_0 + 1} + \cdots + \frac{1}{a_{n-1} + 1} \\ &> 1996 - n. \end{aligned}$$

For $1 \le n \le 999$,

$$\begin{aligned} \frac{1}{a_0 + 1} + \cdots + \frac{1}{a_{n-1} + 1} &< \frac{n}{a_{n-1} + 1} \\ &< \frac{999}{a_{998} + 1} < \frac{999}{1996 - 998 + 1} = 1. \end{aligned}$$

So $[a_n] = 1996 - n$.

Comments: With 1996 replaced by 1994, 999 replaced by 998, this was a problem proposed by USA in the 1994 IMO.

Other commended solver: **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 5), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5), **POON Wing Chi** (La Salle College) and **YAU Kwan Kiu** (Queen's College, Form 7).

Problem 33. Let A, B, C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$. (A problem proposed by Germany in the last IMO.)

Solution: Henry NG Ka Man (S.T.F.A. Leung Kau Kui College, Form 5).

Without loss of generality, we may assume A, B, C have coordinates $(a,0)$, $(b,0)$, $(0,c)$, (where $a \ne b$ and $c \ne 0$) respectively. Let X be a point in the plane of ABC with coordinates (x,y). For X to satisfy the given conditions, the equations on x and y are $ax - cy = a^2 - c^2 - ab$, $bx - cy = b^2 - c^2 - ab$ and $x = a + b$ (after simplification), which has a unique solution $(x,y) = (a+b, c+2ab/c)$.

Other commended solvers: **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 5), **W. H. FOK** (Homantin Government Secondary School), **Alan LEUNG Wing Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College) and **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3).

Problem 34. Let $n > 2$ be an integer, c be a nonzero real number and z be a nonreal

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

root of $X^n + cX + 1$. Show that

$$|z| \geq \frac{1}{\sqrt[n]{n-1}}.$$

Solution 1: W. H. FOK (Homantin Government Secondary School).

Write $z = r(\cos\theta + i\sin\theta)$ with $\sin\theta \neq 0$. Taking the real and imaginary parts of $z^n + cz + 1 = 0$ using De Moivre's theorem, we have

$$r^n\cos n\theta + cr\cos\theta + 1 = 0$$

and $$r^n\sin n\theta + cr\sin\theta = 0.$$

Then

$$\begin{aligned} r^n\sin(n-1)\theta &= r^n\sin n\theta\cos\theta - r^n\cos n\theta\sin\theta \\ &= -cr\sin\theta\cos\theta + (cr\cos\theta + 1)\sin\theta \\ &= \sin\theta. \end{aligned}$$

Since

$$\begin{aligned} |\sin(k+1)\theta| &= |\sin k\theta\cos\theta + \cos k\theta\sin\theta| \\ &\leq |\sin k\theta| + |\sin\theta|, \end{aligned}$$

induction gives $|\sin k\theta| \leq k|\sin\theta|$ for every positive integer k. So

$$|z|^n = r^n = |\sin\theta/\sin(n-1)\theta| \geq 1/(n-1).$$

Solution 2: LEUNG Hoi-Ming (SKH Lui Ming Choi Secondary School).

Let $r = |z|$ and $w = z/r$. Then $|w| = 1$ and $w\overline{w} = 1$. Since $(rw)^n + crw + 1 = 0$, multiplying by $\overline{w}$, then conjugating, we get

$$r^n w^{n-1} + cr + \overline{w} = 0$$

and $$r^n \overline{w}^{n-1} + cr + w = 0.$$

Subtracting these equations and solving for r^n, we get

$$r^n = \frac{w - \overline{w}}{w^{n-1} - \overline{w}^{n-1}} = \frac{1}{\sum_{i=0}^{n-2} w^{n-2-i}\overline{w}^i}.$$

Since r is real and $|w| = 1$, by the triangle inequality,

$$r^n \geq \frac{1}{\sum_{i=0}^{n-2} |w^{n-2-i}\overline{w}^i|} = \frac{1}{n-1}.$$

Other commended solvers: **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 5).

Problem 35. On a blackboard, nine 0's and one 1 are written. If any two of the numbers on the board may both be replaced by their average in one operation, what is the least *positive* number that can appear on the board after a finite number of such operations?

Solution: POON Wing Chi (La Salle College).

Let m be the least positive number on the board and n be the number of zeros on the board after an operation. Consider the number $c = m/2^n$. If two positive numbers are both replaced by their average, then n does not change, but m (and c) may increase. If a 0 is averaged with a positive number r, then n decreases by one and m remains unchanged or becomes $r/2$ ($\geq m/2$.) The new c value will be greater than or equal to $(m/2)/2^{n-1} = m/2^n$, which is the old c value. In the beginning, $c = 1/512$. After a finite number of operations, $c \geq 1/512$ and $m \geq 2^n/512 \geq 1/512$. To obtain exactly 1/512, start with 1 and average with each of the nine 0's.

Comments: This problem comes from an article in the March/April 1994 issue of *Quantum*, published by Springer Verlag. The article dealt with the concept of *monoinvariant*, which is an expression like c in the problem that increases after each operation. Studying such expression often solves the problem.

Olympiad Corner

(*continued from page 1*)

(ii) $|a_i - a_{i+1}| \leq 2, \quad i = 1, \cdots, n-1.$

Determine whether $f(1996)$ is divisible by 3.

Problem 4. In ΔABC, $AB = AC$. Suppose that the bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

Problem 5. Let $r_1, r_2, \ldots, r_m$ be m given positive rational numbers such that

$$\sum_{k=1}^{m} r_k = 1.$$

Define the function f by

$$f(n) = n - \sum_{k=1}^{m} \lfloor r_k n \rfloor$$

for each positive integer n. Determine the minimum and maximum values of $f(n)$.

From the Editors' Desk:

Thanks to our readers for another year of support, especially the submission of articles and problem solutions. If you would like to receive your personal copy for the five issues for the 96-97 academic year, send five stamped self-addressed envelopes to Dr. Kin-Yin Li, Hong Kong University of Science and Technology, Department of Mathematics, Clear Water Bay, Kowloon, Hong Kong.

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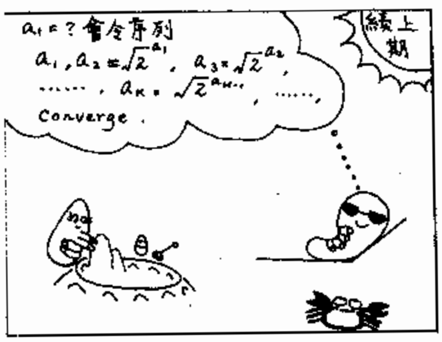
APMO and **IMO**: The Eighth APMO took place on March 16th. The Hong Kong students had a very strong (record setting) performance. The top 8 scorers are as follow. (Note the maximum is 7×5=35 points.)

1. 潘維凱 (**Bobby POON Wai Hoi**), St. Paul's College, 35 points (Perfect score! First time for Hong Kong)
2. 余振陵 (**YU Chun Ling**), Ying Wa College, 33 points
3. 何穎業 (**HO Wing Yip**), Clementi Secondary School, 32 points
4. 莫子韜 (**MOK Tsz Tao**), Queen's College, 31 points
5. 謝珊珊 (**TSE Shan Shan**), Tuen Mun Government Secondary School, 29 points
6. 羅肇龍 (**LAW Siu Lung**), Diocesan Boy's School, 26 points
7. 翁漢威 (**YUNG Hon Wai**), Heep Woh College, 26 points
8. 朱天健 (**CHU Tim Kin**), King's College, 24 points

The first 6 students are invited to be the Hong Kong team members to participate in the 37th International Mathematical Olympiad to be held in India this summer. The selection was based on their outstanding performance in the APMO and throughout the Hong Kong Math Olympiad training program.



Mathematical Excalibur

Volume 2, Number 4 Sept-Oct, 1996

Olympiad Corner

37th International Mathematical Olympiad, July 5-17, 1996, Mumbai, India.

First Day (10 July, 1996)
Time: 4½ hours
(Each problem is worth 7 points.)

Problem 1. Let $ABCD$ be a rectangular board with $|AB| = 20$, $|BC| = 12$. The board is divided into 20×12 unit squares. Let r be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the two squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square which has A as a vertex to the square which has B as a vertex.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task can be done if $r = 73$.

(c) Can the task be done when $r = 97$?

(*continued on page 4*)

Acknowledgment: Thanks to NG Hok Lin, POON Wing Chi and WAN Yung Chun for their help in typesetting.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Nov 15, 1996.

For individual subscription for the remaining four issues for the 96-97 academic year, send us four stamped self-addressed envelopes. Send all correspondence to:

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Clear Water Bay, Kowloon, Hong Kong

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Ptolemy's Theorem

Kin-Yin Li

If four points are chosen from a plane, the chance that they are collinear or concyclic is extremely small. So there should be some special conditions for this to happen. Such a condition is given by the famous theorem of Ptolemy.

Ptolemy's Theorem. *For distinct points A, B, C, D on a plane, we have $AB \cdot CD + AD \cdot BC \ge AC \cdot BD$. Equality happens if and only if A, B, C, D are collinear or concyclic with A, C separating B, D.*

A simple proof using complex numbers can be given as follows. Let a, b, c, d be the complex numbers corresponding to the points A, B, C, D respectively. Since

$$(b-a)(d-c) + (d-a)(c-b) = (c-a)(d-b),$$

taking absolute values and applying the triangle inequality, we get

$$\begin{aligned} AB \cdot CD + AD \cdot BC &= |b-a||d-c| + |d-a||c-b| \\ &\ge |c-a||d-b| = AC \cdot BD. \end{aligned}$$

From the triangle inequality, we have equality if and only if

$$(b-a)(d-c) = t(d-a)(c-b) \text{ for some } t>0.$$

In such case, $(d-a)/(b-a)$ is a positive multiple of $(d-c)/(c-b)$. So

$$\arg\{(d-a)/(b-a)\} = \arg\{(d-c)/(c-b)\},$$

i.e., $\angle DAB = 180° - \angle DCB$. This means A, B, C, D are collinear or concyclic with A, C separating B, D.

Next we will give two simple and useful corollaries.

Corollary 1. *For a cyclic quadrilateral $ABCD$ with ΔABC equilateral, we have $BD = AD + CD$.*

Corollary 2. *For a cyclic quadrilateral $ABCD$ with $\angle ABC = \angle ADC = 90°$, we have $BD = AC\sin\angle BAD$.*

The first corollary follows because $AB = BC = CA$ and thus

$$\begin{aligned} & AB \cdot CD + AD \cdot BC = AC \cdot BD \\ \Rightarrow\ & CA \cdot CD + AD \cdot CA = AC \cdot BD \\ \Rightarrow\ & CD + AD = BD. \end{aligned}$$

The second corollary also follows easily because

$$\begin{aligned} AC\sin\angle A &= AC\sin(\angle BAC + \angle DAC) \\ &= (BC \cdot AD + AB \cdot CD)/AC = BD \end{aligned}$$

using the compound angle formula. (Actually, corollary 2 is true if A, B, C, D are just concyclic, not necessarily in that order, and $\angle ABC = \angle ADC = 90°$, since by the sine law, $BD/\sin\angle BAD$ equals the diameter AC of the circumcircle of ΔBAD.)

Example 1. (*IMO 1995*) Let $ABCDEF$ be a convex hexagon with $AB=BC=CD$, $DE=EF=FA$ and $\angle BCD = \angle EFA = 60°$. Let G and H be two points inside the hexagon such that $\angle AGB = \angle DHE = 120°$. Show that

$$AG + GB + GH + DH + HE \ge CF.$$

Solution. Let X, Y be points outside the hexagon such that ΔABX and ΔDEY are equilateral. Then $ABCDEF$ is congruent to $DBXAEY$ and $CF = XY$. Now

$$\angle AXB + \angle AGB = \angle DYE + \angle DHE = 180°.$$

Thus $AXBG$ and $DHEY$ are cyclic quadrilaterals. By corollary 1, $XG = AG + GB$ and $HY = DH + HE$. So

$$\begin{aligned} &AG+GB+GH+DH+HE \\ &\quad = XG+GH+HY \ge XY = CF. \end{aligned}$$

Example 2. (*IMO 1996*) Let P be a point inside ΔABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D, E be the incenters of ΔAPB, ΔAPC, respectively.

(*continued on page 2*)

Ptolemy's Theorem

(*continued from page 1*)

Show that AP, BD and CE meet at a point.

Solution. Equivalently we have to show the angle bisectors BD, CE of $\angle ABP$, $\angle ACP$, respectively, meet at the same point on AP. Let the feet of the perpendiculars from P to BC, CA, AB be X, Y, Z respectively. Then $AZPY$, $BXPZ$, $CYPX$ are cyclic quadrilaterals. Now

$$\begin{aligned}\angle APB - \angle ACB &= \angle YAP + \angle XBP\\ &= \angle YZP + \angle XZP\\ &= \angle YZX.\end{aligned}$$

Similarly $\angle APC - \angle ABC = \angle XYZ$. So $XZ = XY$. By corollary 2,

$$BP\sin\angle B = XZ = XY = CP\sin\angle C.$$

Then $BP/CP = \sin\angle C/\sin\angle B = AB/AC$. So $AB/BP = AC/CP$. By the angle bisector theorem, this implies BD and CE meet at the same point on AP.

Example 3. (*Erdös-Mordell Inequality*) Let P be a point inside ΔABC and let d_a, d_b, d_c be the distances from P to BC, CA, AB respectively. Show that

$$PA + PB + PC \geq 2(d_a + d_b + d_c)$$

with equality if and only if ΔABC is equilateral and P is the incenter.

Solution. Let X, Y, Z be the feet of perpendiculars from P to BC, CA, AB respectively. By corollary 2 or sine law and cosine law,

$$\begin{aligned}PA\sin\angle A &= YZ\\ &= \sqrt{d_b^2 + d_c^2 - 2d_b d_c\cos(180°-\angle A)}.\end{aligned}$$

Since $180°-\angle A = \angle B+\angle C$, expanding and regrouping, we get

$$\begin{aligned}PA\sin\angle A &= \{(d_b\sin\angle C+d_c\sin\angle B)^2\\ &\quad + (d_b\sin\angle C+d_c\sin\angle B)^2\}^{1/2}\\ &\geq d_b\sin\angle C + d_c\sin\angle B.\end{aligned}$$

Using inequalities like the last one and the fact $x + 1/x \geq 2$, we have

$$\begin{aligned}PA+PB+PC &\geq \sum \frac{d_b\sin\angle C + d_c\sin\angle B}{\sin\angle A}\\ &= \sum d_a\left(\frac{\sin\angle B}{\sin\angle C} + \frac{\sin\angle C}{\sin\angle B}\right)\\ &\geq 2(d_a + d_b + d_c).\end{aligned}$$

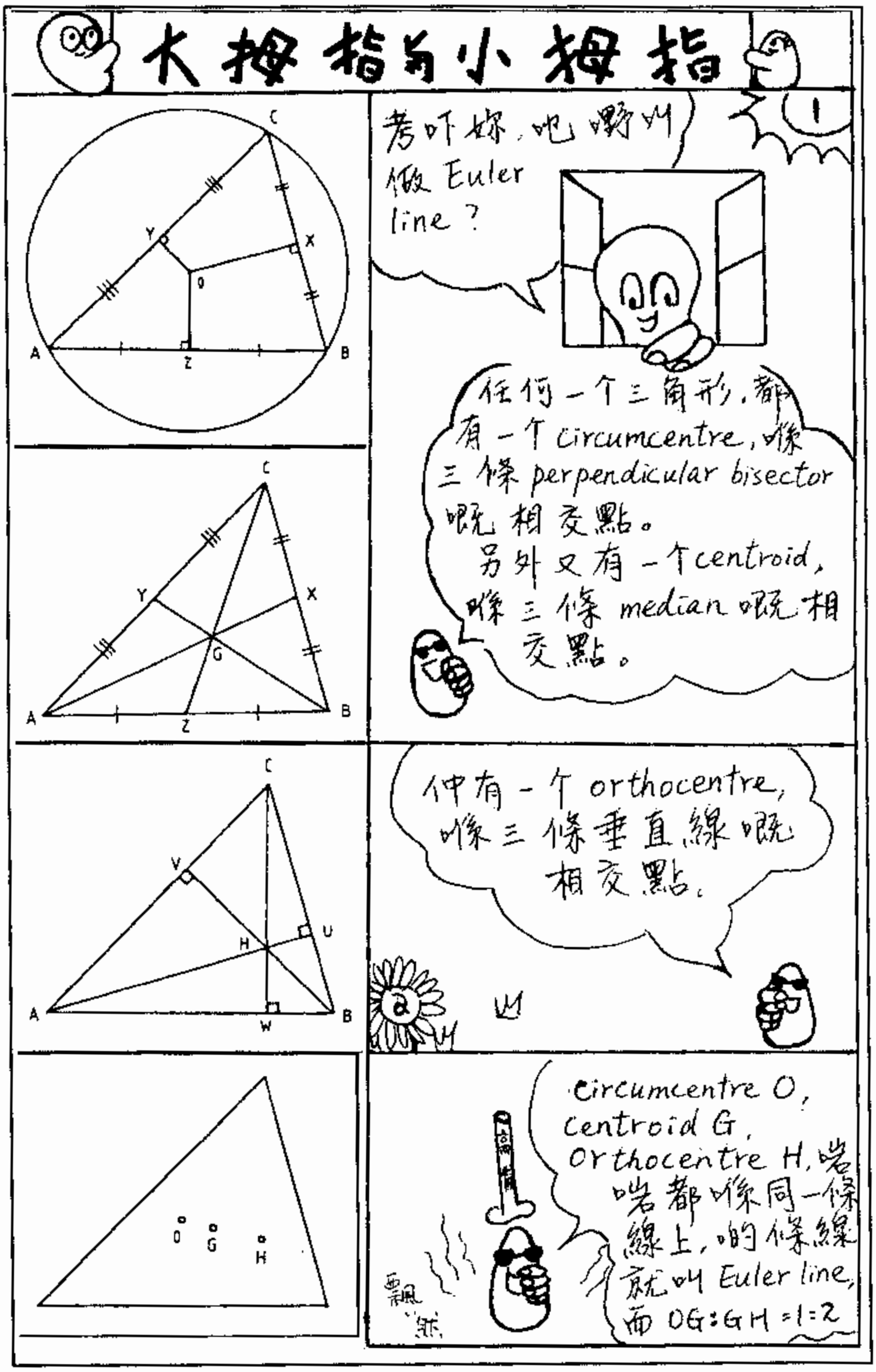
where the middle equation was obtained by rearranging terms. Finally, equality occurs if and only if $\angle A = \angle B = \angle C$ and $d_a = d_b = d_c$, i.e., ΔABC is equilateral and P is the incenter.

Example 4. (*IMO 1991*) Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.

Solution. Suppose none of the three angles is less than or equal to 30°. If one of them is at least 150°, then the other two will be at most 30°, a contradiction. So we may assume the three angles are greater than 30° and less than 150°. Let d_a be the distance from P to BC, then

$$\begin{aligned}2d_a &= 2PB\sin\angle PBC\\ &> (2\sin 30°)PB = PB.\end{aligned}$$

Three such inequalities added together will yield $2(d_a + d_b + d_c) > PA + PB + PC$, contradicting the Erdös-Mordell inequality. So one of the three angles is at most 30°.



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Nov 15, 1996.

Problem 41. Find all nonnegative integers x, y satisfying $(xy-7)^2 = x^2 + y^2$.

Problem 42. What are the possible values of $\sqrt{x^2+x+1}-\sqrt{x^2-x+1}$ as x ranges over all real numbers?

Problem 43. How many 3-element subsets of the set $X = \{1, 2, 3, ..., 20\}$ are there such that the product of the 3 numbers in the subset is divisible by 4?

Problem 44. For an acute triangle ABC, let H be the foot of the perpendicular from A to BC. Let M, N be the feet of the perpendiculars from H to AB, AC, respectively. Define L_A to be the line through A perpendicular to MN and similarly define L_B and L_C. Show that L_A, L_B and L_C pass through a common point O. (This was an unused problem proposed by Iceland in a past IMO.)

Problem 45. Let a, b, $c > 0$ and $abc=1$. Show that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1$$

(This was an unused problem in IMO96.)

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Solutions

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Problem 36. Let a, b and c be positive numbers such that $a^2 + b^2 - ab = c^2$. Prove that $(a-c)(b-c) \le 0$.

Solution: POON Wing Chi (La Salle College, Form 6).

Without loss of generality, we may assume $a \le b$. Since a, $b > 0$, so

$$a \le \sqrt{a^2+b(b-a)} = c = \sqrt{b^2-a(b-a)} \le b.$$

Therefore $(a-c)(b-c) \le 0$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 5), **CHAN Wing Sum** (HKUST), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 54), **KWOK Wing Yin** (St. Clare's Girls' School), **LEE Ho Fai Vincent** (Queen's College, Form 6), **Alan LEUNG Wing Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5), **NG Pui Keung** (St. Paul's Co-educational College, From 5), **PAI Hung Ming Tedward** (S.K.H. Tang Shiu Kin Secondary School, Form 6), **SZE Hoi Wing Holman** (St. Paul's Co-educational College , Form 5) and **YU Kit Wing** (HKUST).

Problem 37. Two non-intersecting circles λ_1 and λ_2 have centres O_1 and O_2 respectively. A_1 and A_2 are points on λ_1 and λ_2 respectively, such that A_1A_2 is an external common tangent of the circles. The segment O_1O_2 intersects λ_1 and λ_2 at B_1 and B_2 respectively. The lines A_1B_1 and A_2B_2 intersect at C, and the line through C perpendicular to B_1B_2 intersects A_1A_2 at D. Prove that D is the midpoint of A_1A_2.

Solution: Independent solution by **CHAN Ming Chiu** (La Salle College, Form 5), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **PAI Hung Ming Tedward** (S.K.H. Tang Shiu Kin Secondary School, Form 6).

We have $\angle DA_1C = 90° - \angle O_1A_1B_1 = 90° - \angle O_1B_1A_1 = 90° - \angle CB_1O_2 = \angle A_1CD$, which implies $A_1D = CD$. Similarly, $A_2D = CD$. So $A_1D = A_2D$.

Problem 38. Prove that from any sequence of 1996 real numbers, one can choose a block of consecutive terms whose sum differs from an integer by at most 0.001.

Solution: LIU Wai Kwong (Pui Tak Canossian College).

Let the numbers be x_1, x_2, ..., x_{1996} and let $s_i = x_1 + x_2 + ... + x_i$ for $i = 1, 2, ...,$ 1996. Define $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x. Consider the 1995 intervals $[0, \frac{1}{1995})$, $[\frac{1}{1995}, \frac{2}{1995})$, ..., $[\frac{1994}{1995}, 1)$ and the 1996 numbers $\{s_1\}$, $\{s_2\}$, ..., $\{s_{1996}\}$. By the pigeon-hole principle, there is a pair s_i, s_j with $\{s_i\}$, $\{s_j\}$ in the same interval. By cancelling the common terms in s_i, s_j, we get a block of consecutive terms whose sum differ from an integer by at most $\frac{1}{1995} < 0.001$.

Problem 39. Eight students took part in a contest with eight problems.

(a) Each problem was solved by 5 students. Prove that there were two students who between them solved all eight problems.

(b) Prove that this is not necessarily the case if 5 is replaced by 4. (A counterexample is enough.)

Solution: Independent solution by **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **POON Wing Chi** (La Salle College, Form 6).

(a) If a pair of students together did not solve all 8 problems, then there was at least 1 problem they both missed. Among 8 students, there are 28 pairs. However, for each problem, there were only 3 students (which give 3 pairs) missed the problem. For the 8 problems, there were at most 24 pairs missing at least 1 problem. Since 28 > 24, by the pigeonhole principle, there was a pair together solved all 8 problems. (In fact, there were at least 28 − 24 = 4 such pairs!)

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

(b) Here is a counter example:

students 1, 2 solved problems 1, 2, 3, 4
students 3, 4 solved problems 3, 4, 5, 6
students 5, 6 solved problems 1, 6, 7, 8
students 7, 8 solved problems 2, 5, 7, 8

Other commended solvers: **CHAN Wing Sum** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 40. ABC is an equilateral triangle. For a positive integer $n \geq 2$, D is the point on AB such that $AD = \frac{1}{n} AB$. $P_1, P_2, \ldots, P_{n-1}$ are points on BC which divide it into n equal segments. Prove that $\angle AP_1D + \angle AP_2D + \ldots + \angle AP_{n-1}D = 30^\circ$.
[*Hint*: Consider Q_i such that ADP_iQ_i is a parallelogram.]

Solution: CHAN Ming Chiu (La Salle College, Form 5).

Let P_0 be B and P_n be C. Let Q_0 be the point on AB such that $Q_0B = \frac{1}{n} AB$ and Q_i (i = 1, 2, ..., $n-1$) be such that ADP_iQ_i forms a parallelogram. Then $\angle AP_iD = \angle Q_iAP_i$. Now P_iQ_i is parallel to and has the same length as AD and P_0Q_0 (i = 1, 2, ..., $n-1$). Also, $P_iQ_i = P_iP_{i+1}$ (i = 0, 1, ..., $n-1$). These imply $\Delta P_iQ_iP_{i+1}$ is equilateral (i = 0, 1, ..., $n-1$). By symmetry with respect to the perpendicular bisector of BC, we get $\angle Q_iAP_i = \angle Q_{n-i-1}AP_{n-i}$. Thus

$$\begin{aligned}&(\angle Q_1AP_1 + \angle Q_{n-2}AP_{n-1})\\ &\quad + (\angle Q_2AP_2 + \angle Q_{n-3}AP_{n-2})\\ &\quad + \ldots + (\angle Q_{n-1}AP_{n-1} + \angle Q_0AP_1)\\ &\quad = \angle BAC\end{aligned}$$

and

$$\angle AP_1D + \angle AP_2D + \ldots + \angle AP_{n-1}D = \frac{1}{2}\angle BAC = 30^\circ.$$

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3) and **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5).

Olympiad Corner

(*continued from page 1*)

Problem 2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

Problem 3. Let $S = \{0, 1, 2, 3, \ldots\}$ be the set of non-negative integers. Find all functions f defined on S and taking their values in S such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all m, n in S.

Second Day (11 July, 1996)
Time: 4½ hours
(Each problem is worth 7 points.)

Problem 4. The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

Problem 5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to ED, BC is parallel to FE and CD is parallel to AF. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

Problem 6. Let n, p, q be positive integers with $n > p + q$. Let $x_0, x_1, \ldots, x_n$ be integers satisfying the following conditions:

(a) $x_0 = x_n = 0$;

(b) for each integer i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exists a pair (i,j) of indices with $i < j$ and $(i,j) \neq (0,n)$ such that $x_i = x_j$.

IMO96, Mumbai, India Facts and Statistics

Number of Participating Teams: 75
Informal Rank for the Hong Kong Team: 25
Medals for the Hong Kong Team: 1 silver and 4 bronze medals.

Below: A photo of the Hong Kong Team taken at the Kai Tak Airport before departure. From left to right are: Bobby POON Wai Hoi, MOK Tze Tao, HO Wing Yip, Roger NG Keng Po (observer), TSE Shan Shan, LAM Sze Ho (Deputy Leader), YU Chun Ling, LAW Siu Lung.

Mathematical Excalibur

Volume 2, Number 5 Nov-Dec, 1996

Olympiad Corner

25th United States of America Mathematical Olympiad:

Part I (9am-noon, May 2, 1996)

Problem 1. Prove that the average of the numbers $n \sin n°$ ($n = 2, 4, 6, \ldots, 180$) is $\cot 1°$.

Problem 2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

Problem 3. Let ABC be a triangle. Prove that there is a line l (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in l has area more than 2/3 the area of triangle ABC.

(*continued on page 4*)

Acknowledgment: Thanks to Amy LAU, EEE Dept, HKUST for her help in typesetting; and Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Jan 31, 1997.

For individual subscription for the remaining three issues for the 96-97 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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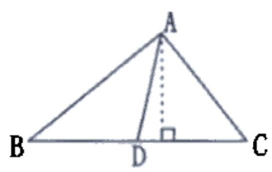
Email: makyli@uxmail.ust.hk

老師不教的幾何 (一)

張百康

香港中學課程不重視幾何，許多漂亮而有意義的幾何性質都被摒諸課堂以外。我在這裏嘗試逐期介紹一些重要的幾何定理，增加同學對幾何的認識。

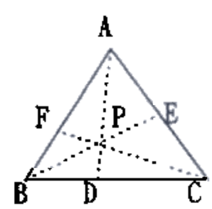
讓我們先從三角形的一個簡單而重要的性質談起：



（圖一）

圖一的兩個三角形有公共邊AD，它們是等高的，因此

$$\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = \frac{BD}{DC} \text{。}$$



（圖二）

再看圖二，圖中的線段AD、BE和CF相交於同一點P。利用上述共邊三角形的性質可知

$$\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = \frac{BD}{DC} = \frac{\text{Area of } \Delta PBD}{\text{Area of } \Delta PDC}$$

運用分數特性：

$$若 \frac{a}{b} = \frac{c}{d}，則 \frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}$$

可得

$$\frac{BD}{DC} = \frac{\text{Area of } \Delta ABP}{\text{Area of } \Delta ACP}$$

同理

$$\frac{CE}{EA} = \frac{\text{Area of } \Delta BCP}{\text{Area of } \Delta BAP}$$

及

$$\frac{AF}{FB} = \frac{\text{Area of } \Delta ACP}{\text{Area of } \Delta BCP}$$

將上述三等式同側相乘和約項可簡化爲

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \text{。}$$

這定理是意大利人西瓦 (Giovanni Ceva)在十七世紀時發現的，所以後人稱之爲「西瓦(Ceva)定理」。西瓦定理可用不同方法證明，但上述證法巧妙地利用了重疊圖形面積相減的手法，值得大家借鏡。

西瓦定理的逆命題是否也眞確？所謂逆命題，就是把原命題的因果對調。就西瓦定理來說，它的逆命題是:

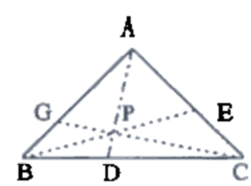
設三角形ABC的西瓦線(Cevians)[連頂點至對邊的線段]AD、BE和CF滿足條件

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \qquad (*)$$

則此三西瓦線共點。

此逆定理的證法特點是可借用原定理:雖然三西瓦線不一定共點，但其中兩條西瓦線必然共點。我們可設AD和BE交於P（圖三）過P作西瓦線CG，則

$$\frac{AG}{GB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \qquad (**)$$



（圖三）

(*continued on page 2*)

老師不教的幾何 (一)

(*continued from page 1*)

比較 (*) 和 (**) 可知

$$\frac{AF}{FB}=\frac{AG}{GB}$$

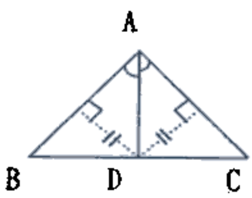
因此F和G是AB上的同一點。

西瓦定理的逆定理看似陌生，但事實上它概括了三角形的三個重要性質：三中線 (medians) 共點、三高 (altitudes) 共點、三分角線 (angle-bisectors) 共點。這三個點分別名爲重心 (centroid)、垂心 (orthocentre) 和內切圓心 (incentre)。

三中線共點這一事實可以很容易地從西瓦定理的逆定理推出，同學們請試試。大家更可利用共邊三角形的面積關係，得出《重心把中線分成兩段長度2:1的線段》這一著名性質。

三高共點此性質可利用直角三角形邊長的三角函數關係輕易導出，也留待同學們自己試試。

三角形的分角線有一個簡單而重要的性質：



(圖四)

圖四的共邊三角形ABD和ACD的面積和邊長有下列關係

$$\frac{BD}{DC}=\frac{\text{Area of }\Delta ABD}{\text{Area of }\Delta ACD}=\frac{AB}{AC},$$

因此

$$\frac{BD}{DC}=\frac{AB}{AC}。$$

三分角線共點這一特性可借助此等式得出。

(下期續)

Fermat Point

On the outside of ΔABC draw equilateral triangles BCA', CAB' and ABC'. The three lines AA', BB' and CC' meet at a point called the Fermat Point.

Error Correcting Codes (Part I)

Tsz-Mei Ko

Suppose one would like to transmit a message, say "HELLO...", from one computer to another. One possible way is to use a table to encode the message into binary digits. Then the receiver would be able to decode the message with a similar table. One such table is the American Standard Code for Information Interchange (ASCII) shown in Figure 1. The letter H would be encoded as 1001000, the letter E would be encoded as 1000101, etc. (Figure 2).

A	1000001	S	1010011	a	1100001	s	1110011
B	1000010	T	1010100	b	1100010	t	1110100
C	1000011	U	1010101	c	1100011	u	1110101
D	1000100	V	1010110	d	1100100	v	1110110
E	1000101	W	1010111	e	1100101	w	1110111
F	1000110	X	1011000	f	1100110	x	1111000
G	1000111	Y	1011001	g	1100111	y	1111001
H	1001000	Z	1011010	h	1101000	z	1111010
I	1001001	0	0110000	i	1101001	.	0101110
J	1001010	1	0110001	j	1101010	,	0101100
K	1001011	2	0110010	k	1101011	?	0111111
L	1001100	3	0110011	l	1101100	)	0101001
M	1001101	4	0110100	m	1101101	(	1111011
N	1001110	5	0110101	n	1101110	/	0101111
O	1001111	6	0110110	o	1101111	&	0100110
P	1010000	7	0110111	p	1110000	+	0101011
Q	1010001	8	0111000	q	1110001	-	0101101
R	1010010	9	0111001	r	1110010	=	0111101

Figure 1. ASCII code

Figure 2. Two computers talking

The receiver will be able to decode the message correctly if there is no error during transmission. However, if there are transmission errors, the receiver may decode the message incorrectly. For example, the letter H (1001000) would be received as J (1001010) if there is an error at position 6.

Figure 3. Error at position 6.

One possible way to detect transmission errors is to add redundant bits, i.e., append extra bits to the original message. For an even parity code, a 0 or 1 is appended so that the total number of 1's is an even number. The letters H and E would be represented by 10010000 and 10001011 respectively. With an even parity code, the receiver can detect one transmission error, but unable to correct it. For example, if 10010000 (for the letter H) is received as 10010100, the receiver knows that there is at least one error during transmission since the received bit sequence has an odd parity, i.e., the total number of 1's is an odd number.

Figure 4. Even parity code

Is there an encoding method so that the receiver would be able to correct transmission errors? Figure 5 shows one such method by arranging the bit sequence (e.g., 1001) into a rectangular block and add parity bits to both rows and columns. For the example shown, 1001 would be encoded as 10011111 (by first appending the row parities and then the column parities). If there is an error during transmission, say at position 2, the receiver can similarly arrange the received sequence 11011111 into a rectangular block and detect that there is an error in row 1 and column 2.

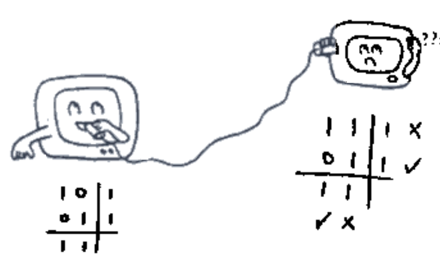


Figure 5. A code that can correct 1 error.

The above method can be used to correct one error but rather costly. For every four bits, one would need to transmit an extra four redundant bits. Is there a better way to do the encoding? In 1950, Hamming found an ingenious method to

(*continued on page 4*)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Jan 31, 1997.

Problem 46. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

Problem 47. If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \le xyz + 2$.

Problem 48. Squares *ABDE* and *BCFG* are drawn outside of triangle *ABC*. Prove that triangle *ABC* is isosceles if *DG* is parallel to *AC*.

Problem 49. Let u_1, u_2, u_3, ... be a sequence of integers such that $u_1 = 29$, $u_2 = 45$ and $u_{n+2} = u_{n+1}^2 - u_n$ for $n = 1, 2, 3, \ldots$. Show that 1996 divides infinitely many terms of this sequence. (Source: 1986 Canadian Mathematical Olympiad with modification)

Problem 50. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes? (Source: unused problem in the 1996 IMO.)

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Solutions

****

Problem 41. Find all nonnegative integers x, y satisfying $(xy - 7)^2 = x^2 + y^2$.

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 4).

Suppose x, y are nonnegative integers such that $(xy - 7)^2 = x^2 + y^2$. Then $(xy - 6)^2 + 13 = (x + y)^2$ by algebra. So

$$13 = [(x+y) + (xy-6)][(x+y) - (xy-6)].$$

Since 13 is prime, the factors on the right side can only be ±1 or ±13. There are four possibilities yielding $(x,y) = (0,7), (7,0), (3,4), (4,3)$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 5), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHING Wai Hung** (S.T.F.A. Leung Kau Kui College, Form 5), **CHUI Yuk Man** (Queen Elizabeth School, Form 7), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College, Form 7), **TING Kwong Chi & David GIGGS** (SKH Lam Woo Memorial Secondary School, Form 5), **YU Chun Ling** (HKU) and **YUNG Fai** (CUHK).

Problem 42. What are the possible values of $\sqrt{x^2+x+1} - \sqrt{x^2-x+1}$ as x ranges over all real numbers?

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6).

Let $A=(x,0)$, $B = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The expression $\sqrt{x^2+x+1} - \sqrt{x^2-x+1}$ is just $AB - AC$. As x ranges over all real numbers, A moves along the real axis and the triangle inequality yields

$$-1 = -BC < AB - AC < BC = 1.$$

All numbers on the intergal $(-1,1)$ are possible.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 5), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College, Form 7), **YU Chun Ling** (HKU) and **YUNG Fai** (CUHK).

Problem 43. How many 3-element subsets of the set $X = \{1, 2, 3, \ldots, 20\}$ are there such that the product of the 3 numbers in the subset is divisible by 4?

Solution: CHAN Ming Chiu (La Salle College, Form 6), **CHAN Wing Sum** (HKUST), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 5), **CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 6), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHUI Yuk Man** (Queen Elizabeth School, Form 7), **FUNG Tak Kwan** (La Salle College, Form 7), **LEUNG Wing Lun** (STFA Leung Kau Kui College, Form 6), **LIU Wai Kwong** (Pui Tak Canossian College), **Henry NG Ka Man** (STFA Leung Kau Kui College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Wing Chi** (La Salle College, Form 7), **TSANG Sai Wing** (Valtorta College, Form 6), **YU Chun Ling** (HKU), **YUEN Chu Ming** (Kiangsu-Chekiang College (Shatin), Form 6) and **YUNG Fai** (CUHK).

There are $C_3^{20} = 1140$ 3-element subsets of X. For a 3-element subset whose 3 numbers have product not divisible by 4, the numbers are either all odd (there are $C_3^{10} = 120$ such subsets) or two odd and one even, but the even one is not divisible by 4 (there are $C_2^{10} \times 5 = 225$ such subsets). So the answer to the problem is $1140 - 120 - 225 = 795$.

Problem 44. For an acute triangle *ABC*, let H be the foot of the perpendicular from A to BC. Let M, N be the feet of the perpendiculars from H to AB, AC, respectively. Define L_A to be the line through A perpendicular to MN and similarly define L_B and L_C. Show that L_A, L_B and L_C pass through a common point O. (This was an unused problem proposed by Iceland in a past IMO.)

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6).

Let L_A intersect the circumcircle of ΔABC at A and E. Since $\angle AMH = 90° = \angle ANH$, A, M, H, N are concyclic. So $\angle MAH = \angle MNH = 90° - \angle ANM = \angle NAE = \angle CBE$. Now $\angle ABE = \angle CBE + \angle ABC = \angle MAH + \angle ABC = 90°$. So AE is a diameter of the circumcircle and

(continued on page 4)

Problem Corner

(*continued from page 3*)

L_A passes through the circumcenter O. Similarly, L_B and L_C will pass through O.

Other commended solvers: **Calvin CHEUNG Cheuk Lun** (STFA Leung Kau Kui College, Form 5), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College, Form 7) and **YU Chun Ling** (HKU).

Problem 45. Let $a, b, c > 0$ and abc=1. Show that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1$$

(This was an unused problem in IMO96.)

Solution: YUNG Fai (CUHK).

Expanding $(a^3-b^3)(a^2-b^2)\ge 0$, we get $a^5+b^5\ge a^2b^2(a+b)$. So using this and $abc=1$, we get

$$\frac{ab}{a^5+b^5+ab}\le\frac{ab}{a^2b^2(a+b)+ab}\times\frac{c^2}{c^2}$$

$$=\frac{c}{a+b+c}.$$

Adding 3 such inequalities, we get the desired inequality. In fact, equality can occur if and only if $a=b=c=1$.

Other commended solvers: **POON Wing Chi** (La Salle College, Form 7) and **YU Chun Ling** (HKU).

Olympiad Corner

(*continued from page 1*)

Part II (1pm-4pm, May 2, 1996)

Problem 4. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length n*. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1}=2a_n$ for all positive integers n.

Problem 5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10°$, $\angle PBA = 20°$, $\angle PCA = 30°$, $\angle PAC = 40°$. Prove that triangle ABC is isosceles.

Problem 6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

Error Correcting Codes (Part I)

(*continued from page 2*)

add the redundancy. To encode a four-bit sequence $p_1p_2p_3p_4$ (say 1001), one would first draw three intersecting circles A, B, C and put the information bits p_1, p_2, p_3, p_4 into the four overlapping regions $A\cap B$, $A\cap C$, $B\cap C$ and $A\cap B\cap C$ (Figure 6). Then three parity bits p_5, p_6, p_7 are generated so that the total number of 1's in each circle is an even number. For the example shown, 1001 would be encoded as 1001001.

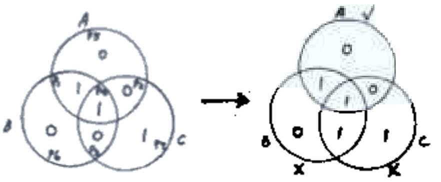
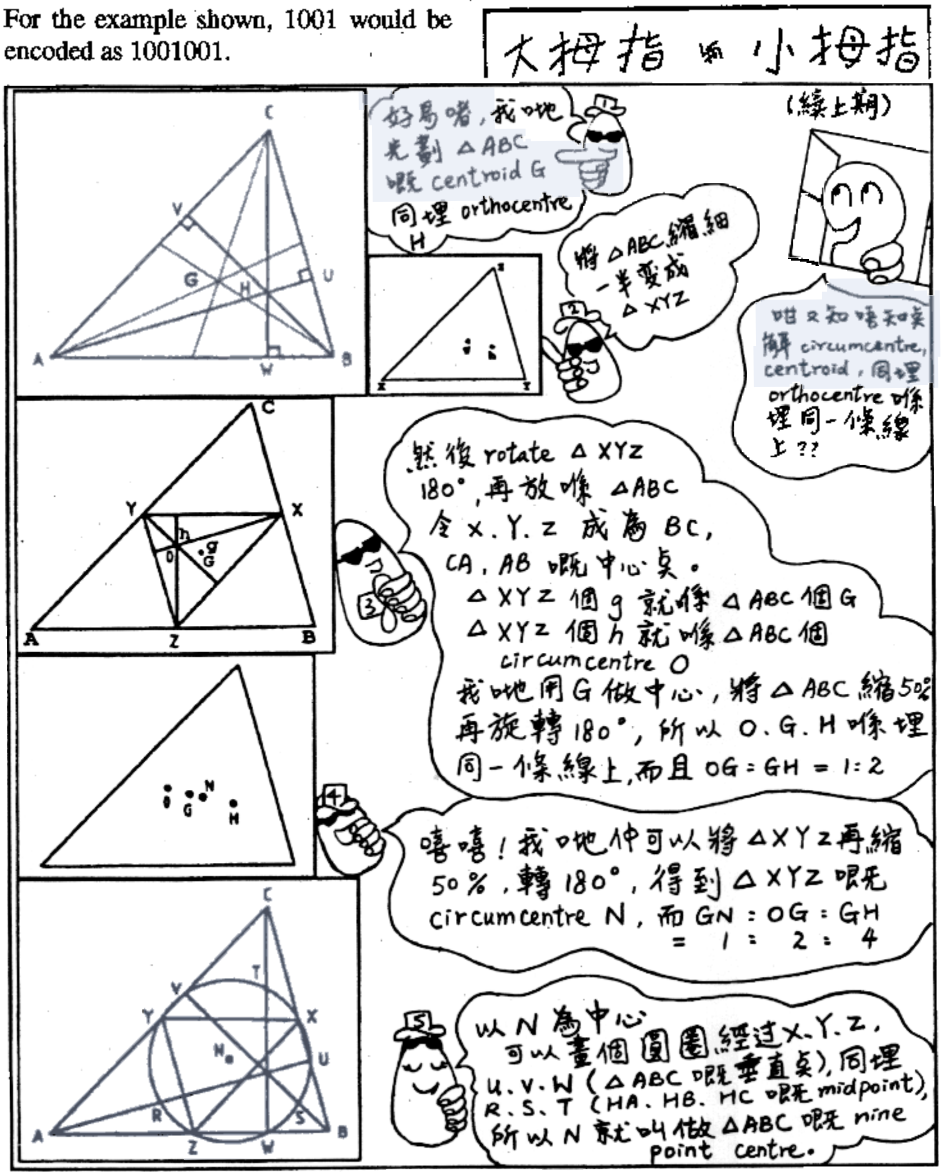


Figure 6. Hamming code

If there is one error during transmission, say 1001001 received as 1011001, the receiver can check the parities of the three circles to find that the error is in circles B and C but not in A. This (7,4) Hamming code (the notation (7,4) means that every 4 information bits are encoded as a 7 bit sequence) can be generalized. For example, one may draw 4 intersecting spheres in a three-dimensional space to obtain a (15,11) Hamming code. Hamming has also proved that his coding method is optimum for single error correction.

(... to be continued)



Mathematical Excalibur

Volume 3, Number 1 Jan-Feb, 1997

Olympiad Corner

1997 Chinese Mathematical Olympiad:

Part I (8:00-12:30, January 13, 1997)

Problem 1. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following two conditions:

(1) $-\frac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ $(i = 1, 2, \ldots, 1997)$;

(2) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

Find the maximum value of

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}.$$

Problem 2. Let $A_1B_1C_1D_1$ be an arbitrary convex quadrilateral. Let P be a point inside the quadrilateral such that the segments from P to each vertex form acute angles with the two sides through the vertex. Recursively define A_k, B_k, C_k and D_k as the points symmetric to P with respect to the lines $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$, respectively $(k = 2, 3, \cdots)$.

(*continued on page 4*)

Editors: CHEUNG Pak-Hong, Curr. Studies, HKU
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Acknowledgment: Thanks to Teresa FUNG, EEE Dept, HKUST for her help in typesetting; and Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Apr. 5, 1997.

For individual subscription for the remaining three issues for the 96-97 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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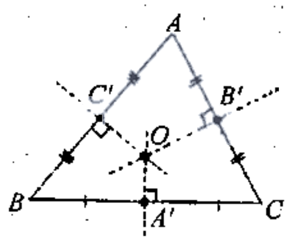
老師不教的幾何（二）

張百康

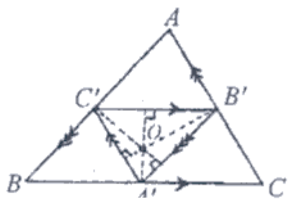
十八世紀時，瑞士出了一位大數學家歐拉 (Euler)。他雖然在二十多歲時已一目失明，但畢生努力從事數學研究，著作豐富。數學上不少定理、公式和方法都是他發現、證明或發明的。我們現在要介紹的是歐拉線 (Euler Line) —— 一條貫穿三角形幾個重要的點的直線。

對一任意的三角形 ABC，通過它的三條邊的中點 (mid-points) A'，B' 和 C' 分別作出這三條邊的垂直平分線 (perpendicular bisectors)。我們知道：這三條垂直平分線相交於同一點，即圖一的點 O。這點 O 就是三角形 ABC 的外接圓心 (circumcentre)，道理相信大家已知道。



（圖一）

另一方面，三角形 $A'B'C'$ 和 ABC 不但相似，而且對應邊平行。這個邊長縮小一半的三角形 $A'B'C'$ 稱爲三角形 ABC 的中點三角形 (medial triangle)。它的三條高 (altitudes) 剛好就落在 OA'，OB' 和 OC' 上，因此 O 點也扮演了中點三角形 $A'B'C'$ 的垂心 (orthocentre) 角色（圖二）。



（圖二）

歐拉發現任何三角形的外接圓心 (O)、重心 (G) 和垂心 (H) 共線，他的證明如下（圖三）：

由於三角形 ABC 的高 AH 和邊 BC 的垂直平分線 OA' 平行，因此

$$\angle HAG = \angle OA'G$$

並且，AH 和 $A'O$ 分別是相似三角形 ABC 和 $A'B'C'$ 的對應線，所以

$$AH : A'O = BC : B'C' = 2:1$$

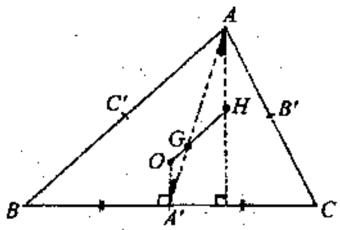
恰巧地，重心 G 也把中線 AA' 分成

$$AG : A'G = 2:1$$

因此，三角形 HAG 和 $OA'G$ 相似。由此推知

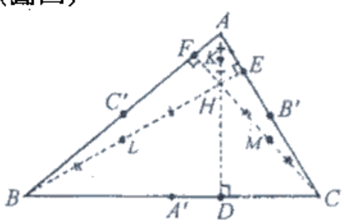
$$\angle HGA = \angle OGA'$$

所以 O、G、H 成一直線，稱爲歐拉線，並且 $OG : GH = 1:2$。



（圖三）

歐拉線 OH 的中點絕不平凡，它是著名的九點圓 (Nine-point circle) 的圓心。所謂九點圓是指一個通過三角形 ABC 的三邊的中點 A'、B'、C'，三高的垂足 D、E、F 以及三頂點和垂心間的中點 K、L、M 的圓（圖四）。



（圖四）

有關這九點爲甚麼共圓的完整證明是數學家彭賽列 (Poncelet) 於 1821 年首先給出的，他將 A'，B'，C'，K，L，M 六點分成互有重覆四點組合，然後證明每個組合的四點共圓，再利用這三個組合的重覆性證明這三個圓實質上是同一個圓，最後證明 D，E，F 也在這圓上。讓我們看看他的證法：

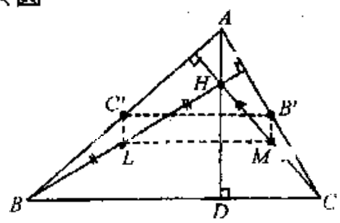
先考慮 B'，C'，L，M 四點（圖五）。

(*continued on page 2*)

老師不教的幾何 (二)

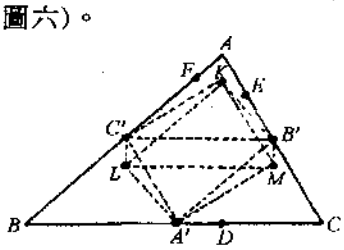
(continued from page 1)

在三角形 ABH 中， C' 和 L 分別是邊 AB 和 HB 的中點，因此 $C'L$ 平行 AH。同理，在三角形 ACH 中，$B'M$ 平行 AH。所以 $C'L$ 平行 $B'M$。再考慮三角形 ABC 和 HBC ，利用同樣的中點定理，可知 $B'C'$ 平行 ML 和 CB。由於 AD 垂直 BC，因此 $B'C'LM$ 是個矩形。矩形的頂點當然共圓。



(圖五)

重覆同樣的論証於 $A'C'KM$ 和 $A'B'KL$ 可推證它們也是矩形，因此分別共圓。但這三個矩形兩兩有共同對角線，即外接圓 (circumcircle) 的直徑 (圖六)。

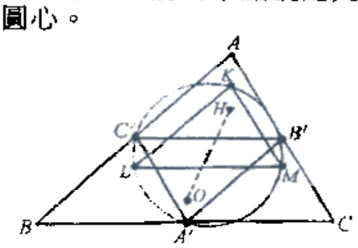


(圖六)

不同的圓不可能有共同直徑，因此 A'，B'，C'，K，L，M 六點共圓。另一方面，$\angle A'DK$ 是直角(圖四)，而 $A'K$ 是前述六點圓的直徑，因此 D 也在此六點圓上。同理，E 和 F 也在此六點圓上，所以九點共圓。

九點圓和歐拉線有甚麼關係？

大家不妨細心比較兩個頂點都在九點圓上的三角形 $A'B'C'$ 和 KLM (圖七)。由於 KA'，LB' 和 MC' 是九點圓的直徑，因此三角形 KLM 繞九點圓的圓心旋轉 180° 可得三角形 $A'B'C'$。三角形 ABC 的歐拉線 OH 兩端恰巧正分別是三角形 $A'B'C'$ 和 KLM 的垂心(可參考圖二及圖四)，因此是全等三角形 $A'B'C'$ 和 KLM 的對應點，它們的中點就是九點圓的圓心。



(圖七)

歐拉線眞不簡單，它一線穿四心，說它是三角形的脊骨一點也不過份。

$\sqrt{2}$ 是無理數的六個證明

香港大學數學系

蕭文強

『如何證明 $\sqrt{2}$ 是無理數呢？』

『那還不容易！設 $\sqrt{2}=m/n$，可當 m 和 n 不全是偶數。由於 $m^2=2n^2$，m 必是偶數，寫作 $2k$，則 $4k^2=2n^2$，$2k^2=n^2$，故 n 亦是偶數，矛盾！』

上述證明，只用到奇偶性質，來源已不可稽考。亞里士多德（ARISTOTLE）在公元前 330 年左右把它（以幾何形式）寫下來，用作反證法的示範，可見在那個時候這回事已是衆所週知了。不過由於這證明是如此簡潔，很多數學史家都相信那不是這回事的發現經過，而是『事後孔明』的解釋。

在這個證明中，2 沒有什麼特別，換了是另一個質數，同樣的思路仍可沿用，只是單憑奇偶性質不足夠，需要用到質因子唯一分解性質。再推廣少許，我們還能夠證明若 P_1、…、P_s 是 s 個不同的質數，則 $\sqrt{P_1\cdots P_s}$ 是無理數。因此，若 H 不是完全平方，則 $\sqrt{H}$ 是無理數。其實，如果我們願意運用質因子唯一分解性質，還有另一個證明辦法，即是數一數 $m^2=Hn^2$ 兩邊中某質因子出現的次數，一奇一偶，矛盾！

讓我們來看第三個證明。設 $\sqrt{H}=m/n$，可當 m 和 n 無公共因子。由於 $m^2=Hn^2=n(Hn)$，n 必須是 1 或 −1，即是說 H 是個完全平方，矛盾！這個證明跟前兩個證明有一點不相同，它能推廣至頗一般的情況，證明了若有理數是代數整數，則它必是整數。（代數整數是指首一整數系數多項式方程 $x^N+c_{N-1}x^{N-1}+\cdots+c_1x+c_0=0$ 的根，例如 $\sqrt{H}$ 是 $x^2-H=0$ 的根。請讀者試自行證明這回事吧。）

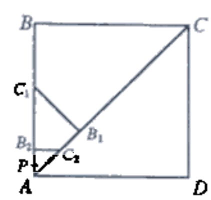
現在再看一個十分簡捷的證明：若 $\sqrt{2}$ 是有理數，取最小正整數 k 使 $k\sqrt{2}$ 是整數，則 $m=k\sqrt{2}-k=k(\sqrt{2}-1)$ 是一個較 k 更小的正整數，但 $m\sqrt{2}=2k-k\sqrt{2}$ 仍是整數，這與 k 的選取矛盾！（把 2 換作一個非完全平方 H，類似的證明適用。）

上述證明是數論專家埃斯特曼（THEODOR ESTERMANN）在 1975 年一則短文的內容，巧妙簡捷，兼而有之。後來有人讚曰：『如同所有精采念頭，一經指出即明顯不過，但這個精采念頭卻要等到畢達哥拉斯 (PYTHAGORAS) 二千多年後才給指出來！』如果我們試圖追尋如何選取 m 的線索，自然會問到它的幾何詮釋，這個幾何詮釋，說不定正是二千多年前希臘數學家發現正方形的對角線和邊是不可公度量的經過呢！不可公度量，是指不存在一公共度量，使對角線和邊各自是該公共度量的若干整數倍，也就是說，$\sqrt{2}$ 不是有理數。（以下敘述，取材於 H. EVES 的著作 "AN INTRODUCTION TO THE HISTORY OF MATHEMATICS" 的第 3 章，3rd edition，1969。）在下圖中設 AP 是正方形的對角線 AC 和邊 AB 的公共度量，即有 $AC=jAP$ 和 $AB=kAP$。構作 B_1C_1 使 B_1C_1 垂直於 AC，也使 $CB=CB_1$。不難知道 $BC_1=B_1C_1=AB_1$，因此

$$AC_1=AB-AB_1=AB-(AC-AB)=2AB-AC=(2k-j)AP\ ,$$

$$AB_1=AC-AB=(j-k)AP\ 。$$

注意：AC_1 和 AB_1 是一個較小的正方形的對角線和邊，那個較小的正方形的邊 AB_1 小於原正方形的邊 AB 的一半。按此步驟重複下去，必得到一個足夠小的正方形，它的邊 AB_t 小於 AP，但 AB_t 卻仍然是 AP 的若干整數倍，豈非矛盾！（有些數學史家認爲古代希臘數學家曾企圖以此方法研究不可公度量理論，相當於企圖發展今天稱作連分數展開式的研究。可惜當時的數學家無功而退，只遺留下來蛛絲馬跡，在古希臘數學名著《歐幾里得原本》(EUCLID'S ELEMENTS) 的章節間依稀可見！）



請注意：$AC_1/AB_1=(2k-j)/(j-k)$，而 $m=j-k$ 正是埃斯特曼的短小精悍證明中的 m。因爲 $AC_1/AB_1=\sqrt{2}$，便有 $(2k-j)/m=\sqrt{2}$，即是 $m\sqrt{2}=2k-j$ 是整數了。當我們了解埃斯特曼證明的背後的幾何詮釋，我們可以把它重寫成第六個證明：若 $\sqrt{2}=j/k$ 是最簡的分數式，則有 $\sqrt{2}=(2k-j)/(j-k)$（這是因爲 $j\sqrt{2}-k\sqrt{2}=2k-j$），但 $k<j<2k$（因爲 $1<\sqrt{2}<2$），故 $2k-j<j$ 和 $j-k<k$，這與 j 和 k 的選取矛盾！

請讀者想一想，上面討論的六個證明，眞的是六個不同的證明嗎？還是六個相同的證明呢？

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Apr. 5, 1997.

Problem 51. Is there a positive integer n such that $\sqrt{n-1}+\sqrt{n+1}$ is a rational number?

Problem 52. Let a, b, c be distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the value of abc.

Problem 53. For ΔABC, define A' on BC so that $AB + BA' = AC + CA'$ and similarly define B' on CA and C' on AB. Show that AA', BB', CC' are concurrent. (The point of concurrency is called the *Nagel point* of ΔABC.)

Problem 54. Let R be the set of real numbers. Find all functions $f : R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

for all $x, y \in R$. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Problem 55. In the beginning, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

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Solutions

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Problem 46. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$? (Source: 1963 Putnam Exam.)

Solution: CHEUNG Tak Fai (Valtorta College, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

Suppose

$$x^{13} + x + 90 = (x^2 - x + a)q(x),$$

where $q(x)$ is a polynomial with integer coefficients. Taking $x = -1, 0, 1$, we get

$$88 = (2+a)q(-1),$$
$$90 = aq(0)$$
and
$$92 = aq(1).$$

Since a divides 90, 92 and $a+2$ divides 88, a can only be 2 or -1. Now $x^2 - x - 1$ has a positive root, but $x^{13} + x + 90$ cannot have a positive root. So a can only be 2. We can check by long division that $x^2 - x + 2$ divides $x^{13} + x + 90$ or observe that if w is any of the two roots of $x^2 - x + 2$, then $w^2 = w - 2$, $w^4 = -3w + 2$, $w^8 = -3w - 14$, $w^{12} = 45w - 46$ and $w^{13} + w + 90 = 0$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST) and **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6).

Problem 47. If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \le xyz + 2$.

Solution: CHAN Ming Chiu (La Salle College, Form 6).

If one of x, y, z is nonpositive, say z, then

$$2 + xyz - x - y - z = (2-x-y) - z(1-xy) \ge 0$$

because

$$x + y \le \sqrt{2(x^2+y^2)} \le 2$$

and

$$xy \le (x^2 + y^2)/2 \le 1$$

So we may assume x, y, z are positive, say $0 < x \le y \le z$. If $z \le 1$, then

$$2 + xyz - x - y - z = (1-x)(1-y) + (1-z)(1-xy) \ge 0.$$

If $z > 1$, then

$$(x + y) + z \le \sqrt{2\left((x+y)^2 + z^2\right)} = 2\sqrt{xy+1} \le xy + 2 \le xyz + 2.$$

Comments: This was an unused problem in the 1987 IMO and later appeared as a problem on the 1991 Polish Mathematical Olympiad.

Problem 48. Squares $ABDE$ and $BCFG$ are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC.

Solution: Henry NG Ka Man (STFA Leung Kau Kui College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **YUNG Fai** (CUHK).

From B, draw a perpendicular line to AC (and hence also perpendicular to DG.) Let it intersect AC at X and DG at Y. Since $\angle ABX = 90° - \angle DBY = \angle BDY$ and $AB = BD$, the right triangles ABX and BDY are congruent and $AX = BY$. Similarly, the right triangles CBX and BGY are congruent and $BY = CX$. So $AX = CX$, which implies $AB = CB$.

Comments: This was a problem on the 1988 Leningrad Mathematical Olympiad. Most solvers gave solutions using pure geometry or a bit of trigonometry. The editor will like to point out there is also a simple vector solution. Set the origin O at the midpoint of AC. Let $\vec{OC} = m$, $\vec{OB} = n$ and k be the unit vector perpendicular to the plane. Then $\vec{AB} = n + m$, $\vec{CB} = n - m$, $\vec{BD} = -(n + m) \times k$, $\vec{BG} = (n - m) \times k$ and $\vec{DG} = \vec{BG} - \vec{BD} = 2n \times k$. If DG is parallel to AC, then $n \times k$ is a multiple of m and so $m = \vec{OC}$ and $n = \vec{OB}$ are perpendicular. Therefore, triangle ABC is isosceles.

Other commended solvers: **CHAN Wing Chiu** (La Salle College, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 5), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHING Wai Hung** (S.T.F.A. Leung Kau Kui College, Form 5), **Alan LEUNG Wing Lun** (STFA Leung Kau Kui College, Form 5), **OR Fook Sing** & **WAN Tsz Kit** (Valtorta College, Form 6), **TSANG Sai Wing** (Valtorta College, Form 6), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 5), **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Problem 49. Let u_1, u_2, u_3, … be a sequence of integers such that $u_1 = 29$, $u_2 = 45$ and $u_{n+2} = u_{n+1}^2 - u_n$ for $n = 1, 2, 3, \ldots$. Show that 1996 divides infinitely many terms of this sequence. (Source: 1986 Canadian Mathematical Olympiad with modification)

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6) and **YUNG Fai** (CUHK).

Let U_n be the remainder of u_n upon division by 1996, i.e.,

$$U_n \equiv u_n \pmod{1996}.$$

Consider the sequence of pairs (U_n, U_{n+1}). There are at most 1996^2 distinct pairs. So let $(U_p, U_{p+1}) = (U_q, U_{q+1})$ be the first repetition with $p < q$. If $p > 1$, then the recurrence relation implies $(U_{p-1}, U_p) = (U_{q-1}, U_q)$ resulting in an earlier repetition. So $p = 1$ and the sequence of pairs (U_n, U_{n+1}) is periodic with period $q - 1$. Since $u_3 = 1996$, we have $0 = U_3 = U_{3+k(q-1)}$ and so 1996 divides $u_{3+k(q-1)}$ for every positive integer k.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

Problem 50. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes? (Source: unused problem in the 1996 IMO.)

Solution 1: Henry NG Ka Man (STFA Leung Kau Kui College, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

If the initial numbers are $a = w$, $b = x$, $c = y$, $d = z$, then after 4 steps, the numbers will be

$$a = 2(w - 2x + 3y - 2z),$$
$$b = 2(x - 2y + 3z - 2w),$$
$$c = 2(y - 2z + 3w - 2x),$$
$$d = 2(z - 2w + 3y - 2z).$$

From that point on, a, b, c, d will always be even, so $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ will always be divisible by 4.

Solution 2: Official Solution.

After $n \geq 1$ steps, the sum of the integers will be 0. So $d = -a - b - c$. Then

$$\begin{aligned} bc - ad &= bc + a(a + b + c) \\ &= (a + b)(a + c). \end{aligned}$$

Similarly,

$$ac - bd = (a + b)(b + c)$$

and

$$ab - cd = (a + c)(b + c).$$

Finally $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ cannot all be prime because their product is the square of $(a+b)(a+c)(b+c)$.

Other commended solvers: **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 5) and **William CHEUNG Pok-man** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(*continued from page 1*)

Consider the sequence of quadrilaterals

$$A_jB_jC_jD_j \quad (j = 1, 2, \cdots).$$

(1) Determine which of the first 12 quadrilaterals are similar to the 1997$^{\text{th}}$ quadrilateral.

(2) If the 1997$^{\text{th}}$ quadrilateral is cyclic, determine which of the first 12 quadrilaterals are cyclic.

Problem 3. Prove that there are infinitely many natural numbers n such that

$$1, 2, \cdots, 3n$$

can be put into an array

$$\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{matrix}$$

satisfying the following two conditions:

(1) $a_1+b_1+c_1 = a_2+b_2+c_2 = \cdots = a_n+b_n+c_n$ and the sum is a multiple of 6;

(2) $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n = c_1+c_2+\cdots+c_n$ and the sum is a multiple of 6.

Part II (8:00-12:30, January 14, 1997)

Problem 4. Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

Problem 5. Let $A = \{1, 2, 3, \cdots, 17\}$ For a mapping $f : A \to A$, denote

$$f^{[1]}(x) = f(x),$$
$$f^{[k+1]}(x) = f\left(f^{[k]}(x)\right) \ (k = 1, 2, 3,$$

Consider one-to-one mappings f from A to A satisfying the condition: there exists a natural number M such that

(1) for $m < M$, $1 \leq i \leq 16$,
$$f^{[m]}(i+1) - f^{[m]}(i) \not\equiv \pm 1 \pmod{17},$$
$$f^{[m]}(1) - f^{[m]}(17) \not\equiv \pm 1 \pmod{17};$$

(2) for $1 \leq i \leq 16$,
$$f^{[M]}(i+1) - f^{[M]}(i) \equiv 1 \text{ or } -1 \pmod{17},$$
$$f^{[M]}(1) - f^{[M]}(17) \equiv 1 \text{ or } -1 \pmod{17}.$$

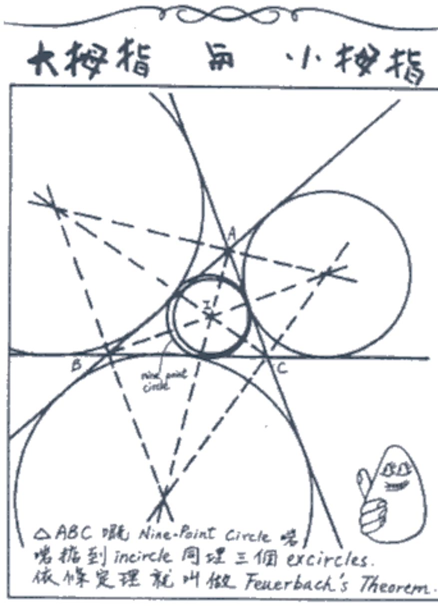
For all mappings f satisfying the above condition, determine the largest possible value of the corresponding M's.

Problem 6. Consider a sequence of nonnegative real numbers a_1, a_2, … satisfying the condition

$$a_{n+m} \leq a_n + a_m, \quad m, n \in N.$$

Prove that for any $n \geq m$,

$$a_n \leq ma_1 + \left(\frac{n}{m} - 1\right)a_m.$$



Mathematical Excalibur

Volume 3, Number 2 March-May, 1997

Olympiad Corner

The Ninth Asian Pacific Mathematics Olympiad, March 1997:

Time Allowed: 4 hours.
Each question is worth 7 points.

Problem 1. Given

$$S=1+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{1}{3}+\frac{1}{6}}+\cdots+\frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}},$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

Problem 2. Find an integer n, with $100 \le n \le 1997$, such that $\frac{2^n+2}{n}$ is also an integer.

Problem 3. Let ABC be a triangle inscribed in a circle and let

$$l_a=\frac{m_a}{M_a},\ l_b=\frac{m_b}{M_b},\ l_c=\frac{m_c}{M_c},$$

(*continued on page 4*)

Editors: CHEUNG Pak-Hong, Curr. Studies, HKU
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Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is July 10, 1997.

For individual subscription for the remaining issue for the 96-97 academic year, send us a stamped self-addressed envelope. Send all correspondence to:

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由圓周率到四年一閏

香港道教聯合會青松中學

梁子傑

有時，一個簡單的電腦程序，就可以令我們發現不少有趣的數學現象，以下便是一個好例子：

```
PROGRAM rational_real; {written in MS QuickPascal}
VAR
   numer, denom : LongInt;
   devi, min, real_no : Double;
BEGIN
   min := 10; real_no := pi;
   FOR denom := 1 TO 50000 DO
      BEGIN
         numer := round(real_no * denom);
         devi := abs(numer/denom - real_no);
         IF devi < min THEN
            BEGIN
               min := devi;
               writeln (numer, '/', denom)
            END
      END
END.
```

這個程序是用來尋找圓周率π的有理數近似值的。程序令分母(denom)由1開始，先計算出最接近的分子(numer)的數值，然後計算出這個有理數近似值(numer/denom)跟圓周率的偏差(devi)，如果偏差比以前的小，就將該有理數印出來。

不出兩秒鐘，電腦就會計算出結果：

3/1, 13/4, 16/5, 19/6, 22/7, 179/57, 201/64, 223/71, 245/78, 267/85, 289/92, 311/99, 333/106, 355/113, 52163/16604, 52518/16717, 52873/16830, 53228/16943, ⋯

從以上的結果，我們不難發現這個現象：並不是每當分母增加時，所計算出來的有理數近似值就一定較準確，好似當分母介乎於 8 至 56 之間時，以 7 作爲分母的近似值就比它們準確了。

如果大家細心地觀察一會，相信亦會發現在這些分母之中，出現了一些「跳躍」的現象，好似由 7 跳至57、由113 跳至 16604 等。再留心看看，每次跳躍之後，分母增加的幅度亦有關係：7之後的57、64、71等，就相隔7；113 之後的16604、16717等，就相隔113。同時分子亦有相類似的關係。奇怪嗎？爲甚麼會有這個現象出現呢？

要解釋以上的現象，我們就需要認識一個很特別的表達數值的方法，它就是「連分數」(continued fraction)。所謂「連分數」就是利用一連串的倒數來表達一個數的數值，例如：

$$\frac{1057}{498}=2+\frac{61}{498}=2+\cfrac{1}{\cfrac{498}{61}}$$

$$=2+\cfrac{1}{8+\cfrac{10}{61}}=2+\cfrac{1}{8+\cfrac{1}{6+\cfrac{1}{10}}}.$$

我們並且用這個記號來表示以上的結果：$\frac{1057}{498}=[2;8,6,10]$。

到了今天，數學家經已發現了很多有關連分數的性質，其中一項就是「漸近分數」的現象。以上述數字爲例，如果我們逐次選取連分數中部份的數字，即 $[2]=2$，$[2;8]=2+\frac{1}{8}=\frac{17}{8}$，$[2;8,6]=2+\cfrac{1}{8+\cfrac{1}{6}}=\frac{104}{49}$，$[2;8,6,10]=\frac{1057}{498}$，所得到的一列分數，就叫做「部份連分數」。我們可以證明「部份連分數」是一列越來越漸近原本數值的分數，換句話講，在數列中每一個分數都可以表示爲原本數值的約數，而且後者的準確性會比前者佳；當然，後者分母的數值卻比前者的大，應用起來就不及前者方便了。

回到上面電腦程序的結果，我們發現如果將圓周率π表示成連分數的話，我們有 $\pi=[3;7,15,1,292,1,1,\ldots]$。（因爲圓周率是一個無理數，它的連分數表達式自然是無窮盡的。）寫出π的部份連分數，我們有 $[3]=3$，$[3;7]=\frac{22}{7}$，$[3;7,15]=\frac{333}{106}$，$[3;7,15,1]=\frac{355}{113}$，$[3;7,15,1,292]=\frac{103993}{33102}$，……等等。而這些部份連分數，不是和電腦程序計算出來的結果相同嗎？

不過，電腦程序計算出來的結果卻比「部份連分數」爲多，這是因爲部份連分數的現象祇是一個充分條件，而不是一個必要條件，所以「部份連分數」並不是一個完整的漸近分數的數列。雖然如此，一個完整的數列亦可以利用「連分數」來表達出來。留意 $\frac{13}{4}=[3;4]$，$\frac{16}{5}=[3;5]$，$\frac{19}{6}=[3;6]$，$\frac{22}{7}=[3;7]$；另外，$\frac{179}{57}=[3;7,8]$，$\frac{201}{64}=[3;7,9]$……等等；不難看出，每當分母增加至和「部份連分數」相同的數值時，下一個漸近分數

就將會由下一個「部份連分數」的一半開始。例如，$\frac{355}{113}=[3;7,15,1]$，下一個部份連分數是[3;7,15,1,292]，而292的一半等於146，故此$\frac{355}{113}$之後的漸近分數應等於$[3;7,15,1,146]=\frac{52163}{16604}$。

以上的現象同時解釋了，爲何在數列中，分母數值的間隔會和「跳躍」前的分母數值相等的現象。大家祇要將[3;7,8]和[3;7,9]計算一次，就會明白爲何分子的差距和分母的差距，剛好等於[3;7]的分子和分母了。

在未討論下一個問題前，值得指出的是，在電腦程序計算出的漸近分數中，有幾個數值是十分「著名的」。例如：$\frac{3}{1}$，這是人類對圓周率最早期的約數，中國古籍中就有「徑一周三」的記載。另外$\frac{22}{7}$更毋須多介紹了。$\frac{223}{71}$是古希臘數學家阿基米德提出的約數；而$\frac{355}{113}$就最先由中國南北朝時代的數學家祖沖之提出的。當知道下一個漸近分數的分母將會高達16604時，相信大家都會明白爲何祖沖之計算出圓周率的七位小數約數之後，要經過多達一千年的時間，才有人能夠計到更佳的圓周率近似值！

其實漸近分數並不是一些數字的玩意，它有不少實際的應用價值，其中一項就是閏年的計算。跟據資料顯示，地球環繞太陽一周需要365日5小時48分46秒，化爲分數就即是有$365\frac{10463}{43200}$日。其中的365日當然沒有問題，但剩下的$\frac{10463}{43200}$日又如何處理呢？完全不理，那麼祇要每過5年，就會有一整天的時間差距了。（又或者每43200年，就會相差了10463日。）因此我們在曆法上就訂立了閏年的制度：即每隔一些年份就在那年增加一日，藉此保持曆法上不會出現偏差。但是，到底要經過多少年才有一次閏年呢？我們不可能經過4萬多年後，才增加萬多日來補償。不過我們可以通過計算$\frac{10463}{43200}$的漸近分數來獲得答案。

首先我們將本文開始的電腦程序中的real_no句子改爲：real_no := 10463/43200; 執行後，可得到結果：

0/1, 1/3, 1/4, 4/17, 5/21, 6/25, 7/29, 8/33, 23/95, 31/128, 101/417, 132/545, ...

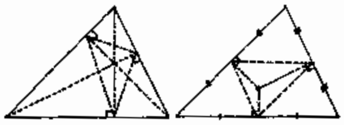
從結果中的$\frac{1}{4}$可知，我們應該每4年就要多加一日。按此比例，每100年就應有25個閏年。但由$\frac{23}{95}$和$\frac{31}{128}$可以知道，每100年其實祇需要24個閏年。所以如果我們祇跟著「四年一閏」的方法來編寫曆

(*continued on page 4*)

老師不教的幾何（三）

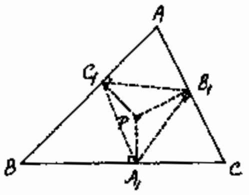
張百康

我們較早前蹓過的垂足三角形 (orthic triangle) 和中點三角形 (medial triangle) 有甚麼共同的性質呢？大家請重溫一下這兩個三角形（圖一）。



（圖一）

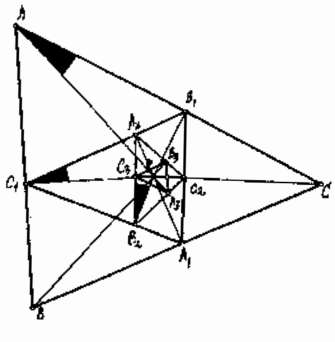
其實，它們可以看成是更一般情況的兩個特例：令P爲已知三角形ABC內任意一點。從P作垂直線垂直於這三角形的三邊，連這三垂線的垂足可得另一三角形$A_1B_1C_1$，稱爲三角形ABC相對於踏板點 (pedal point) P的踏板三角形 (pedal triangle)（圖二）。



（圖二）

分別以三角形ABC的垂心和外心作踏板點便可得垂足三角形和中點三角形作爲ABC的踏板三角形。踏板三角形有甚麼有趣的性質呢？在1892年出版的一本幾何書中，編輯J. Neuberg提出及證明了踏板三角形的一個周期性現象：

以P作爲三角形$A_1B_1C_1$的踏板點，可以得到$A_1B_1C_1$的踏板三角形$A_2B_2C_2$。繼續這作法以P爲踏板點，可以得到另一個踏板三角形$A_3B_3C_3$，如此類推(圖三)。



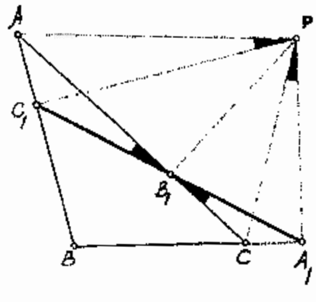
(圖三)

由於各有一對對角是直角，因此下列的四邊形都是外接四邊形：AC_1PB_1、$C_1B_2PA_2$和$B_2A_3PC_3$。因此$\angle B_1AP$、$\angle B_1C_1P$ $(=\angle A_2C_1P)$、$\angle A_2B_2P$ $(=\angle C_3B_2P)$和$\angle C_3A_3P$依次兩兩爲等弧上的圓周角。所以$\angle B_1AP = \angle C_3A_3P$；同理，$\angle C_1AP = \angle B_3A_3P$。這兩結果告訴我們：$\angle BAC = \angle B_3A_3C_3$；同理，$\angle ACB = \angle A_3C_3B_3$和$\angle ABC = \angle A_3B_3C_3$。因此三角形$ABC$和$A_3B_3C_3$相似。顯而易見，踏板三角形有下列周期性：

$$\Delta ABC \sim \Delta A_3B_3C_3 \sim \Delta A_6B_6C_6 \sim \cdots,$$
$$\Delta A_1B_1C_1 \sim \Delta A_4B_4C_4 \sim \Delta A_7B_7C_7 \sim \cdots$$
和 $\Delta A_2B_2C_2 \sim \Delta A_5B_5C_5 \sim \Delta A_8B_8C_8 \sim \cdots$。

如果踏板點在三角形ABC外部時，這周期性還成立嗎？答案在一般情況下是肯定的，大家可參考上述證明加以修改便可。有沒有例外情況？如果大家懂得用互動幾何軟件如 Cabri Geometry 或 Geometer's Sketchpad，這是一個有意義的探究活動。通過探究，大家應該發現：如果踏板點P在三角形ABC的外接圓上，則踏板三角形$A_1B_1C_1$退化爲一直線，也沒有其他的踏板點使踏板三角形退化爲直線。這直線稱爲辛姆生線 (Simson Line)。辛姆生 (Robert Simson) 是十七、八世紀的數學家，但後人在他的著作中找不到這性質的證明，反而是 William Wallace 在 1797 年發表了下列證明：

設A_1、B_1、C_1共線（圖四），則$\angle AB_1C_1$和$\angle A_1B_1C$爲對頂角，所以相等。已知$\angle PA_1C = \angle PB_1C = 90°$及$\angle PB_1A = \angle PC_1A = 90°$，所以$P$、$A_1$、$C$、$B_1$四點共圓，且$P$、$B_1$、$C_1$、$A$也共圓。由此推出$\angle A_1PC = \angle A_1B_1C = \angle AB_1C_1 = \angle APC_1$。並且，$\angle PA_1B = \angle PC_1B = 90°$，所以$P$、$A_1$、$B$、$C_1$四點也共圓，因此，$\angle A_1PC_1$和$\angle C_1BA_1$互補。但$\angle A_1PC_1 = \angle A_1PC + \angle CPC_1 = \angle APC_1 + \angle CPC_1 = \angle APC$，所以$\angle APC$和$\angle ABC$ $(=\angle C_1BA_1)$也互補。故A、B、C、P四點共圓，即P點在三角形ABC的外接圓上。



(圖四)

(*continued on page 4*)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is July 10, 1997.

Problem 56. Find all prime numbers p such that $2^p + p^2$ is also prime.

Problem 57. Prove that for real numbers $x, y, z > 0$,

$$\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+x}\geq\frac{x+y+z}{2}.$$

Problem 58. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$. (*Source*: unused problem in the 1996 IMO.)

Problem 59. Let n be a positive integer greater than 2. Find all real number solutions $(x_1, x_2, \cdots, x_n)$ to the equation

$$(1-x_1)^2 + (x_1-x_2)^2 + \cdots + (x_{n-1}-x_n)^2 + x_n^2 = \frac{1}{n+1}.$$

(Source: 1975 British Mathematical Olympiad)

Problem 60. Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$.

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Solutions

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Problem 51. Is there a positive integer n such that $\sqrt{n-1}+\sqrt{n+1}$ is a rational number?

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 4).

Assume there is a positive integer n such that

$$\sqrt{n-1}+\sqrt{n+1}=r$$

is rational. Squaring and simplifying, we get

$$\sqrt{n^2-1}=\frac{r^2-2n}{2}$$

is also rational. However, for $n > 1$, if $\sqrt{n^2-1} = a/b$ for some positive integers a, b having no common factor greater than 1, then $a^2 = b^2(n^2-1)$, which implies b also divides a. So b must be 1. Now for $n > 1$,

$$n^2 > n^2 - 1 = a^2 > (n - 1)^2$$

is impossible. So $n = 1$, but then

$$\sqrt{n-1}+\sqrt{n+1}=\sqrt{2}$$

is irrational. Therefore, no such n exists.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, Form 6), **CHOI Wing Shan Winnie** (St. Stephen's Girls' College, Form 6), **LEUNG Shun Ming** (La Salle College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **TSE Wing Ho** (Ho Fung College, Form 5), **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4) and **YUNG Fai** (CUHK).

Problem 52. Let a, b, c be distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the value of abc.

Solution: CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6), **YEUNG Yi Pok** (Pui Shing Catholic Secondary School, Form 7) and **YUNG Fai** (CUHK).

Let a, b, c be roots of

$$x^3 - px^2 + qx - r = 0.$$

Then $p = a + b + c$, $q = ab + bc + ca$ and $r = abc$. Since $a^2 + b^2 + c^2 = p^2 - 2q$, so

$$a^3 = 3(b^2 + c^2) - 25 = 3(p^2-2q-a^2) - 25.$$

This is equivalent to $a^3 + 3a^2 + (25 + 6q - 3p^2) = 0$. Then a is a root of $x^3 + 3x^2 + (25 + 6q - 3p^2) = 0$. Similarly, b and c are roots of this equation. Comparing coefficients of the two equations, we get $p = -3$, $q = 0$ and $abc = r = -(25 + 6q - 3p^2) = 2$.

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College), **TSE Wing Ho** (Ho Fung College, Form 5) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4) ∘

Problem 53. For ΔABC, define A' on BC so that $AB + BA' = AC + CA'$ and similarly define B' on CA and C' on AB. Show that AA', BB', CC' are concurrent. (The point of concurrency is called the *Nagel point* of ΔABC.)

Solution: CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6), **LIU Wai Kwong** (Pui Tak Canossian College) and **YEUNG Yi Pok** (Pui Shing Catholic Secondary School, Form 7)

Let $a = BC$, $b = CA$, $c = AB$ and $s = (AB + BC + CA)/2$. Since $AB + BA' = s = AC + CA'$, we have $BA' = s - c$ and $CA' = s - b$. Similarly, $CB' = s - a$, $AB' = s - c$, $AC' = s - b$ and $BC' = s - a$. Then

$$(CA'/BA')(AB'/CB')(BC'/AC') = 1.$$

So by the converse of Ceva's theorem, AA', BB', CC' are concurrent.

Other commended solvers: **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Problem 54. Let R be the set of real numbers. Find all functions $f : R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

for all $x, y \in R$. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Solution: YUNG Fai (CUHK).

Putting $y = 0$, we get

$$f(f(x)) = [1 + f(0)]f(x).$$

Replacing x by $x + y$, we get

$$[1+f(0)]f(x+y) = f(f(x+y)) = f(x+y) + f(x)f(y) - xy,$$

which simplifies to

$$f(0)f(x+y) = f(x)f(y) - xy.$$

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Putting $y = 1$, we get

$$f(0)f(x+1) = f(x)f(1) - x.$$

Putting $y = -1$ and replacing x by $x+1$, we get

$$f(0)f(x) = f(x+1)f(-1) + x + 1.$$

Eliminating $f(x+1)$ in the last two equations, we get

$$[f^2(0) - f(1)f(-1)]f(x) = [f(0) - f(-1)]x + f(0).$$

If $f^2(0) - f(1)f(-1) \neq 0$, then $f(x)$ is linear. If $f^2(0) - f(1)f(-1) = 0$, then putting $x = 0$ in the last equation, we get $f(0) = 0$. In this case, the displayed equation above implies $f(x)f(y) = xy$. Then $f(x)f(1) = x$ for all $x \in R$. So $f(1) \neq 0$ and $f(x)$ is linear.

Finally, substituting $f(x) = ax + b$ into the original equation, since $f(x)$ cannot be constant, we find $a = 1$ and $b = 0$, i.e., $f(x) = x$ for all $x \in R$.

Other commended solvers: **CHAN Wing Sum** (HKUST) and **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, Form 6).

Problem 55. In the beginning, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6) and **YUNG Fai** (CUHK).

Assign an ordered pair (a,b) to each square with a, $b = 1, 2, \ldots, 9$. Divide the 81 squares into 3 types. Type A consists of squares with both a and b odd, type B consists of squares with both a and b even and type C consists of the remaining squares. The numbers of squares of the types A, B and C are 25, 16 and 40, respectively.

Assume no collision occurs. After two successive moves, beetles in type A squares will be in type B squares. So the number of beetles in type A squares are at most 16 at any time. Then there are at most 32 beetles in type A or type B squares at any time. Also, after one move, beetles in type C squares will go to type A or type B squares. So there are at most 32 beetles in type C squares at any time. Hence there are at most 64 beetles on the board, a contradiction.

Other commended solvers: **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4.

Olympiad Corner

(*continued from page 1*)

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is equilateral.

Problem 4. Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n $(n \geq 3)$, a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

(a) Prove that if this process were continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \geq 3$.

(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

Problem 5. Suppose that n persons A_1, A_2, …, A_n $(n \geq 3)$ are seated in circle and that A_i has a_i objects such that

$$a_1 + a_2 + \cdots + a_n = nN$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}, where A_{n+1} means A_1 and A_0 means A_n. How should this distribution be performed so that the total numbers of objects transferred is minimum?

由圓周率到四年一閏

(*continued from page 1*)

法，一百年後就會多了一日。因此在今天我們使用的曆法之中，年份能夠被4整除的，例如1996年，就定爲閏年，但如果年份能夠被100整除的話，例如1900年，就不是閏年了。

再算一算，就知道每400年就有96個閏年，416年就應有96+4=100個閏年。不過，這結果又不乎合 $\frac{101}{417}$ 這個條件！故此，曆法上又需要在每400年中增加一日，就好似2000年，因爲這數字能被400整除，這年又變回一年閏年了！

公元2000年快到了，大家渴望見一見這400年才有一次的閏年嗎？

老師不教的幾何（三）

(*continued from page 1*)

把上述推理逆轉過來，恰巧也成立，因此只有外接圓上的點能使踏板三角形退化爲辛姆生線。

踏板三角形的周期性可否推廣到n邊形呢？大家不妨先用四邊形來試試。B. M. Stewart在1940年證明了：n邊形的第n個踏板n邊形相似於原n邊形（刊於American Mathematical Monthly第七卷第462-466頁）。

台灣師範大學附屬中學初中二年級的孫君儀同學最近以踏板多邊形作爲研究課題，獲得1997年台灣科學展覽第三名。她借助Geometer's Sketchpad發現了一些有趣性質並加以證明，大家不妨試試探討，甚至再推廣。這些性質包括：

- 對於凹n邊形和自交n邊形，第n個踏板n邊形是否和原n邊形相似？
- 踏板點在n邊形外部，類似性質是否存在？有甚麼條件會使踏板n邊形不存在？
- 第n個踏板n邊形和原n邊形的面積比是多少？
- 垂足改爲夾$x°$角時，類似性質是否存在？
- 踏板點在何處可使第三垂足三角形的面積最大？

Mathematical Excalibur

Volume 3, Number 3 June-August, 1997

Olympiad Corner

The 38th International Mathematical Olympiad, Mar del Plata, Argentina:

First day (July 24, 1997)
Each problem is worth 7 points.
Time Allowed: $4\frac{1}{2}$ hours.

Problem 1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m,n) = |S_1 - S_2|.$$

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

(*continued on page 4*)

Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is September 30, 1997.

For individual subscription for the five issues for the 97-98 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Error Correcting Codes (Part II)

Tsz-Mei Ko

In Part I, we introduced the family of Hamming codes. In particular, the (7,4) Hamming code encodes 4-bit messages $p_1p_2p_3p_4$ into 7-bit codewords $p_1p_2p_3p_4p_5p_6p_7$ by appending three parity bits

$$p_5 = p_1 + p_2 + p_4 \pmod 2,$$
$$p_6 = p_1 + p_3 + p_4 \pmod 2,$$
$$p_7 = p_2 + p_3 + p_4 \pmod 2,$$

to the original message. Figure 1 shows the 16 possible codewords for the (7,4) Hamming code. To convey the message 0100, as an example, the sender would send 0100101. If there is a transmission error in position 4 so that the received sequence becomes 0101101, the receiver would still be able to recover the error by decoding the received sequence as the closest codeword. (Note that 0100101 is different from 0101101 in only one position while all other codewords are different from 0101101 in more than one position.)

Now, if we group the first six bits of a (7,4) Hamming codeword into two-bit pairs (p_1p_2, p_3p_4, p_5p_6) and use an arithmetic system called a 4-element field (Figure 2), we observe something interesting: the three points (1, p_1p_2), (2, p_3p_4) and (3, p_5p_6) form a straight line! For example, the first 6 bits of the codeword 0100101 forms the ordered triple (01, 00, 10) = (1, 0, 2) and (1,1), (2,0), (3,2) are three consecutive points on the straight line $f(x) = 2x + 3$ since

$$f(1) = 2(1) + 3 = 2 + 3 = 1;$$
$$f(2) = 2(2) + 3 = 3 + 3 = 0;$$
$$f(3) = 2(3) + 3 = 1 + 3 = 2;$$

by using the addition and multiplication tables given in Figure 2. This fact is also true for the other 15 codewords and their corresponding straight lines $f(x)$ are listed in Figure 3.

This "straight line" property can be utilized for decoding. As an example, assume that the received sequence is 0101101. The first 6 bits form the ordered triple (01, 01, 10) = (1, 1, 2). We observe that a straight line passing through (1,1) and (2,1) should pass through (3,1). That is (1,1), (2,1) and (3,2) do not lie on a straight line and thus there is a transmission error. For the (7,4) Hamming code which is capable of correcting one error, we

message $p_1p_2p_3p_4$	codeword $p_1p_2p_3p_4p_5p_6p_7$
0000	0000000
0001	0001111
0010	0010011
0011	0011100
0100	0100101
0101	0101010
0110	0110110
0111	0111001
1000	1000110
1001	1001001
1010	1010101
1011	1011010
1100	1100011
1101	1101100
1110	1110000
1111	1111111

Figure 1. The (7,4) Hamming Code.

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Figure 2. Arithmetic Tables for a 4-Element Field.

codeword	p_1p_2	p_3p_4	p_5p_6	$f(x)$
0000000	0	0	0	0
0001111	0	1	3	$2x+2$
0010011	0	2	1	$3x+3$
0011100	0	3	2	$x+1$
0100101	1	0	2	$2x+3$
0101010	1	1	1	1
0110110	1	2	3	x
0111001	1	3	0	$3x+2$
1000110	2	0	3	$3x+1$
1001001	2	1	0	$x+3$
1010101	2	2	2	2
1011010	2	3	1	$2x$
1100011	3	0	1	$x+2$
1101100	3	1	2	$3x$
1110000	3	2	0	$2x+1$
1111111	3	3	3	3

Figure 3. The (7,4) Hamming Codewords form Straight Lines $f(x)$.

assume that only one of the three points is incorrect. That is, the original "straight line" $f(x)$ should pass through (1,1) and (2,1); (1,1) and (3,2); or (2,1) and (3,2) corresponding to $f(x) = 1$; $f(x) = 2x + 3$; or $f(x) = 3x$ respectively. Then the first 6 bits for the original codeword should be 010101, 010010 or 110110. Among these three possible solutions, only 010010 satisfies the equation for the last parity bit $p_7 = p_2 + p_3 + p_4 \pmod 2$. Thus we decode the received sequence 0101101 as 0100101 corresponding to the message 0100.

The above decoding procedure seems to be quite complicated. However, it can be generalized to construct (and decode) multiple-error correcting codes by using "polynomials" instead of "straight lines". Suppose we would like to transmit a message that contains k symbols $s_1 s_2 \cdots s_k$. We may use these k symbols to form a kth degree polynomial $f(x)$ such that $f(i) = s_i$ $(1 \le i \le k)$. To construct a code that can correct t errors, we may append $2t$ symbols $f(k+1)$, $f(k+2)$, ..., $f(k+2t)$ to the original message so that the encoded sequence contains $k + 2t$ symbols corresponding to $k + 2t$ consecutive points on a kth degree polynomial (Figure 4). If there are less than or equal to t errors during transmission, at least $k + t$ symbols would be received correctly. Then the receiver may simply check which $k + t$ symbols lie on a kth degree polynomial to decode the received sequence.

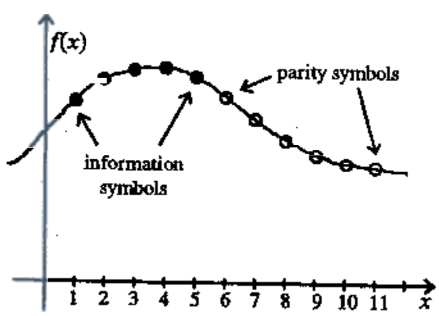

Figure 4. A Polynomial Code.

We use a (21,9) double error correcting code to illustrate the idea. Assume we would like to send a 9 bit message, say 101010100. We may first group the information bits into 3-bit symbols as (101, 010, 100) = (5, 2, 4). (In general, we may group the information bits into m-bit symbols where m cannot be too small. Otherwise, we cannot construct the polynomial $f(x)$. Why? Also m should not be too large to reduce the number of parity bits.) Then we use the three message symbols (5, 2, 4) to form a second degree polynomial $f(x)$ such that $f(1) = 5, f(2) = 2$ and $f(3) = 4$. That is

$$f(x) = \frac{5(x-2)(x-3)}{(1-2)(1-3)} + \frac{2(x-1)(x-3)}{(2-1)(2-3)} + \frac{4(x-1)(x-2)}{(3-1)(3-2)}.$$

Note that we have 8 kinds of symbols (since we group the bits into 3-bit symbols) and thus we need an 8-element field for our arithmetic. (Basically, a field is an arithmetic system that allows us to add, subtract, multiply and divide.) By using the 8-element field given in Figure 5, we can simplify $f(x)$ to obtain

$$f(x) = x^2 + 7x + 5.$$

Note that $f(1) = 5$, $f(2) = 2$ and $f(3) = 4$ as desired.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	3	1	7	5
3	0	3	6	5	7	4	1	2
4	0	4	3	7	6	2	5	1
5	0	5	1	4	2	7	3	6
6	0	6	7	1	5	3	2	4
7	0	7	5	2	1	6	4	3

Figure 5. Arithmetic Tables for an 8-Element Field.

Now suppose we would like to construct a code that can correct two errors. We can append

$$f(4) = 4^2 + 7(4) + 3 = 6 + 1 + 3 = 4;$$
$$f(5) = 5^2 + 7(5) + 3 = 7 + 6 + 3 = 2;$$
$$f(6) = 6^2 + 7(6) + 3 = 2 + 4 + 3 = 5;$$
$$f(7) = 7^2 + 7(7) + 3 = 3 + 3 + 3 = 3;$$

to the message symbols. That is, we would transmit a 21 bit sequence (5,2,4,4,2,5,3) = 101010100100010101011. If there are transmission errors, say at positions 5 and 15, the received sequence becomes 101000100100011101011 = (5,0,4,4,3,5,3). (This code is actually capable of correcting two symbol errors instead of two bit errors.) Then the receiver would search for the 5 received symbols that are not corrupted. Among the $\binom{7}{5} = 21$ cases, only $f(1) = 5$, $f(3) = 4$, $f(4) = 4$, $f(6) = 5$, $f(7) = 3$, form a second degree polynomial. So the receiver uses these five points to reconstruct $f(x) = x^2 + 7x + 5$ and decode the received message as $(f(1), f(2), f(3)) = (5, 2, 4) = 101010100$.

The above idea, using polynomials to construct codes, was first proposed by Reed and Solomon in 1960. It is now widely used in electronics and communication systems including our compact discs.

38th IMO

Kin-Yin Li

For the first time in history, the International Mathematical Olympiad (IMO) was held in the southern hemisphere. Teams representing a record 82 countries and regions participated in the event at Mar del Plata, Argentina this year from July 18 to 31. The site was at a resort area bordered by the beautiful Atlantic Ocean. All through the period, the weather was nice and cool.

The Hong Kong team, like many southeast Asia teams, had to overcome thirty plus hours of flight time to arrive Argentina. With two short days of rest, the team members wrote the exams with jet lag. This year the team consisted of

Chan Chung Lam (*Bishop Hall Jubilee School*)
Cheung Pok Man (*STFA Leung Kau Kui College*)
Lau Lap Ming (*St. Paul's College*)
Leung Wing Chung (*Queen Elizabeth School*)
Mok Tze Tao (*Queen's College*)
Yu Ka Chun (*Queen's College*)

brought home 5 bronze medals and came in one mark behind Canada and one mark ahead of France. The top team was China with 6 gold, followed by Hungary, Iran, USA and Russia. As usual, problem 6 was the most difficult with 73% of the contestants getting zero, 90% getting less than half of the score for the problem.

The excursions were good. The hospitality was superb!!! The team members had a wild time playing the indoor games the day before the closing ceremony. One member of the team even admitted it was the best he has participated in three years. There were many fond memories.

There was a surprise ending on the way back. Due to the typhoon weather in Hong Kong, the team was stranded in Los Angeles for a day. Yes, the team took full advantage to tour the city, Hollywood, Beverly Hills, Rodeo Drive, in particular. The next day the team was stranded again in Taipei. It was unbelievably fortunate to have a chance to see these cities. What a bonus for a year's hard work!

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is September 30, 1997.

Problem 61. Find the smallest positive integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

Problem 62. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ. (*Source*: 1996 Australian Mathematical Olympiad.)

Problem 63. Show that for $n \ge 2$, there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $|a_k - k| = |a_1 - 1| \ne 0$ for $k = 2, 3, \ldots, n$ if and only if n is even.

Problem 64. Show that it is impossible to place 1995 different positive integers along a circle so that for every two adjacent numbers, the ratio of the larger to the smaller one is a prime number.

Problem 65. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

****

Solutions

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Problem 56. Find all prime numbers p such that $2^p + p^2$ is also prime.

Solution: CHAN Lung Chak (St. Paul's Co-ed. College, Form 4), **CHAN Wing Sum** (HKUST), **LAW Ka Ho** (Queen Elizabeth School, Form 4), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4), **TAM Siu Lung** (Queen Elizabeth School, Form 4), **WONG Chun Wai** (SKH Kei Hau Secondary School, Form 4), **Alan WONG Tak Wai** (University of Waterloo, Canada), **WONG Sui Kam** (Queen Elizabeth School, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

For $p = 2$, $2^p + p^2 = 8$ is not prime. For $p = 3$, $2^p + p^2 = 17$ is prime. For prime $p = 3n \pm 1 > 3$, we see that

$$2^p + p^2 = (3-1)^p + (3n \pm 1)^2$$

is divisible by 3 (after expansion) and is greater than 3. So $p = 3$ is the only such prime.

Problem 57. Prove that for real numbers $x, y, z > 0$,

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x} \ge \frac{x+y+z}{2}.$$

Solution 1: Note that

$$\begin{aligned} 4x^2 &= ((x+y)+(x-y))^2 \\ &= (x+y)^2 + 2(x+y)(x-y) + (x-y)^2 \\ &\ge (x+y)^2 + 2(x+y)(x-y). \end{aligned}$$

Dividing both sides by $4(x+y)$, we obtain

$$\frac{x^2}{x+y} \ge \frac{x+y}{4} + \frac{x-y}{2}$$

In place of x, y, similar inequalities for y, z and z, x can be obtained. Adding these inequalities give the desired inequality.

Solution 2: Venus CHU Choi Yam (St. Paul's Co-ed. College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4), **Alan WONG Tak Wai** (University of Waterloo, Canada).

The Cauchy-Schwarz inequality asserts that

$$\left(a_1^2 + a_2^2 + \cdots + a_k^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2\right) \ge \left(a_1b_1 + a_2b_2 + \cdots + a_kb_k\right)^2$$

with equality if and only if $a_ib_j = a_jb_i$ for all i, j such that $1 \le i < j \le k$. Taking $k = 3$,

$$a_1 = \sqrt{x+y},\ a_2 = \sqrt{y+z},\ a_3 = \sqrt{z+x},$$

$$b_1 = \frac{x}{\sqrt{x+y}},\ b_2 = \frac{y}{\sqrt{y+z}},\ b_3 = \frac{z}{\sqrt{z+x}},$$

then dividing both sides by $2(x + y + z)$, we get the desired inequality.

Other commended solvers: **CHAN Wing Sum** (HKUST), **Alex CHUENG King Chung** (Po Leung Kuk 1983 Board of Director's College, Form 6), **Yves CHEUNG Yui Ho** (STFA Leung Kau Kui College, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 4), and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Problem 58. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$. (*Source*: unused problem in the 1996 IMO.)

Solution: Official Solution.

Let Y be the midpoint of AC. Since $\angle OFP = \angle OYP = 90°$, points F, P, Y, O lie on a circle Γ_1 with center at the midpoint Q of OP. Now the nine point circle Γ_2 of ΔABC also passes through F and Y and has center at the midpoint N of OH. So FY is perpendicular to NQ. Since NQ is parallel to HP by the midpoint theorem, FY is perpendicular to HP. Then $\angle FHP = 90° - \angle YFH = 90° - \angle YCH = \angle BAC$.

Problem 59. Let n be a positive integer greater than 2. Find all real number solutions $(x_1, x_2, \cdots, x_n)$ to the equation

$$(1-x_1)^2 + (x_1-x_2)^2 + \cdots + (x_{n-1}-x_n)^2 + x_n^2 = \frac{1}{n+1}$$

(Source: 1975 British Mathematical Olympiad)

Solution 1: Official Solution.

Let

$$1 - x_1 = \frac{1}{n+1} + z_1,$$
$$x_1 - x_2 = \frac{1}{n+1} + z_2, \ \ldots$$
$$x_{n-1} - x_n = \frac{1}{n+1} + z_n,$$
$$x_n = \frac{1}{n+1} + z_{n+1}.$$

Adding the above $n + 1$ equations, we get

$$z_1 + z_2 + \cdots + z_{n+1} = 0.$$

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

In terms of z_i, the given equation can then be simplified to

$$z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0.$$

So all $z_i = 0$, which implies

$$x_i = \frac{n+1-i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

Solution 2: Venus CHU Choi Yam (St. Paul's Co-ed. College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **POON Man Wai** (St. Paul's College, Form 4).

We use the Cauchy-Schwarz inequality as stated in Problem 57 Solution 2. Taking $k = n + 1$,

$$a_1 = 1 - x_1, a_2 = x_1 - x_2, \ldots, a_n = x_{n-1} - x_n, a_{n+1} = x_n,$$

$$b_1 = b_2 = \cdots = b_{n+1} = 1,$$

we see that we have equality. So $a_1 = a_2 = \cdots = a_{n+1}$ yielding the unique solution

$$x_i = \frac{n+1-i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

Problem 60. Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$.

Solution: LAW Ka Ho (Queen Elizabeth School, Form 4), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4) and **TAM Siu Lung** (Queen Elizabeth School, Form 4).

Note that $(x - 1)^3$ divides $p(x) + 1$ and $p(-x) - 1$; so $(x - 1)^3$ divides their sum $p(x) + p(-x)$. Also $(x + 1)^3$ divides $p(x) - 1$ and $p(-x) + 1$; so $(x + 1)^3$ divides $p(x) + p(-x)$. Then $(x - 1)^3(x + 1)^3$ divides $p(x) + p(-x)$, which is of degree at most 5. So $p(x) + p(-x) = 0$ for all x. Then the even degree term coefficients of $p(x)$ are zero. Now

$$p(x) + 1 = (x - 1)^3(Ax^2 + Bx - 1).$$

Comparing the degree 2 and 4 coefficients, we get $3 + 3B - A = 0$ and $B - 3A = 0$, which implies $A = -3/8$ and $B = -9/8$. This yields

$$p(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x.$$

Other commended solvers: **CHAN Wing Sum** (HKUST), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School, Form 3), **SIN Ka Fai** (STFA Leung Kau Kui College, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Olympiad Corner

(*continued from page 1*)

Problem 2. Angle A is the smallest in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

Problem 3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and $|x_i| \le \dfrac{n+1}{2}$ for $i = 1, 2, \ldots, n$.

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

Second day (July 25, 1997)
Each problem is worth 7 points.
Time Allowed: $4\frac{1}{2}$ hours.

Problem 4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a *silver* matrix if, for each $i = 1, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Problem 5. Find all pairs (a,b) of integers $a \ge 1$, $b \ge 1$ that satisfy the equation

$$a^{b^2} = b^a$$

Problem 6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways:

4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.

Prove that, for any integer $n \ge 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

Above: A photo of the Hong Kong Team taken in front of the IMO97 score board. From left to right are: LEUNG Wing Chung, CHEUNG Pok Man, YU Ka Chun, LAU Lap Ming, CHAN Chung Lam, MOK Tze Tao, LUK Mee Lin (La Salle College, Deputy Leader), LI Kin Yin (HKUST Math Dept, Team Leader).

Mathematical Excalibur

Volume 3, Number 4 September-November, 1997

Olympiad Corner

British Mathematical Olympiad:

Round 1 (January 15, 1997)
Time Allowed: $3\frac{1}{2}$ hours.

Problem 1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$. Determine all such integers N for which $R(N) = 4N + 3$.

Problem 2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by

$$a_1 = 1,\ a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1}),\ n > 1.$$

Determine the value of a_{1997}.

Problem 3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

(*continued on page 4*)

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Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is January 10, 1998.

For individual subscription for the four remaining issues for the 97-98 academic year, send us four stamped self-addressed envelopes. Send all correspondence to:

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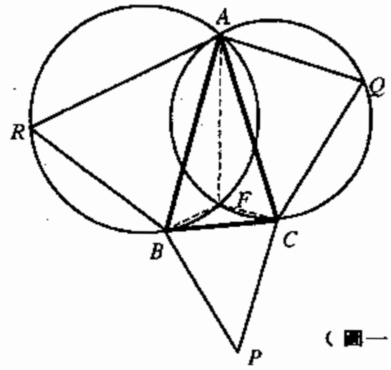
老師不教的幾何（四）

張百康

在任意的三角形的三邊上作另一些三角形，只要滿足一些簡單的條件，卻常常可以得到一些美妙的結果。

圖一的三角形 ABC 的三條邊外側分別隨意作了三個三角形 ABR、BCP 和 CAQ，並同時作三角形 ABR 和 CAQ 的外接圓。連此兩外接圓的一交點 F 至 A、B 及 C。



（圖一）

$$\begin{aligned}\angle BFC &= 360° - \angle AFB - \angle AFC \\ &= 360° - (180° - \angle ARB) \\ &\qquad - (180° - \angle AQC) \\ &= \angle ARB + \angle AQC.\end{aligned}$$

如果條件

$$\angle BPC + \angle ARB + \angle CQA = 180°$$

成立，$\angle BPC$ 和 $\angle BFC$ 互補，因此三角形 BCP 的外接圓也通過點 F。這個條件並不難得，下列兩種情況都是它的特例：

(1) 三角形 ABR、CPB 和 QCA 相似；

(2) A、B 和 C 分別是三角形 PQR 的邊 QR、RP 和 PQ 上的點。

如果三角形 ABR、CPB 和 QCA 相似，則它們的外接圓心 O_3、O_1 和 O_2 所組成的三角形也和它們相似（圖二），道理如下：

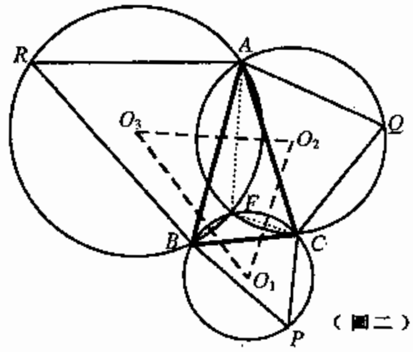
由於 O_1O_2 和 O_1O_3 是圓心連線，所以它們分別垂直公共弦 CF 和 BF，因此

$$\begin{aligned}\angle O_2O_1O_3 &= 360° - 90° - 90° - \angle BFC \\ &= 180° - \angle BFC \\ &= \angle CPB\ (=\angle ABR = \angle QCA)。\end{aligned}$$

同理

$\angle O_1O_3O_2 = \angle BRA\ (=\angle CAQ = \angle PBC)$,
$\angle O_3O_2O_1 = \angle AQC\ (=\angle RAB = \angle BCP)$,

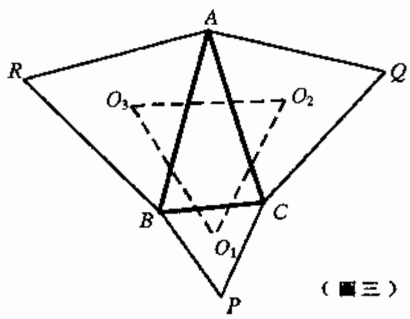
證畢。



（圖二）

拿破侖(Napoleon)是一位大家都知道的大將軍，但你可知道他對數學，尤其是幾何，有濃厚興趣？我現在要介紹的一類三角形，據說是他發現的，所以後人將這種三角形命名為拿破侖三角形。

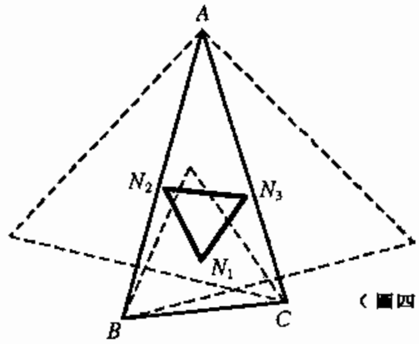
在任意的一個三角形 ABC 的三條邊上，分別向外側作三等邊三角形 ABR、BCP 和 CAQ（圖三）。這三個等邊三角形的心 O_1、O_2、O_3 可連成一三角形 $O_1O_2O_3$，稱



（圖三）

為外拿破侖三角形。由前述結果可推知外拿破侖三角形也是等邊三角形。

如果我們改變一下上述作法，把三個等邊三角形作於三角形 ABC 三條邊的內側，可以得到如圖四所示的另一三角形 $N_1N_2N_3$，稱為內拿破侖三角形。



（圖四）

俄羅斯數學家I.M. Yaglom 巧妙地證明內拿破侖三角形也是等邊三角形：

應用餘弦公式於圖三的三角形 AO_3O_2 可得

$$(O_2O_3)^2=\frac{b^2}{3}+\frac{c^2}{3}-2\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}\cos(A+60°),$$

這裏我們利用了

$$AO_2=\frac{b}{\sqrt{3}}\text{，}AO_3=\frac{c}{\sqrt{3}}$$

和 $\angle O_3AO_2 = A + 60°$

等簡單事實，請同學們自行驗證。

類似手法再應用於三角形 AN_3N_2 可得

$$(N_2N_3)^2=\frac{b^2}{3}+\frac{c^2}{3}-2\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}\cos(60°-A).$$

將上述兩等式同側相減可得

$$\begin{aligned}(O_2O_3)^2-(N_2N_3)^2 &=\frac{2bc}{3}\left(\cos(60°-A)-\cos(A+60°)\right)\\ &=\frac{2}{\sqrt{3}}bc\sin A\\ &=\frac{4}{\sqrt{3}}\times\Delta ABC\text{ 的面積。}\end{aligned}$$

此處的簡化過程從略。

由於 $O_2O_3 = O_3O_1 = O_1O_2$，因此 $N_2N_3 = N_3N_1 = N_1N_2$。這證明的巧

(*continued on page 4*)

Inverse Sequences and Complementary Sequences

Yau Kwan Kiu Garry
Form 7, Queen's College

Editor's Note: This article is modified and shortened by the editors.

Consider the sequence

$$f(n) = 0, 0, 0, 1, 2, 3, 3, 4, 5, 6, 7, 10, \ldots,$$

i.e., $f(1) = 0, f(2) = 0, f(3) = 0, f(4) = 1$, etc. We can construct another sequence $f^*(n)$ according to the definition

$$f^*(n) = k, \text{ where } f(k) < n \le f(k+1).$$

For our example,

$$f^*(n) = 3, 4, 5, 7, 8, 9, 10, 11, 11, 11, \ldots.$$

Note that $f^*(n)$ can also be referred as the "frequency distribution function" of $f(n)$ since $f^*(n)$ is the number of terms in the sequence f that are less than n.

Figure 1 shows the two functions $f(n)$ and $f^*(n)$. We note something interesting: f^* is a mirror image of f. If we compute the frequency distribution of $f^*(n)$, we obtain $f(n)$ again. That is, $f^{**}(n) = f(n)$. The sequences $f(n)$ and $f^*(n)$ are called inverse sequences.

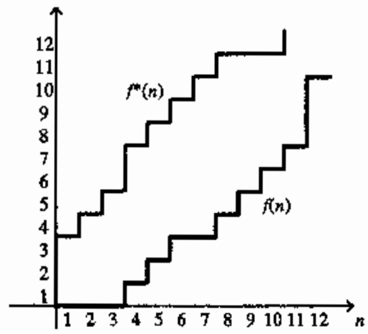


Figure 1. The functions $f(n)$ and $f^*(n)$.

Now we construct two other sequences

$$F(n) = f(n) + n \text{ and } G(n) = f^*(n) + n.$$

For our example,

$$F(n) = 1, 2, 3, 5, 7, 9, 10, 12, 14, \ldots;$$
$$G(n) = 4, 6, 8, 11, 13, 15, 17, 19, 20, \ldots.$$

Notice anything? The two sequences $F(n)$ and $G(n)$ together contain each natural number exactly once. This fact and its converse were first discovered and proved by mathematicians Lambek and Moser in 1954 (c.f. American Mathematical Monthly, vol. 61, p. 454, 1954). The sequences $F(n)$ and $G(n)$ are called complementary sequences.

Theorem (Lambek and Moser). $f(n)$ and $f^*(n)$ are inverse sequences if and only if $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ are complementary sequences (with the minor conditions that (i) $f(n)$ and $f^*(n)$ are non-decreasing sequences of non-negative integers; (ii) $F(n)$ and $G(n)$ are strictly increasing sequences of positive integers.)

If a formula for the nth term of a sequence is known, the theorem of Lambek and Moser can be used to find a general formula for the complementary sequence. The following example illustrates the idea.

Example. We can separate the natural numbers into two sequences $F(n)$ and $G(n)$ that contain squares and non-squares as follows.

$$F(n) = 1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots,$$
$$G(n) = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, \ldots.$$

We know that a formula for the nth square is $F(n) = n^2$. Can we find a formula for the nth non-square $G(n)$?

We note that $F(n)$ and $G(n)$ are complementary and thus the sequences

$$f(n) = F(n) - n = 0, 2, 6, 12, 20, \ldots,$$
$$f^*(n) = G(n) - n = 1, 1, 2, 2, 2, 2, 3, \ldots,$$

are inverse sequences. Now

$$f(n) = F(n) - n = n^2 - n.$$

Therefore, $f^*(n) = k$ where

$$f(k) < n \le f(k+1),$$
$$k^2 - k < n \le (k+1)^2 - (k+1) = k^2 + k.$$

Since both k and n are integers,

$$k^2 - k + \tfrac{1}{4} < n < k^2 + k + \tfrac{1}{4},$$
$$(k - \tfrac{1}{2})^2 < n < (k + \tfrac{1}{2})^2,$$
$$k - \tfrac{1}{2} < \sqrt{n} < k + \tfrac{1}{2},$$
$$\sqrt{n} - \frac{1}{2} < k < \sqrt{n} + \frac{1}{2}.$$

Consequently,

$$f^*(n) = k = \left[\sqrt{n} + \tfrac{1}{2}\right]$$

and

$$G(n) = f^*(n) + n = n + \left[\sqrt{n} + \tfrac{1}{2}\right].$$

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is January 10, 1998.

Problem 66.

(a) Find the first positive integer whose square ends in three 4's.

(b) Find all positive integers whose squares end in three 4's.

(c) Show that no perfect square ends with four 4's.

(*Source:* 1995 British Mathematical Olympiad.)

Problem 67. Let Z and R denote the integers and real numbers, respectively. Find all functions $f: Z \to R$ such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{2}$$

for all integers x, y such that $x + y$ is divisible by 3. (*Source:* a modified problem from the 1995 Iranian Mathematical Olympiad.)

Problem 68. If the equation

$$ax^2 + (c-b)x + (e-d) = 0$$

has real roots greater than 1, show that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has at least one real root. (*Source*: 1995 Greek Mathematical Olympiad.)

Problem 69. *ABCD* is a quadrilateral such that $AB = AD$ and $\angle B = \angle D = 90°$. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$. (*Source:* 1995 Russian Mathematical Olympiad.)

Problem 70. Lines $l_1, l_2, \cdots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \cdots, k-1$ so that in each of the lines $l_1, l_2, \cdots, l_k$ the numbers $1, 2, \cdots, k-1$ appear exactly once if and only if k is even. (*Source:* a modified problem from the 1995 Greek Mathematical Olympiad.)

****

Solutions

****

Due to the large number of solutions received by the editors, we will first acknowledge the solvers by their schools and grade levels. The numbers following a solver's name are the number of the problems which the solver submitted correct solutions.

Bishop Hall Jubilee School: (Form 4) CHAN Kin Hang (61, 63, 64, 65). **Cheung Chuk Shan College**: (Form 5) CHOW King Fun (61). **Heep Woh College**: (Form 7) KU Wah Kwan (61, 63). **Ho Fung College**: (Form 6) TSE Wing Ho (61, 64). **HK Taoist Association Ching Chung Secondary School**: (Form 7) LI Fung (61, 62). **HKUST**: CHAN Wing Sum (61, 63). **La Salle College**: (Form 3) CHAN Ernest Eason (61); (Form 5) Vincent LUNG (61). **N.T. Heung Yee Kuk Yuen Long District Secondary School**: (Form 7) CHU Kai Mun (61, 63, 64). **Queen Elizabeth School**: (Form 4) LAI Chi Fung Brian (61), LAW Ka Ho (61, 62, 63, 64, 65). **Saint Louis School**: (Form 7) SHAM Wing Hang (61). **St. Paul's Co-educational College**: (Form 5) CHAN Lung Chak (61, 62), MAK Shiu Ting (61), NGAN Chung Wai Hubert (61, 62, 63, 64, 65), SHEK Ka Wai Wilson (62); (Form 7) CHU Choi Yam Venus (61). **St. Stephen's Girls' College**: (Form 6) WAN Hoi Wah (61). **SKH Kei Hau Secondary School**: (Form 4) WONG Chun Wai (61, 62, 63, 64, 65). **Shi Hui Wen Secondary School**: (Form 6) Jimmy KONG Ka Ho (61, 62, 64, 65). **STFA Leung Kau Kui College**: (Form 5) CHU Chun Yiu (61, 63), IP Man Wai (61), Gary NG Ka Wing (61, 62, 63, 64, 65), SIN Ka Fai (61, 62, 64), YUEN Man Long (61, 62, 63, 64, 65); (Form 6) Yves CHEUNG Yui Ho (61, 62, 63, 64), CHING Wai Hung (61, 62, 64), WONG Hau Lun (61, 62, 63, 64, 65); (Form 7) William CHEUNG Pok Man (62, 63, 64). **Valtorta College**: (Form 6) CHANG Pui Kwan (61), KO Tsz Wan (61), Ryan LAI (61), LAM Wai Hung (61), LIN Kai Shuen (61), NG Lai Ha (61), TAM Ka Kwong (61), TANG Ka Wai (61), WONG Shu Fai (61); (Form 7) KWAN Yee Kin (61), LEUNG Pak Keung (62), TSANG Sai Wing (62), WAN Tsz Kit (61, 62, 64).

Problem 61. Find the smallest positive integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

Solution:

Let n be the smallest such positive integer. Then

$$n = a + (a+1) + \cdots + (a+8) = 9a + 36,$$
$$n = b + (b+1) + \cdots + (b+9) = 10b + 45,$$
$$n = c + (c+1) + \cdots + (c+10) = 11c + 55.$$

These imply n is divisible by

$$9 \times 5 \times 11 = 495.$$

So $n \ge 495$. Letting $a = 51$, $b = 45$, $c = 40$, we see that 495 is possible. So $n = 495$.

Problem 62. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ. (*Source*: 1996 Australian Mathematical Olympiad.)

Solution: WONG Chun Wai.

Let $[XYZ]$ denote the area of ΔXYZ. Then

$$\frac{MP}{MQ} = \frac{[PAC]}{[QAC]} = \frac{\frac{AP}{AB}[ABC]}{\frac{AQ}{AD}[ADC]} = \frac{CD \cdot AD \cdot [ABC]}{AB \cdot BC \cdot [ADC]} = \frac{[ADC] \cdot [ABC]}{[ABC] \cdot [ADC]} = 1.$$

Problem 63. Show that for $n \ge 2$, there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $|a_k - k| = |a_1 - 1| \ne 0$ for $k = 2, 3, \ldots, n$ if and only if n is even.

Solution: LAW Ka Ho.

Suppose for some n, the condition is possible. Let $d = |a_1 - 1|$, p be the number of times $a_k > k$ and q be the number of times $a_k < k$. Then $p + q = n$ and

$$0 = (a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = pd - qd.$$

So $p = q$ and n is even. If n is even, then the permutation $2, 1, 4, 3, \ldots, n, n-1$ satisfies the condition with $|a_1 - 1| = 1$.

Comments: This was a problem on the 1996 Australian Mathematical Olympiad.

Problem 64. Show that it is impossible to place 1995 different positive integers

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

along a circle so that for every two adjacent numbers, the ratio of the larger to the smaller one is a prime number.

Solution: William CHEUNG Pok Man.

Suppose this is possible. Let $a_1, a_2, \ldots, a_{1995}$ be the numbers in the clockwise direction. Then a_{k-1}/a_k is a prime or the reciprocal of a prime for $k = 1, 2, \ldots, 1995$ with $a_0 = a_{1995}$. Suppose m of these are primes and $1995 - m$ of these are reciprocals of primes. Since

$$\left(\frac{a_0}{a_1}\right)\left(\frac{a_1}{a_2}\right)\cdots\left(\frac{a_{1994}}{a_{1995}}\right)=1,$$

this means the product of m primes will equal to a product of $1995 - m$ primes. Unique prime factorization implies $m = 1995 - m$, which is impossible as 1995 is odd.

Comments: This was a problem on the 1995 Russia Mathematical Olympiad.

Problem 65. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

Solution: LAW Ka Ho.

There are 12 sides and 54 diagonals. With 12 colors, there is a color, say X, which is used to paint at most 5 of these segments. For each X colored segment, 10 triangles can be formed having this segment as a side (using the remaining 10 vertices). So there are at most 50 triangles with at least one side colored X. However, if any three colors are the colors of the sides of a triangle, there would be $C_2^{11} = 55$ triangles having at least one side colored X, a contradiction.

Comments: This was also a problem on the 1995 Russia Mathematical Olympiad.

Olympiad Corner

(*continued from page 1*)

Problem 4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

Problem 5. Let x, y and z be positive real numbers.

(i) If $x + y + z \geq 3$, is it necessarily true that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq 3$?

(ii) If $x + y + z \leq 3$, is it necessarily true that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3$?

Round 2 (February 27, 1997)
Time Allowed: $3\frac{1}{2}$ hours.

Problem 1. Let M and N be two 9-digit positive integers with the property that if any one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

Problem 2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

Problem 3. Find the number of polynomials of degree 5 with distinct coefficients from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are divisible by $x^2 - x + 1$.

Problem 4. The set

$$S = \{1/r : r = 1, 2, 3, \ldots\}$$

of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, 1/20, 1/8, 1/5 is such a progression, of length 3 (and common difference 3/40). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S ($-1/40$ and 11/40 not being members of S).

(i) find a maximal progression in S of length 1996.

(ii) Is there a maximal progression in S of length 1997?

老師不教的幾何（四）

(*continued from page 2*)

妙處在於它帶給我們另一個美麗而意想不到的結果：

外拿破侖三角形的面積
－ 內拿破侖三角形的面積
＝ 三角形ABC的面積，

同學們請自己驗證便可。

在任意三角形ABC的三邊外側作等邊三角形後，還有另一個美妙的特性是十七世紀數學家費馬(Fermat) 所發現的：

圖五中三角形ABR 、BCP 和CAQ 都是等邊的，所以ΔARC 繞點A 旋轉$60°$可得ΔABQ。因此

$$RC = BQ$$

及 $$\angle RFB = 60°。$$

同理，$PA = CR$。所以

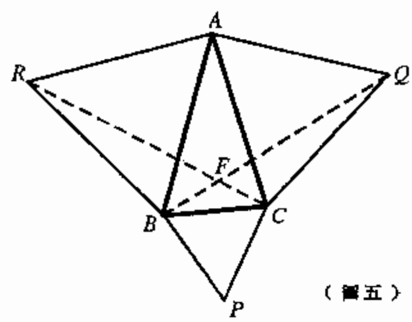
$$AP = BQ = CR 。$$

再者，

$$\angle RFB = 60° = \angle RAB$$

及

$$\angle CFQ = 60° = \angle CAQ 。$$



（圖五）

因此$ARBF$ 和$CQAF$ 都是圓外接四邊形。由於$\angle BFC = 120°$，而$\angle CPB = 60°$，可以推知$BPCF$ 也是圓外接四邊形。這三個圓於F 共點，稱為費馬點。由F 原是BQ 和CR 的交點，從對稱觀點可知F 也在AP上。

Mathematical Excalibur

Volume 4, Number 1 December, 1997 - February, 1998

Olympiad Corner

International Mathematics Tournament of the Towns, Spring 1997:

Junior A-Level Paper

Problem 1. One side of a triangle is equal to one third of the sum of the other two. Prove that the angle opposite the first side is the smallest angle of the triangle. (3 points)

Problem 2. You are given 25 pieces of cheese of different weights. Is it always possible to cut one of the pieces in two parts and put the 26 pieces in two packets so that

(i) each packet contains 13 pieces;

(ii) the total weights of the two packets are equal;

(iii) the two parts of the piece which has been cut are in different packets?

(5 points)

Problem 3. In a chess tournament, each of $2n$ players plays every other player once in each of two rounds. A win is worth 1 point and a draw is worth ½point. Prove that if for every player, the total score in the first round differs from that in the second round by at least n points, then the difference is exactly n points for every player. (5 points)

(continued on page 4)

Editors: 張百康 (CHEUNG Pak-Hong), Curr. Studies, HKU
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Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

On-line: http: //www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is April 15, 1998.

For individual subscription for the three remaining issues for the 97-98 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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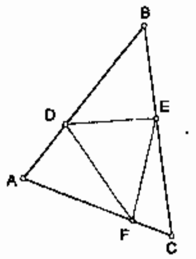
老師不教的幾何（五）

張百康

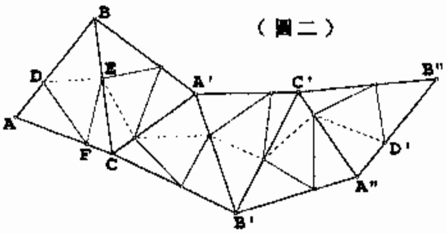
圖一顯示了一個銳角三角形ABC和它的一個內切三角形DEF。



（圖一）

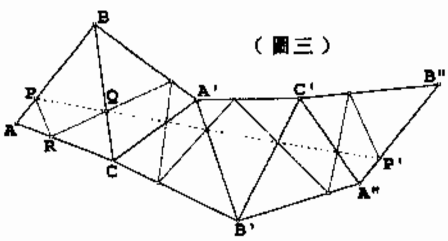
大家想一想：隨意在ΔABC上作內切三角形，哪一個的周界最短？這問題早在十八世紀時，由數學家Fagnano最先提出，並且用微分方法，經過繁複的運算和簡化，求得一個最短周長的內切三角形。因此這問題又名Fagnano問題。

經過整整一個世紀，才有另一位數學家Schwarz找到一個漂亮的初等幾何解法：如圖二所示，Schwarz將ΔABC以它的邊輪流作鏡面反射，得到六個相連的全等三角形。



（圖二）

圖二的虛折線$DE\ldots D'$全長剛好是ΔDEF周長的兩倍。將D、E、F沿ΔABC的三邊移動，是否可以找到另一個內切三角形PQR，使虛折線成一直線PP'（圖三）？



（圖三）

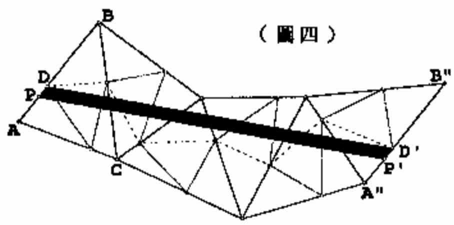
如果我們用一線段連D和D'，這線段肯定較折線$DE\ldots D'$短。但PP'和DD'的長度又如何比較呢？破綻正正在於為甚麼Schwarz要作不多不少的五次鏡面反射。大家細心看一看圖二的AB和$A''B''$，它們不但等長，而且好像互相平行呢。Schwarz巧妙地用旋轉的觀念來證明AB平行$A''B''$：

AB繞點B旋轉$2\angle B$得$A'B$，再繞點A'旋轉$2\angle A$得$A'B'$；$A'B'$繞點B'旋轉$-2\angle B$得$A''B$，再繞點A''旋轉$-2\angle A$得$A''B''$。因此AB和$A''B''$的夾角是

$$2\angle B+2\angle A-2\angle B-2\angle A=0,$$

也就是說，AB平行$A''B''$。

由於$AD=A''D'$和$AP=A''P'$，因此$PDD'P'$是平行四邊形（圖四）。換言之，ΔPQR的周長$(=\frac{1}{2}PP')$是所有ΔABC的內切三角形中最短的。

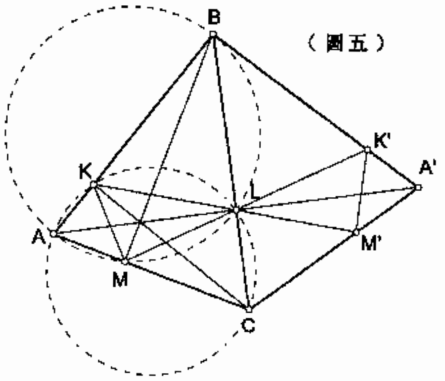


（圖四）

這ΔPQR究竟有甚麼特性？大家不妨再看一遍圖三，不難發現ΔPQR好像是ΔABC的垂足三角形(orthic triangle)。各同學可利用圖五證明圖中的垂足三角形KLM的角$\angle KLM$被高AL平分，關鍵在於圖中的一些四點共圓特性，留待各同學自行理解。

由於$\Delta A'BC$是ΔABC以BC為鏡面的反射影象，所以高AL和$A'L$成一直線，並且$A'L$也平分角$K'LM$。由此推知，KL和LM也成一直線。餘此類推，如按圖二

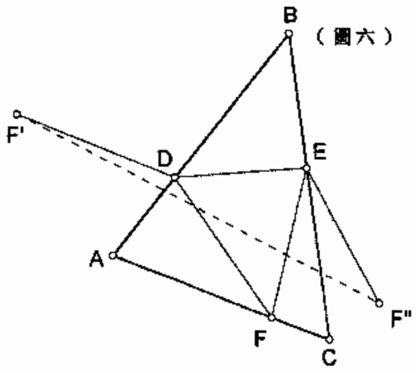
連續進行鏡面反射，則ΔKLM就是我們要找尋的最短周長內切三角形。



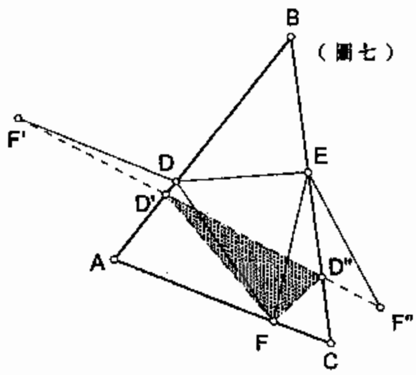
（圖五）

Schwarz 的證明固然巧妙，但好戲還在後頭。在1900年，當時還在柏林唸書的匈牙利數學家 Fejér 找到一個比 Schwarz 的證明還精簡的證法：

分別以ΔABC的邊BA和BC作鏡面，找到F的影像F'和F''（圖六）。由鏡面反射的性質可知：$FD = F'D$及$FE = F''E$，因此折線$F'DEF''$全長等於內切三角形DEF的周長。明顯地，祇要F點不變，不管其餘兩點D和E在AB和BC上如何移動，所得的內切三角形周長肯定大於直線$F'F''$的長度。設$F'F''$與AB及BC分別交於點D'和D''（圖七），則$\Delta FD'D''$的周長是所有一頂點在F的內切三角形中最短的。

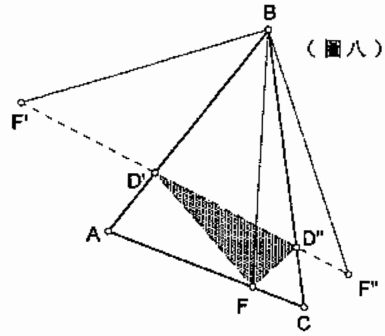


（圖六）



（圖七）

接著，我們改變F的位置，找尋上述這種$\Delta FD'D''$中周長最短者，便是我們要找的最短周長內切三角形。



（圖八）

利用鏡面反射的對稱性質可知：圖八中的$BF' = BF = BF''$，$\angle F'BD' = \angle FBD'$及$\angle FBD'' = \angle F''BD''$，因此

$$F'F'' = 2BF\sin B。$$

其中只有BF可改變，而BF長度的最小值是當它是ΔABC的高，即F是垂足。同理可知另外兩頂點D和E也必定是垂足方可使ΔDEF成爲周長最短的內切三角形。

青出於藍勝於藍，Schwarz 看過學生 Fejér 的證明後，也讚賞不已。

A Proof for The Lambek and Moser Theorem

Two sequences $f(n)$ and $f^*(n)$ are called inverse sequences if

$$f^*(n) = k, \text{ where } f(k) < n \le f(k+1).$$

Two sequences $F(n)$ and $G(n)$ are called complementary sequences if $F(n)$ and $G(n)$ together contain each natural number exactly once. (c.f. vol. 3 no. 4)

Theorem: $f(n)$ and $f^*(n)$ are inverse sequences if and only if $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ are complementary sequences (with the minor conditions that (i) $f(n)$ and $f^*(n)$ are non-decreasing sequences of non-negative integers; (ii) $F(n)$ and $G(n)$ are strictly increasing sequences of positive integers.)

Proof: We will first prove the converse. Let $F(n)$ and $G(n)$ be strictly increasing sequences of positive integers such that F and G are complementary. For example,

$$\begin{array}{ll} F(n)= & \overbrace{1,2,3,\quad 6,\ 8,}^{r}\quad 10,\ 11,\ \cdots \\ G(n)= & \qquad\underbrace{4,5,\ 7,\ 9,}_{s}\qquad 12,\ \cdots \end{array}$$

(Note the inserted spaces in the above illustration so that the natural numbers are in increasing order from left to right in relative position.) Let N be a natural number. Let r and s be the number of terms in $F(n)$ and $G(n)$ that are $\le N$ respectively. (In the above illustration, $N = 9$, $r = 5$ and $s = 4$.) Note that $r + s = N$.

Now consider $f(n) = F(n) - n$ and $f^*(n) = G(n) - n$.

$$\begin{array}{ll} f(n)= & \overbrace{0,0,0,\quad 2,\ 3,}^{r}\quad 4,\ 4,\ \cdots \\ f^*(n)= & \qquad\underbrace{3,3,\ 4,\ 5,}_{s}\qquad 7,\ \cdots \end{array}$$

We observe that

$$f^*(s) = G(s) - s = N - s = r.$$

That is,

$f^*(s) =$ the number of terms in f appear on the left hand side (in position) of the term $f^*(s)$.

Likewise,

$f(r) =$ the number of terms in f^* appear on the left hand side (in position) of the term $f(r)$.

Since the term $f(r)$ appear on the left hand side of $f^*(s)$, $f(r) < s$. We may similarly show that $f(r+1) \ge s$ and thus

$$f(r) < s \le f(r+1).$$

That is, $f^*(n)$ is the frequency distribution of $f(n)$ and thus $f(n)$ and $f^*(n)$ are inverse sequences. The fact that $f(r) < s \le f(r+1)$ can also be proved formally as follows.

$$f(r) = F(r) - r < N - r = s;$$

$$f(r+1) = F(r+1) - (r+1) > N - (r+1) = s - 1,$$
$$f(r+1) \ge s.$$

We will now show that if $f(n)$ is a non-decreasing sequence of non-negative integers and $f^*(n)$ is the frequency distribution function of $f(n)$, then $F(n) = f(n) + n$ and $G(n) = g(n) + n$ are complementary. Given the sequence $f(n)$, we can first construct the sequence $F(n) = f(n) + n$. Let $H(n)$ be the complementary sequence of $F(n)$ and let $h(n) = H(n) - n$. From the converse proof, $h(n)$ must be the frequency distribution of $f(n)$. Since the frequency distribution of a given function is unique, $h(n) = f^*(n)$ and thus

$$G(n) = f^*(n) + n = h(n) + n = H(n)$$

is the complementary sequence of $F(n)$.

Q.E.D.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is April 15, 1998.

Problem 71. Find all real solutions of the system

$$x+\log\left(x+\sqrt{x^2+1}\right)=y,$$
$$y+\log\left(y+\sqrt{y^2+1}\right)=z,$$
$$z+\log\left(z+\sqrt{z^2+1}\right)=x.$$

(*Source:* 1995 Israel Math Olympiad.)

Problem 72. Is it possible to write the numbers 1, 2, … , 121 in an 11×11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column? (*Source:* 1995 Russian Math Olympiad.)

Problem 73. Prove that if a and b are rational numbers satisfying the equation $a^5+b^5=2a^2b^2$, then $1-ab$ is the square of a rational number. (*Source:* 26th British Math Olympiad.)

Problem 74. Points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 of acute triangle ABC, respectively. Find the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$, $\angle A_2C_1B_2$. (*Source:* 1995 Russian Math Olympiad.)

Problem 75. Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$, for some integer N, then $N = 26$. (*Source:* 23rd British Math Olympiad.)

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Solutions

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Problem 66.

(a) Find the first positive integer whose square ends in three 4's.

(b) Find all positive integers whose squares end in three 4's.

(c) Show that no perfect square ends with four 4's.

(*Source:* 1995 British Mathematical Olympiad.)

Solution: Andy CHAN Kin Hang (Bishop Hall Jubilee School, Form 4) and **SHUM Ho Keung** (PLK No. 1 W. H. Cheung College, Form 5).

(a) Since $21^2 < 444 < 22^2$ and $1444 = 38^2$, the first such positive integer is 38.

(b) Assume n is such an integer. Then

$$n^2 - 1444 = (n - 38)(n + 38)$$

is divisible by $1000 = 2^35^3$. This implies at least one of $n - 38$, $n + 38$ is divisible by 4. Since their difference is 76, hence both must be divisible by 4. Since 76 is not divisible by 5, hence one of $n - 38$, $n + 38$ is divisible by $4\cdot5^3 = 500$. Then $n = 500k \pm 38$ for some nonnegative integer k. Conversely, for such n,

$$n^2 = 1000(250k^2 \pm 38k) + 1444$$

always ends in three 4's.

(c) Since $250k^2 \pm 38k$ is even, no perfect square ends with four 4's.

Other commended solvers: **KWOK Chi Hang** (Valtorta College, Form 6), **LAI Chi Fung, Brian** (Queen Elizabeth School, Form 5), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **LI Fung** (HK Taoist Association Ching Chung Secondary School, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5) and **WONG Shu Fai** (Valtorta College, Form 6).

Problem 67. Let Z and R denote the integers and real numbers, respectively. Find all functions $f: Z \to R$ such that

$$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2}$$

for all integers x, y such that $x + y$ is divisible by 3. (*Source:* a modified problem from the 1995 Iranian Mathematical Olympiad.)

Solution: CHAN Wing Sum (City U) and **TSANG Sai Wing** (Valtorta College, Form 7).

For all integer n,

$$f(0) + f(3n) = 2f(n) = f(n) + f(2n).$$

This implies

$$f(n)=f(2n)=\frac{f(3n)+f(3n)}{2}=f(3n).$$

So $f(n) = f(0)$ for all integer n. It is also clear that all constant functions are solutions.

Other commended solvers: **Andy CHAN Kin Hang** (Bishop Hall Jubilee School, Form 4), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **LI Fung** (HK Taoist Association Ching Chung Secondary School, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5) and **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6).

Problem 68. If the equation

$$ax^2 + (c - b)x + (e - d) = 0$$

has real roots greater than 1, show that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has at least one real root. (*Source*: 1995 Greek Mathematical Olympiad.)

Solution: CHAN Wing Chiu (La Salle College, Form 5).

Suppose

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

has no real root. Let $y > 1$ be a root of $ay^2 + (c - b)y + (e - d) = 0$ and $z=\sqrt{y}$. Since

$$p(x) = ax^4+(c-b)x^2+(e-d)+(x-1)(bx^2+d),$$

we get

$$p(z) = (z - 1)(bz^2 + d)$$

and

$$p(-z) = (-z - 1)(bz^2 + d).$$

Now $z > 1$ implies one of $p(z)$, $p(-z)$ is positive, while the other is negative. Therefore, $p(x)$ has a root between z and $-z$, a contradiction.

Problem 69. $ABCD$ is a quadrilateral such that $AB = AD$ and $\angle B = \angle D = 90°$. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$. (*Source:* 1995 Russian Mathematical Olympiad.)

Solution 1: WONG Hau Lun (STFA Leung Kau Kui College, Form 6).

Let E' be the mirror image of E with

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

respect to AC. Let X be the intersection of DF and AE. Let Y be the intersection of AF and BE. Since $\angle ADE = 90° = \angle AXD$, we have $\angle ADF = \angle DEA = \angle BE'A = 180° - \angle AE'F$. So A, D, F, E' are concyclic. Then $\angle AFD = \angle AE'D = \angle AEB$. So X, E, F, Y are concyclic. Therefore $\angle EYF = \angle EXF = 90°$.

Solution 2: CHING Wai Hung (STFA Leung Kau Kui College, Form 6).

Since $DF \perp AE$ and $DA \perp DE$, so

$$\begin{aligned} 0 &= \overrightarrow{DF}\cdot\overrightarrow{AE} \\ &= (\overrightarrow{DA}+\overrightarrow{AF})\cdot\overrightarrow{AE} \\ &= \overrightarrow{DA}\cdot(\overrightarrow{AD}+\overrightarrow{DE})+\overrightarrow{AF}\cdot\overrightarrow{AE} \end{aligned}$$

which simplifies to

$$\overrightarrow{AF}\cdot\overrightarrow{AE} = |\overrightarrow{AD}|^2.$$

Since $BF \perp BA$, so

$$\begin{aligned} \overrightarrow{AF}\cdot\overrightarrow{BE} &= \overrightarrow{AF}\cdot(\overrightarrow{BA}+\overrightarrow{AE}) \\ &= (\overrightarrow{AB}+\overrightarrow{BF})\cdot\overrightarrow{BA}+\overrightarrow{AF}\cdot\overrightarrow{AE} \\ &= -\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{AD}\right|^2 \\ &= 0 \end{aligned}$$

which implies $AF \perp BE$.

Other commended solver: **TSANG Kam Wing** (Valtorta College, Form 5).

Problem 70. Lines $l_1, l_2, \cdots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \cdots, k{-}1$ so that in each of the lines $l_1, l_2, \cdots, l_k$ the numbers $1, 2, \cdots, k{-}1$ appear exactly once if and only if k is even. (*Source:* a modified problem from the 1995 Greek Mathematical Olympiad.)

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 5).

If such labeling exists for an integer k, then the label 1 must occur once on each line and each point labeled 1 lies on exactly 2 lines. Hence there are $k/2$ 1's, i.e. k is even.

Conversely, if k is even, then the following labeling works: for $1 \le i < j \le k-1$, give the intersection of lines l_i and l_j the label $i + j - 1$ when $i + j \le k$, the label $i + j - k$ when $i + j > k$. For the intersection of lines l_k and l_i ($i = 1, 2, \ldots, k-1$), give the label $2i - 1$ when $2i \le k$ the label $2i - k$ when $2i > k$.

Comments: The official solution made use of the special symmetry of an odd number sided regular polygon to construct the labeling as follow: for k even, consider the $k - 1$ sided regular polygon with the vertices labeled $1, 2, \ldots, k-1$. For $1 \le i < j \le k-1$, the perpendicular bisector of the segment joining vertices i and j passes through a unique vertex, give the intersection of lines l_i and l_j the label of that vertex. For the intersection of lines l_k and l_i ($i = 1, 2, \ldots, k-1$), give the label i.

Other commended solver: **LAW Ka Ho** (Queen Elizabeth School, Form 5).

Olympiad Corner

(*continued from page 1*)

Problem 4. $AC'BA'CB'$ is a convex hexagon such that $AB' = AC'$, $BC' = BA'$ and $CA' = CB'$. Moreover, $\angle A + \angle B + \angle C = \angle A' + \angle B' + \angle C'$. Prove that the area of triangle ABC is half of the area of the hexagon. (6 points)

Problem 5. Prove that the number

(a) 97^{97}; (4 points)

(b) 1997^{17} (4 points)

is not representable as a sum of cubes of several consecutive integers.

Problem 6. Let P be a point inside the triangle ABC with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$, and $\angle ACP = 30°$. Find $\angle BPC$. (7 points)

Problem 7. You are given a balance and one copy of each ten weights of 1, 2, 4, 8, 16, 32, 64, 128, 256 and 512 grams. An object weighing M grams, where M is a positive integer, may be balanced in different ways by placing various combinations of the given weights on either pans of the balance.

(a) Prove that no object may be balanced in more than 89 ways. (5 points)

(b) Find a value of M such that an object weighing M grams can be balanced in 89 ways. (4 points)

Senior A-Level Paper

Problem 1. same as Junior A-Level Paper Problem 2. (4 points)

Problem 2. D is the point on BC and E is the point on CA such that AD and BE are the bisectors of $\angle A$ and $\angle B$ of triangle ABC. If DE is the bisector of $\angle ADC$, find $\angle A$. (5 points)

Problem 3. You are given 20 positive weights such that any object of integer weight m, $1 \le m \le 1997$, can be balanced by placing in it one pan of a balance and a subset of the weights on the other pan. What is the minimal value of the largest of the 20 weights if the weights are

(a) all integers; (3 points)

(b) not necessarily integers? (3 points)

Problem 4. A convex polygon G is placed inside a convex polygon F so that their boundaries have no common points. A segment s containing two points on the boundary of F is called a support chord for G if s contains a side or only a vertex of G. Prove that

(a) there exists a support chord for G whose midpoint lies on the boundary of G; (6 points)

(b) there exist at least two such chords. (2 points)

Problem 5. Prove that

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \le 1,$$

where a, b and c are positive numbers such that $abc = 1$. (8 points)

Problem 6. Prove that if $F(x)$ and $G(x)$ are polynomials with coefficients 0 and 1 such that

$$F(x)G(x) = 1 + x + x^2 + \ldots + x^{n-1}$$

holds for some $n > 1$, then one of them is representable in the form

$$(1 + x + x^2 + \ldots + x^{k-1})T(x)$$

for some $k > 1$ and some polynomial $T(x)$ with coefficients 0 and 1. (8 points)

Problem 7. Several strips and a circle of radius 1 are drawn on the plane. The sum of the widths of the strips is 100. Prove that one can translate each strip parallel to itself so that together they cover the circle. (8 points)

Mathematical Excalibur

Volume 4, Number 2 March, 1998 - December, 1998

Olympiad Corner

Tenth Asian Pacific Mathematics Olympiad, March, 1998:

Each question is worth 7 points.

Problem 1. Let F be the set of all n-tuples $(A_1, A_2,\ldots, A_n)$ where each A_i, i = 1, 2, ..., n is a subset of $\{1,2,\ldots,1998\}$. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1,A_2,\ldots,A_n)} \left|A_1 \cup A_2 \cup \ldots A_n\right|.$$

Problem 2. Show that for any positive integers a and b, $(36a+b)$ $(a+36b)$ cannot be a power of 2.

Problem 3. Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Problem 4. Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math Dept, Catherine NG, EEE Dept, HKUST and Tam Siu Lung for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 31, 1998.

For individual subscription for the three remaining issues for the 98-99 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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A Taste of Topology

Wing-Sum Chan

Beauty is the first test: there is no permanent place in the world for ugly mathematics.
(G. H. Hardy)

In topology, there are many abstractions of geometrical ideas, such as continuity and closeness. 'Topology' is derived from the Greek words $\tau o\pi o\sigma$, a place and $\lambda o\gamma o\sigma$, a discourse. It was introduced in 1847 by Johann Benedict Listing (1808-1882), who was a student of Carl Friedrich Gauss (1777-1855). In the early days, people called it *analysis situs*, that is, analysis of position. Rubber-sheet geometry is a rather descritpive term to say what it is. (Just think of properties of objects drawn on a sheet of rubber which are not changed when the sheet is being distorted.) Hence, topologists could not distinguish a triangle from a rectangle and they may even consider a basketball as a ping-pong ball.

Topologists consider two objects to be the same (homeomorphic) if one can be continuously deformed to look like the other. Continuous deformations include bending, stretching and squashing without gluing or tearing points.

Example 1. The following are homeomorphic: (See Figure 1.)

Fig1.

Example 2. The following are non-homeomorphic: (See Figure 2.)

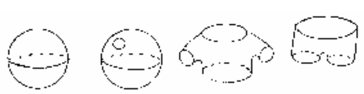


Fig 2.

In practise, continuous deformations may not be easy to carry out. In fact, there is a simple method to see two objects are non-homeomorphic, by seeking their Poincaré-Euler characteristics, (in short, Euler numbers). In order to see what the Euler number is, we need to introduce the concept of subdivision on an n-manifold (here $n \le 2$ throughout). (An n-manifold is roughly an n dimensional object in which each point has a neighborhood homeomorphic to an open interval (if n = 1) or an open disk (if n = 2). For example, a circle is a 1-manifold and a sphere is a 2-manifold.)

Basically, we start with an n manifold and subdividing it into a finite number of vertices, edges and faces. A vertex is a point. An edge is a curve with endpoints that are vertices. A face is a region with boundary that are edges.

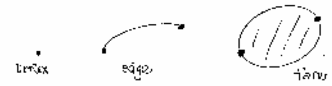


Fig 3

Here are typical pictures of vertex, edge and face, (see Figure 3.)

The Euler number (χ) of a compact (loosely speaking, bounded) 1-manifold is defined to be the number of vertices(v) minus the number of edges(e), and for a compact 2-manifold (surface), it is defined to be the number of vertices(v) minus the number of edges (e) plus the number of

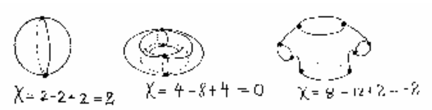


Fig 4

faces (f) (see Figure 4.) The following theorem is a test to distinguish non-homeomorphic objects.

Theorem 1. *If two n-manifolds are homeomorphic, then they have the same Euler number.*

So figure 4 and theorem 1 imply the sphere and the torus are not homeomorphic, i.e. the sphere cannot be continuously deformed to look like the torus and vice versa.

Here are two terms we need before we can state the next theorem. A connected manifold is one where any two points on the manifold can be connected by a curve on the manifold. The manifold is orientable if it has 2 sides, an inside and an outside.

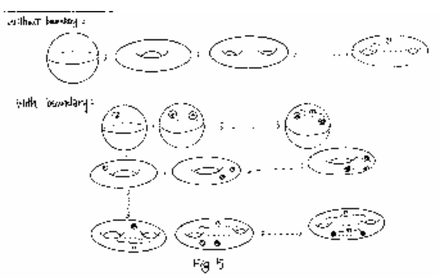
Theorem 2. *Two connected orientable n-manifolds* ($n \le 2$) *with the same number of boundary components are homeomorphic if and only if they have the same Euler number.*

Here are some important results that tell us the general pictures of one and two manifolds.

Classification I. *Any connected compact one-manifold is either homeomorphic to an open interval or a circle.*

Classification II. *Any connected, orientable and compact two-manifolds is homeomorphic to one of the followings:* (see Figure 5.)

Finally, we mention a famous open problem (the Poincaré conjecture), which is to show that every compact, simply connected three-manifold is homeomorphic to a three-sphere, where simply connected means any circle on the manifold can be



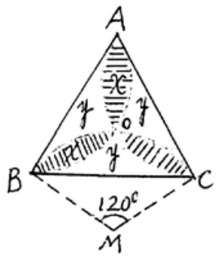
shrunk to a point on the manifold.

巧列一次方程組　妙解陰影面積題

于志洪

江蘇泰州橡膠總廠中學

有一類關於求陰影部分面積的問題，我們可根據題意適當設元，通過一次方程組求得結果。這種數形結合，將幾何面積問題轉化爲解一次方程組代數問題的方法，由於方法新穎、思路清晰，因而頗受師生重視。現舉三例分析說明如下：



一．列二元一次方程組求陰影面積

例一：如上圖，O 爲正三角形 ABC 的中心，$AB = 8\sqrt{3}\text{cm}$，則 $\overparen{AOB}$、$\overparen{BOC}$、$\overparen{COA}$ 所圍成的陰影部分的面積是____ cm^2。(1996 年陝西省中考題)

分析：上圖中含有形狀不同的兩類圖形，分別爲 x 和 y，由圓形特徵得知，2 個 x 和 1 個 y 組成一個圓心角爲 $120°$ 的弓形，而 3 個 x 和 3 個 y 組成一個正三角形 ABC。由於正三角形 ABC 的高 $= \frac{\sqrt{3}}{2} \times 8\sqrt{3} = 12$，又 O 爲正三角形 ABC 的中心，故 $BO = \frac{2}{3} \times 12 = 9 = MB$。

$\therefore 2x + y = S_{扇形MBC} - S_{\Delta MBC}$

$$= \frac{120\pi \cdot 8^2}{360} - \frac{1}{2} \times 8\sqrt{3} \times 4 = \frac{64\pi}{3} - 16\sqrt{3}$$

$$= \frac{1}{2} \times 8\sqrt{3} \times 12 = 48\sqrt{3}$$

解下列方程組

$$\begin{cases} 2x + y = \dfrac{64\pi}{3} - 16\sqrt{3} & \cdots\cdots\cdots(1) \\ 3x + 3y = 48\sqrt{3} & \cdots\cdots\cdots(2) \end{cases}$$

，得 $3x = 64\pi - 93\sqrt{3}$。這就是所求陰影部分的面積。

二．列三元一次方程組求陰影面積

例 2：如下圖，在正方形 $ABCD$ 中，有一個以正方形的中心爲圓心，以邊長一半爲半徑的圓。另分別以 A、B、C、D 爲圓心，以邊長一半爲半徑畫四條弧。若正方形的邊長爲 $2a$，求所圍成的陰影部分的面積。（1997 年泰州市中考模擬題）

分析：圖中含有形狀不同三類圖形，分別爲 x、y、z。由圖形特徵得知：4 個 x 和 1 個 y 組成一個圓；1 個 x 和 1 個 z 組成一個以 a 爲半徑、圓心角爲直角的的扇形；4 個 x、4 個 z 和 1 個 y 組成一個正方形。

故此，可列出方程組

$$\begin{cases} 4x + y = \pi a^2 & \cdots\cdots\cdots(1) \\ x + z = \frac{1}{4}\pi a^2 & \cdots\cdots\cdots(2) \\ 4x + y + 4z = 4a^2 & \cdots\cdots\cdots(3) \end{cases}$$

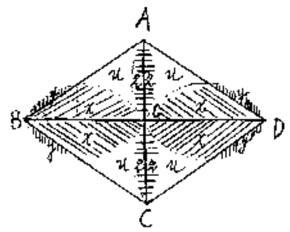
$(3) - (1)$ 得 $z = \frac{1}{4}(4 - \pi)a^2$

再代入 (2) 得 $x = (\frac{\pi}{2} - 1)a^2$。

$\therefore S_{陰影} = 4x = (2\pi - 4)a^2$。

三．列四元一次方程組求陰影面積

例 3：如左圖，菱形 $ABCD$ 的兩條對角線長分別爲 a、b，分別以每邊爲直徑向形內作半圓。求 4 條半圓弧圍成的花瓣形面積(陰影部分的面積)。（人教版九年義務教材初中《幾何》第三冊 P. 212）



分析：圖中含有形狀不同的四類圖形，分別爲 x、y、z、u，則由圖形特徵得知：$2x$、$2y$、z、u 組成一個以邊長爲直徑的半圓；x、z、u 組成直角三角形 BOC。

解:設 x、y、z、u 如圖所示，則依題意得

$$\begin{cases} x + z + u = \dfrac{1}{2} \cdot \dfrac{a}{2} \cdot \dfrac{b}{2} & \cdots(1) \\ 2x + 2z + y + u = \dfrac{1}{2}\pi\left(\dfrac{1}{2}\sqrt{\frac{a^2+b^2}{4}}\right)^2 & \cdots(2) \end{cases}$$

$(2) - (1)$ 再乘 4 得

（續於第四頁）

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *Dec 31, 1998.*

Problem 76. Find all positive integers N such that in base 10, the digits of $9N$ is the reverse of the digits of N and N has at most one digit equal 0. (*Source:* 1977 unused IMO problem proposed by Romania)

Problem 77. Show that if ΔABC satisfies

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C}=2,$$

then it must be a right triangle. (*Source:* 1967 unused IMO problem proposed by Poland)

Problem 78. If $c_1, c_2, \ldots, c_n$ ($n \ge 2$) are real numbers such that

$$(n-1)(c_1^2+c_2^2+\ldots+c_n^2)$$
$$=(c_1+c_2+\ldots+c_n)^2,$$

show that either all of them are nonnegative or all of them are nonpositive. (*Source:* 1977 unused IMO problem proposed by Czechoslovakia)

Problem 79. Which regular polygons can be obtained (and how) by cutting a cube with a plane? (*Source:* 1967 unused IMO problem proposed by Italy)

Problem 80. Is it possible to cover a plane with (infinitely many) circles in such a way that exactly 1998 circles pass through each point? (*Source:* Spring 1988 Tournament of the Towns Problem)

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Solutions

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Problem 71. Find all real solutions of the system

$$x+\log(x+\sqrt{x^2+1})=y,$$
$$y+\log(y+\sqrt{y^2+1})=z,$$
$$z+\log(z+\sqrt{z^2+1})=x.$$

(*Source:* 1995 Israel Math Olympiad)

Solution: CHOI Fun Ieng (Pooi To Middle School (Macau), Form 5).

If $x < 0$, then $0<x+\sqrt{x^2+1}<1$. So $\log(x+\sqrt{x^2+1})<0$, which implies $y < x < 0$. Similarly, we get $z < y < 0$ and $x < z < 0$, yielding the contradiction $x < z < y < x$. If $x > 0$, then $x+\sqrt{x^2+1}>1$. So $\log(x+\sqrt{x^2+1})>0$, which implies $y > x > 0$. Similarly, we get $z > y > 0$ and $x > z > 0$, yielding the contradiction $x > z > y > x$. If $x = 0$, then $x = y = z = 0$ is the only solution.

Other commended solvers: **AU Cheuk Yin** (Ming Kei College, Form 5). **CHEUNG Kwok Koon** (HKUST), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **HO Chung Yu** (Ming Kei College, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 6), **KU Wah Kwan** (Heep Woh College, Form 7), **KWOK Chi Hang** (Valtorta College, Form 6), **LAM Yee** (Valtorta College, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 5), **WONG Chi Man** (Valtoria College, Form 3) and **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6).

Problem 72. Is it possible to write the numbers 1,2,…,121 in an 11x11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column? (*Source:* 1995 Russian Math Olympiad)

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 5).

Suppose such a table exists. The table would be divided into 2 parts by the single column of perfect squares, with one side $11n$ $(0 \le n \le 5)$ cells and the other side $110 - 11n$ cells. Note that numbers between 2 successive perfect squares, say a^2, $(a+1)^2$, lie on one side since they cannot cross over the perfect square column, and those between $(a+1)^2$, $(a+2)^2$ lie on opposite side. Now the number of integers (strictly) between 1, 4, 9, 16, …, 100, 121 is 2, 4, 6, 8, …, 20, respectively. So one side has 2 + 6 + 10 + 14 + 18 = 50 numbers while the other side has 4 + 8 + 12 + 16 + 20 = 60 numbers. Both 50 and 60 are not multiple of 11, a contradiction.

Other commended solvers: **CHEUNG Kwok Koon** (HKUST), **HO Chung Yu** (Ming Kei College, Form 6), **LAI Chi Fung Brian** (Queen Elizabeth School, Form 4), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 5), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6) and **WONG Shu Fai** (Valtorta College, Form 6).

Problem 73. Prove that if a and b are rational numbers satisfying the equation $a^5+b^5=2a^2b^2$, then $1-ab$ is the square of a rational number. (*Source:* 26th British Math Olympiad)

Solution: CHAN Wing Sum (City U).

If $b=0$, then $1-ab=1^2$. If $b\neq 0$, then $a^6+ab^5=2a^3b^2$. So $a^6-2a^3b^2+b^4$ $=b^4-ab^5=b^4(1-ab)$. Therefore, $1-ab$ $=(a^6-2a^3b^2+b^4)/b^4$ is the square of the rational number $(a^3-b^2)/b^2$.

Other recommended solvers: **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **CHOI Fun Ieng** (Pooi To Middle School (Macau), Form 5), **KU Wah Kwan** (Heep Woh College, Form 7) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5).

Problem 74. Points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 of acute triangle ABC, respectively. Find the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$ and $\angle A_2C_1B_2$. (*Source:* 1995 Russian Math Olympiad)

Solution: LAM Po Leung (Ming Kei College, Form 5)

Let A_3, B_3, C_3 be the midpoints of BC, CA, AB, respectively, and H be the orthocenter of ΔABC. Since C_3A_3 is parallel to AC, so $\angle HB_2A_3=90°$ $=\angle HA_1A_3$, which implies H, B_2, A_3, A_1 are concyclic. So $\angle B_2A_1H=\angle B_2A_3H$. Since B_3A_3 is parallel to AB, so

$\angle HC_2A_3 = 90° = \angle HA_1A_3$, which implies H, C_2, A_3, A_1 are concyclic. So $\angle C_2A_1H = \angle C_2A_3H$. Then $\angle B_2A_1C_2$ $= \angle B_2A_1H + \angle C_2A_1H = \angle B_2A_3H + \angle C_2A_3H$ $= \angle C_3A_3B_3 = \angle BAC$ (because $\Delta A_3B_3C_3$ is similar to ΔABC). Similarly, $\angle B_2C_1A_2 = \angle BCA$ and $\angle A_2B_1C_2 = \angle ABC$. Therefore, the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$, $\angle A_2B_1C_2$ is 180°.

Other commended solvers: **HO Chung Yu** (Ming Kei College, Form 6).

Problem 75. Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$ for some integer N, then $N = 26$. (*Source:* 23rd British Math Olympiad)

Solutions: HO Chung Yu (Ming Kei College, Form 6).

If $P(N) = N + 51$ for some integer N, then $P(x) – x – 51 = (x - N)Q(x)$ for some polynomial $Q(x)$ by the factor theorem. Note $Q(x)$ has integer coefficients because $P(x) – x – 51 = P(x) – P(N) – (x – N)$ is a sum of $a_i(x^i – N^i)$ terms (with a_i's integer). Since $Q(21)$ and $Q(37)$ are integers, $P(21) – 21 – 51 = -55$ is divisible by $21 – N$ and $P(3) – 37 – 51 = -55$ is divisible by $37 – N$ is 16, we must have $N = 26$ or 32. However, if $N = 32$, then we get $-247 = P(32) = 32 + 51$, a contradiction. Therefore $N = 26$.

Other commended solvers: **CHEUNG Kwok Koon** (HKUST), **KU Wah Kwan** (Heep Who College, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 5) and **WONG Shu Fai** (Valtorta College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 5. Determine the largest of all integers n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

（續第二頁）

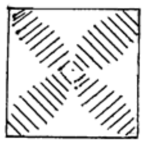
$$4(x+y+z) = \frac{\pi a^2 + \pi b^2 - 4ab}{8}$$ 。

這就是所求陰影部分的面積。

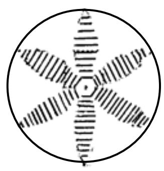
綜上所述可知：一般的陰影圖形大多是由多種規則圖形組成的，所以利用方程式組解決這類問題時，首先要根據圖形的特徵（尤其是對稱性）把圖形分成幾類，用字母表示各類圖形的面積；其次要仔細觀察圖形的組成，分析圖形中各部分之間及各部分與整體圖形的關係，通過規則圖形面積公式列出方程組；最後解方程求出陰影面積。

附練習題

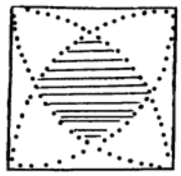
１．如右圖，已知一塊正方形的地瓷磚邊長為 a，瓷磚上的圖案是以各邊為直徑在正方形內畫圓所圍成的（陰影部分），那麼陰影部分的面積是多少？（1997 年寮夏回族自治區中考題）



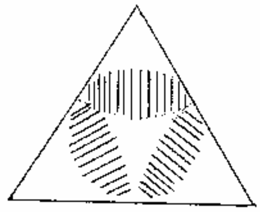
２．如右圖，已知圓形 O 的半徑為 R，求圖中陰影部分的面積。（1998 年泰州市中考模擬題）



３．如右圖，正方形的邊長為 a，分別以正方形的四個頂點為圓心，邊長為半徑，在正方形內畫弧，那麼這四條弧所圍成的陰影部分的面積是多少？（1994 年安徽省中考題）



４．如右圖，圓 O 內切於邊長為 a 的正三角形，分別以三角形的三頂點為圓心，$\frac{a}{2}$ 為半徑畫弧相交成圓中所示的陰影，求陰影部分的面積。（1996 年泰州市中考模擬題）



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《初中生數學園地》
安義人(華南師大主辦) 1997 年 3 月
2. 《列一次方程組解陰影面積題》
《中小學數學》
于志洪(中國教育學會主辦)1997年11月

練習題答案:

1. $(\frac{\pi}{2}-1)a^2$ 2. $2\pi R^2 - 3\sqrt{3}R^2$

3. $(1-\sqrt{3}+\frac{\pi}{3})a^2$ 4. $\frac{5\pi-6\sqrt{3}}{24}a^2$

(Hong Kong team to IMO 98: (from left to right) Lau Wai Tong (Deputy Leader), Law Ka Ho, Chan Kin Hang, Choi Ming Cheung, Lau Lap Ming, Cheung Pok Man, Leung Wing Chung, Liu Kam Moon (Leader).)

Mathematical Excalibur

Volume 4, Number 3 January, 1999 - March, 1999

Olympiad Corner

39th International Mathematical Olympiad, July 1998:

Each problem is worth 7 points.

Problem 1. In the convex quadrilateral *ABCD*, the diagonals *AC* and *BD* are perpendicular and the opposite sides *AB* and *DC* are not parallel. Suppose that the point *P*, where the perpendicular bisectors of *AB* and *DC* meet, is inside *ABCD*. Prove that *ABCD* is a cyclic quadrilateral if and only if the triangles *ABP* and *CDP* have equal areas.

Problem 2. In a competition, there are *a* contestants and *b* judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose *k* is a number such that, for any two judges, their ratings coincide for at most *k* contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Problem 3. For any positive integer *n*, let *d(n)* denote the number of positive divisions of *n* (including 1 and *n* itself).

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math Dept, Catherine NG, EEE Dept, HKUST and Tam Siu Lung for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is April 30, 1999.

For individual subscription for the two remaining issues for the 98-99 academic year, send us two stamped self-addressed envelopes. Send all correspondence to:

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Rearrangement Inequality

Kin-Yin Li

The rearrangement inequality (or the permutation inequality) is an elementary inequality and at the same time a powerful inequality. Its statement is as follow. Suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Let us call

$$A = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

the *ordered sum* of the numbers and

$$B = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$$

the *reverse sum* of the numbers. If $x_1, x_2, ..., x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, ..., b_n$ and if we form the *mixed sum*

$$X = a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

then the rearrangement inequality asserts that $A \geq X \geq B$. In the case the a_i's are strictly increasing, then equality holds if and only if the b_i's are all equal.

We will look at $A \geq X$ first. The proof is by mathematical induction. The case $n = 1$ is clear. Suppose the case $n = k$ is true. Then for the case $n = k + 1$, let $b_{k+1} = x_i$ and $x_{k+1} = b_j$. Observe that $(a_{k+1} - a_i)(b_{k+1} - b_j) \geq 0$. We get

$$a_ib_j + a_{k+1}b_{k+1} \geq a_ib_{k+1} + a_{k+1}b_j.$$

So in *X*, we may switch x_i and x_{k+1} to get a possibly larger sum. After switching, we can apply the case $n = k$ to the first *k* terms to conclude that $A \geq X$. The inequality $X \geq B$ follows from $A \geq X$ using $-b_n \leq -b_{n-1} \leq \cdots \leq -b_1$ in place of $b_1 \leq b_2 \leq \cdots \leq b_n$.

Now we will give some examples.

Example 1. (*Chebysev's Inequality*) Let *A* and *B* be as in the rearrangement inequality, then

$$A \geq \frac{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)}{n} \geq B.$$

Proof. Cyclically rotating the b_i's, we get *n* mixed sums

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n,$$
$$a_1b_2 + a_2b_3 + \cdots + a_nb_1,$$
$$...,$$
$$a_1b_n + a_2b_1 + \cdots + a_nb_{n-1}.$$

By the re-arrangement inequality, each of these is between *A* and *B*, so their average is also between *A* and *B*. This average is just the expression given in the middle of Chebysev's inequality.

Example 2. (*RMS-AM-GM-HM In-equality*) Let $c_1, c_2, ..., c_n \geq 0$. The *root mean square* (RMS) of these numbers is $[(c_1^2 + \cdots + c_n^2)/n]^{1/2}$, the *arithmetic mean* (AM) is $(c_1 + c_2 + \cdots + c_n)/n$ and the *geometric mean* (GM) is $(c_1c_2 \cdots c_n)^{1/n}$. We have *RMS* ≥ *AM* ≥ *GM*. If the numbers are positive, then the *harmonic mean* (HM) is $n/[(1/c_1) + \cdots + (1/c_n)]$. We have *GM* ≥ *HM*.

Proof. Setting $a_i = b_i = c_i$ in the left half of Chebysev's inequality, we easily get *RMS* ≥ *AM*. Next we will show *AM* ≥ *GM*. The case *GM* = 0 is clear. So suppose *GM* > 0. Let $a_1 = c_1/GM$, $a_2 = c_1c_2/GM^2$, ..., $a_n = c_1c_2 \cdots c_n/GM^n = 1$ and $b_i = 1/a_{n-i+1}$ for *i* = 1, 2, ... *n*. (Note the a_i's may not be increasing, but the b_i's will be in the reverse order as the a_i's). So the mixed sum

$$a_1b_1 + a_2b_n + \cdots + a_nb_2 =$$
$$c_1/GM + c_2/GM + \cdots + c_n/GM$$

is greater than or equal to the reverse sum $a_1b_n + \cdots + a_nb_1 = n$. The AM-GM inequality follows easily. Finally *GM* ≥ *HM* follows by applying *AM* ≥ *GM* to the numbers $1/c_1, ..., 1/c_n$.

Example 3. (1974 *USA Math Olympiad*) If $a, b, c > 0$, then prove that

$$a^a b^b c^c \ge (abc)^{(a+b+c)/3}.$$

Solution. By symmetry, we may assume $a \le b \le c$, then $\ln a \le \ln b \le \ln c$. By Chebysev's inequality,

$$a\ln a + b\ln b + c\ln c \ge \frac{(a+b+c)(\ln a + \ln b + \ln c)}{3}.$$

The desired inequality follows from exponentiation.

Example 4. *(1978 IMO)* Let $c_1, c_2, ..., c_n$ be distinct positive integers. Prove that

$$c_1 + \frac{c_2}{4} + \cdots + \frac{c_n}{n^2} \ge 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Solution. Let $a_1, a_2, ..., a_n$ be the c_i's arranged in increasing order. Since a_i's are distinct positive integers, $a_i \ge i$. Since $1 > 1/4 > ... > 1/n^2$, by the re-arrangement inequality,

$$c_1 + \frac{c_2}{4} + \cdots + \frac{c_n}{n^2} \ge a_1 + \frac{a_2}{4} + \cdots + \frac{a_n}{n^2} \ge 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Example 5. *(1995 IMO)* Let $a, b, c > 0$ and $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

Solution. (HO Wing Yip, Hong Kong Team Member) Let $x = bc = 1/a$, $y = ca = 1/b$, $z = ab = 1/c$. The required inequality is equivalent to

$$\frac{x^2}{z+y} + \frac{y^2}{x+z} + \frac{z^2}{y+x} \ge \frac{3}{2}.$$

By symmetry, we may assume $x \le y \le z$, then $x^2 \le y^2 \le z^2$ and $1/(z+y) \le 1/(x+z) \le 1/(y+x)$. The left side of the required inequality is just the ordered sum A of the numbers. By the rearrangement inequality,

$$A \ge \frac{x^2}{y+x} + \frac{y^2}{z+y} + \frac{z^2}{x+z},$$

$$A \ge \frac{x^2}{x+z} + \frac{y^2}{y+x} + \frac{z^2}{z+y}.$$

(continued on page 4)

Power of Points Respect to Circles

Kin-Yin Li

Intersecting Chords Theorem. *Let two lines through a point P not on a circle intersect the inside of the circle at chords AA' and BB', then PA × PA' = PB × PB'. (When P is outside the circle, the limiting case A = A' refers to PA tangent to the circle.)*

This theorem follows from the observation that triangles *ABP* and *A'B'P* are similar and the corresponding sides are in the same ratio. In the case *P* is inside the circle, the product *PA* × *PA*' can be determined by taking the case the chord *AA*' passes through *P* and the center *O*. This gives *PA* × *PA*' = $r^2 - d^2$, where *r* is the radius of the circle and *d* = *OP*. In the case *P* is outside the circle, the product *PA* × *PA*' can be determined by taking the limiting case *PA* is tangent to the circle. Then *PA* × *PA*' = $d^2 - r^2$.

The *power* of a point *P* with respect to a circle is the number $d^2 - r^2$ as mentioned above. (In case *P* is on the circle, we may define the power to be 0 for convenience.) For two circles C_1 and C_2 with different centers O_1 and O_2, the points whose power with respect to C_1 and C_2 are equal form a line perpendicular to line O_1O_2. (This can be shown by setting coordinates with line O_1O_2 as the *x*-axis.) This line is called the *radical axis* of the two circles. In the case of the three circles C_1, C_2, C_3 with noncollinear centers O_1, O_2, O_3, the three radical axes of the three pairs of circles intersect at a point called the *radical center* of the three circles. (This is because the intersection point of any two of these radical axes has equal power with respect to all three circles, hence it is on the third radical axis too.)

If two circles C_1 and C_2 intersect, their radical axis is the line through the intersection point(s) perpendicular to the line of the centers. (This is because the intersection point(s) have 0 power with respect to both circles, hence they are on the radical axis.) If the two circles do not intersect, their radical axis can be found by taking a third circle C_3 intersecting both C_1 and C_2. Let the radical axis of C_1, C_3 intersect the radical axis of C_2, C_3 at *P*. Then the radical axis of C_1, C_2 is the line through *P* perpendicular to the line of centers of C_1, C_2.

We will illustrate the usefulness of the intersecting chords theorem, the concepts of power of a point, radical axis and radical center in the following examples.

Example 1. (*1996 St. Petersburg City Math Olympiad*) Let *BD* be the angle bisector of angle *B* in triangle *ABC* with *D* on side *AC*. The circumcircle of triangle *BDC* meets *AB* at *E*, while the circumcircle of triangle *ABD* meets *BC* at *F*. Prove that *AE* = *CF*.

Solution. By the intersecting chords theorem, *AE* × *AB* = *AD* × *AC* and *CF* × *CB* = *CD* × *CA*, so *AE/CF* = (*AD/CD*)(*BC/AB*). However, *AB/CB* = *AD/CD* by the angle bisector theorem. So *AE* = *CF*.

Example 2. (*1997 USA Math Olympiad*) Let *ABC* be a triangle, and draw isosceles triangles *BCD, CAE, ABF* externally to *ABC*, with *BC, CA, AB* as their respective bases. Prove the lines through *A, B, C,* perpendicular to the lines *EF, FD, DE,* respectively, are concurrent.

Solution. Let C_1 be the circle with center *D* and radius *BD*, C_2 be the circle with center *E* and radius *CE*, and C_3 be the circle with center *F* and radius *AF*. The line through *A* perpendicular to *EF* is the radical axis of C_2, C_3, the line through *B* perpendicular to *FD* is the radical axis of C_3, C_1 and the line through *C* perpendicular to *DE* is the radical axis of C_1, C_2. These three lines concur at the radical center of the three circles.

Example 3. (*1985 IMO*) A circle with center *O* passes through vertices *A* and *C* of triangle *ABC* and intersects side *AB* at *K* and side *BC* at *N*. Let the circumcircles of triangles *ABC* and *KBN* intersect at *B* and *M*. Prove that *OM* is perpendicular to *BM*.

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Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *April 30, 1999.*

Problem 81. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area. (*Source*: 1996 Irish Mathematical Olympiad)

Problem 82. Show that if n is an integer greater than 1, then n^4+4^n cannot be a prime number. (*Source:* 1977 Jozsef Kurschak Competition in Hungary)

Problem 83. Given an alphabet with three letters *a, b, c*, find the number of words of *n* letters which contain an even number of *a*'s. (*Source*: 1996 Italian Mathematical Olympiad)

Problem 84. Let M and N be the midpoints of sides AB and AC of ΔABC, respectively. Draw an arbitrary line through A. Let Q and R be the feet of the perpendiculars from B and C to this line, respectively. Find the locus of the intersection P of the lines QM and RN as the line rotates about A.

Problem 85. Starting at (1, 1), a stone is moved in the coordinate plane according to the following rules:

(a) From any point (a, b), the stone can be moved to $(2a, b)$ or $(a, 2b)$.

(b) From any point (a, b), the stone can be moved to $(a - b, b)$ if $a > b$, or to $(a, b - a)$ if $a < b$.

For which positive integers x, y, can the stone be moved to (x, y)? (*Source:* 1996 German Mathematical Olympiad)

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Solutions

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Problem 76. Find all positive integers N such that in base 10, the digits of $9N$ is the reverse of the digits of N and N has at most one digit equal 0. (*Source:* 1977 unused IMO problem proposed by Romania)

Solution. LAW Ka Ho (Queen Elizabeth School, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6).

Let $[a_1a_2 \ldots a_n]$ denote N in base 10 with $a_1 \neq 0$. Since $9N$ has the same number of digits as N, we get $a_1 = 1$ and $a_n = 9$. Since $9 \times 19 \neq 91$, $n > 2$. Now $9[a_2 \ldots a_{n-1}] + 8 = [a_{n-1} \ldots a_2]$. Again from the number of digits of both sides, we get $a_2 \le 1$. The case $a_2 = 1$ implies $9a_{n-1} + 8$ ends in a_2 and so $a_{n-1} = 7$, which is not possible because $9[1 \ldots 7] + 8 > [7 \ldots 1]$. So $a_2 = 0$ and $a_{n-1} = 8$. Indeed, 1089 is a solution by direct checking. For $n > 4$, we now get $9[a_3 \ldots a_{n-2}] + 8 = [8\,a_{n-2} \ldots a_3]$. Then $a_3 \ge 8$. Since $9a_{n-2} + 8$ ends in a_3, $a_3 = 8$ will imply $a_{n-2} = 0$, causing another 0 digit. So $a_3 = 9$ and $a_{n-2} = 9$. Indeed, 10989 and 109989 are solutions by direct checking. For $n > 6$, we again get $9[a_4 \ldots a_{n-3}] + 8 = [8\,a_{n-3} \ldots a_4]$. So $a_4 = \ldots = a_{n-3} = 9$. Finally direct checking shows these numbers are solutions.

Other recommended solvers: **CHAN Siu Man** (Ming Kei College, Form 6), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 7), **FANG Wai Tong Louis** (St. Mark's School, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 7), **KWOK Chi Hang** (Valtorta College, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 6), **WONG Chi Man** (Valtorta College, Form 4), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 7) and **WONG Shu Fai** (Valtorta College, Form 7).

Problem 77. Show that if $\Delta\,ABC$ satisfies

$$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} = 2,$$

then it must be a right triangle. (*Source*: 1967 unused IMO problem proposed by Poland)

Solution. (All solutions received are essentially the same.)

Using $\sin^2 x = (1-\cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$, the equation is equivalent to

$$\cos 2A + \cos 2B + \cos 2C + 1 = 0.$$

This yields $\cos(A + B)\cos(A - B) + \cos^2 C = 0$. Since $\cos(A + B) = -\cos C$, we get $\cos C\,(\cos(A - B) + \cos(A + B)) = 0$. This simplifies to $\cos C \cos A \cos B = 0$. So one of the angles A, B, C is 90^0.

Solvers: **CHAN Lai Yin, CHAN Man Wai**, **CHAN Siu Man**, **CHAN Suen On**, **CHEUNG Kin Ho**, **CHING Wai Hung**, **CHOI Ching Yu**, **CHOI Fun Ieng**, **CHOI Yuet Kei**, **FANG Wai Tong Louis**, **FUNG Siu Piu, HUNG Kit**, **KEE Wing Tao Wilton**, **KO Tsz Wan**, **KWOK Chi Hang, LAM Tung Man**, **LAM Wai Hung**, **LAM Yee**, **LAW Ka Ho**, **LI Ka Ho**, **LING Hoi Sheung**, **LOK Chan Fai, LUNG Chun Yan**, **MAK Wing Hang**, **MARK Kai Pan**, **Gary NG Ka Wing**, **OR Kin**, **TAM Kwok Cheong**, **TAM Siu Lung**, **TSANG Kam Wing**, **TSANG Pui Man**, **TSANG Wing Kei**, **WONG Chi Man**, **WONG Hau Lun**, **YIM Ka Wing** and **YU Tin Wai**.

Problem 78. If $c_1, c_2, \ldots, c_n\,(n \ge 2)$ are real numbers such that

$$(n-1)(c_1^2 + c_2^2 + \cdots + c_n^2) = (c_1 + c_2 + \cdots + c_n)^2,$$

show that either all of them are non-negative or all of them are non-positive. (*Source*: 1977 unused IMO problem proposed by Czechoslovakia)

Solution. CHOY Ting Pong (Ming Kei College, Form 6).

Assume the conclusion is false. Then there are at lease one negative and one positive numbers, say $c_1 \le c_2 \le \cdots \le c_k \le 0 < c_{k+1} \le \cdots \le c_n$ with $1 \le k < n$, satisfying the condition. Let $w = c_1 + \ldots + c_k$, $x = c_{k+1} + \cdots + c_n$, $y = c_1^2 + \cdots + c_k^2$ and $z = c_{k+1}^2 + \cdots + c_n^2$. Expanding w^2 and x^2 and applying the inequality $a^2 + b^2 \ge 2ab$, we get $ky \ge w^2$ and $(n - k)\,z \ge x^2$. So

$$(w+x)^2 = (n-1)(y+z) \ge ky + (n-k)z \ge w^2 + x^2.$$

Simiplifying, we get $wx \ge 0$, contradicting $w < 0 < x$.

Other commended solvers: **CHAN Siu Man** (Ming Kei College, Form 6), **FANG Wai Tong Louis** (St. Mark's School, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6), **TAM Siu Lung** (Queen Elizabeth School, Form 6), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 7) and **YEUNG Kam Wah** (Valtorta College, Form 7).

Problem 79. Which regular polygons can be obtained (and how) by cutting a cube with a plane? (*Source*: 1967 unused IMO problem proposed by Italy)

Solution. FANG Wai Tong Louis (St. Mark's school, Form 6), **KEE Wing Tao** (PLK Centenary Li Shiu Chung Memorial School, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 6) and **YEUNG Kam Wah** (Valtorta College, Form 7).

Observe that if two sides of a polygon is on a face of the cube, then the whole polygon lies on the face. Since a cube has 6 faces, only regular polygon with 3, 4, 5 or 6 sides are possible. Let the vertices of the bottom face of the cube be A, B, C, D and the vertices on the top face be A', B', C', D' with A' on top of A, B' on top of B and so on. Then the plane through A, B', D' cuts an equilateral triangle. The perpendicular bisecting plane to edge AA' cuts a square. The plane through the mid-points of edges $AB, BC, CC', C'D', D'A', A'A$ cuts a regular hexagon. Finally, a regular pentagon is impossible, otherwise the five sides will be on five faces of the cube implying two of the sides are on parallel planes, but no two sides of a regular pentagon are parallel.

Problem 80. Is it possible to cover a plane with (infinitely many) circles in such a way that exactly 1998 circles pass through each point? (*Source*: Spring 1988 Tournament of the Towns Problem)

Solution. Since no solution is received, we will present the modified solution of Professor Andy Liu (University of Alberta, Canada) to the problem.

First we solve the simpler problem where 1998 is replaced by 2. Consider the lines $y = k$, where k is an integer, on the coordinate plane. Consider every circle of diameter 1 tangent to a pair of these lines. Every point (x, y) lies on exactly two of these circles. (If y is an integer, then (x, y) lies on one circle on top of it and one below it. If y is not an integer, then (x, y) lies on the right half of one circle and on the left half of another.) Now for the case 1998, repeat the argument above 998 times (using lines of the form $y = k + (j/999)$ in the j-th time, $j = 1, 2, ..., 998$.)

Olympiad Corner

(continued from page 1)

Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

Problem 4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Problem 5. Let I be the incentre of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that $\angle RIS$ is acute.

Problem 6. Consider all functions f from the set $\mathbb{N}$ of all positive integers into itself satisfying

$$f(t^2 f(s)) = s(f(t))^2,$$

for all s and t in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

Rearrangement Inequality

(continued from page 2)

So

$$A \geq \frac{1}{2}\left(\frac{y^2+x^2}{y+x} + \frac{z^2+y^2}{z+y} + \frac{x^2+z^2}{x+z}\right).$$

Applying the RMS-AM inequality $r^2 + s^2 \geq (r+s)^2/2$, the right side is at least $(x+y+z)/2$, which is at least $3(xyz)^{1/3}/2 = 3/2$ by the AM-GM inequality.

Power of Points Respect to Circles

(continued from page 2)

Solution. For the three circles mentioned, the radical axes of the three pairs are lines AC, KN and BM. (The centers are noncollinear because two of them are on the perpendicular bisector of AC, but not the third.) So the axes will concur at the radical center P. Since $\angle PMN = \angle BKN = \angle NCA$, it follows that P, M, N, C are concyclic. By power of a point, $BM \times BP = BN \times BC = BO^2 - r^2$ and $PM \times PB = PN \times PK = PO^2 - r^2$, where r is the radius of the circle through A, C, N, K. Then $PO^2 - BO^2 = BP(PM - BM) = PM^2 - BM^2$. This implies OM is perpendicular to BM. (See remarks below.)

Remarks. By coordinate geometry, it can be shown that the locus of points X such that $PO^2 - BO^2 = PX^2 - BX^2$ is the line through O perpendicular to line BP. This is a useful fact.

Example 4. (*1997 Chinese Math Olympiad*) Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the tangents QE and QF to the circle, where E and F are the points of tangency. Prove that P, E, F are collinear.

Solution. Let M be a point on PQ such that $\angle CMP = \angle ADC$. Then D, C, M, Q are concyclic and also, B, C, M, P are concyclic. Let r_1 be the radius of the circumcircle C_1 of $ABCD$ and O_1 be the center of C_1. By power of a point, $PO_1^2 - r_1^2 = PC \times PD = PM \times PQ$ and $QO_1^2 - r_1^2 = QC \times QB = QM \times PQ$. Then $PO_1^2 - QO_1^2 = (PM - QM)PQ = PM^2 - QM^2$, which implies $O_1M \perp PQ$. The circle C_2 with QO_1 as diameter passes through M, E, F and intersects C_1 at E, F. If r_2 is the radius of C_2 and O_2 is the center of C_2, then $PO_1^2 - r_1^2 = PM \times PQ = PO_2^2 - r_2^2$. So P lies on the radical axis of C_1, C_2, which is the line EF.

Mathematical Excalibur

Volume 4, Number 4 April 1999 - September 1999

Olympiad Corner

11th Asian Pacific Mathematical Olympiad, March 1999:

Time allowed: 4 Hours
Each problem is worth 7 points.

Problem 1. Find the smallest positive integer n with the following property: There does not exist an arithmetic progression of 1999 terms of real numbers containing exactly n integers.

Problem 2. Let a_1, a_2, ... be a sequence of real numbers satisfying $a_{i+j} \le a_i + a_j$ for all i, j = 1, 2, ... Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n$$

for each positive integer n.

Problem 3. Let Γ_1 and Γ_2 be two circles interecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, Catherine Ng, EEE Dept, HKUST and Tam Siu Lung for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is September 30, 1999.

For individual subscription for the two remaining issues for the 98-99 academic year, send us two stamped self-addressed envelopes. Send all correspondence to:

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費馬最後定理 (一)

梁子傑
香港道教聯合會青松中學

大約在 1637 年，當法國業餘數學家費馬 (Pierre de Fermat, 1601-1665) 閱讀古希臘名著《算術》時，在書邊的空白地方，他寫下了以下的一段說話：「將個立方數分成兩個立方數，一個四次冪分成兩個四次冪，或者一般地將一個高於二次冪的數分成兩個相同次冪，這是不可能的。 我對這個命題有一個美妙的證明，這裏空白太小，寫不下。」換成現代的數學術語，費馬的意思就即是：「當整數 $n > 2$ 時，方程 $x^n + y^n = z^n$ 沒有正整數解。」

費馬當時相信自己已發現了對以上命題的一個數學證明。 可惜的是，當費馬死後，他的兒子為他收拾書房時，並沒有發現費馬的「美妙證明」。到底，費馬有沒有證實這個命題呢？又或者，費馬這個命題是否正確呢？

費馬這個命題並不難理解，如果大家用計算機輸入一些數字研究一下，（注意：費馬的時代並未發明任何電子計算工具，）那麼就會「相信」費馬這個命題是正確的。由於 費馬在生時提出的其他數學命題，都逐步被證實或否定，就祇剩下這一個看似正確，但無法證明的命題未能獲證，所以數學家就稱它為「費馬最後定理」。

說也奇怪，最先對「費馬最後定理」的證明行出第一步的人，就是費馬本人！有人發現，在費馬的書信中，曾經提及方程 $x^4 + y^4 = z^4$ 無正整數解的證明。費馬首先假設方程 $x^4 + y^4 = z^2$ 是有解的，即是存在 三個正整數 a、b 和 c，並且 $a^4 + b^4$ 剛好等於 c^2。 然後他通過「勾股數組」的通解，構作出另外三個正整數 e、f 和 g，使得 $e^4 + f^4 = g^2$ 並且 $c > g$。 費馬指出這是不可能的，因為如果這是正確的，那麼重覆他的構作方法，就可以構造出一連串遞降的數字，它們全都滿足方程 $x^4 + y^4 = z^2$。 但是 c 是一個有限數， 不可能如此無窮地遞降下去！ 所以前文中假設方程 $x^4 + y^4 = z^2$ 有解這個想法不成立，亦即是說方程 $x^4 + y^4 = z^2$ 無整數解。

又由於方程 $x^4 + y^4 = z^2$ 是無解的，方程 $x^4 + y^4 = z^4$ 亦必定無解。 否則 將後者的解寫成 $x^4 + y^4 = (z^2)^2$ 就會 變成前一個方程的解，從而導出矛盾。 由此可知，當 $n = 4$ 時，「費馬最後定理」成立。

為「費馬最後定理」踏出另一步的人，是瑞士大數學家歐拉（Leonhard Euler, 1707-1783）。他利用了複數 $a + b\sqrt{-3}$ 的性質，證實了方程 $x^3 + y^3 = z^3$ 無解。但由於歐拉在他的證明中，在沒有足夠論據的支持下，認為複數 $a + b\sqrt{-3}$ 的立方根必定可以再次寫成 $a + b\sqrt{-3}$ 的形式， 因此他的證明 未算圓滿。 歐拉證明的缺憾，又過了近半個世紀，才由德國數學家高斯（Carl Friedrich Gauss, 1777-1855）成 功地補充。 同時，高斯更為此而引進了「複整數」的概念，即形如 $a + b\sqrt{-k}$ 的複數，其中 k 為正整數，a 和 b 為整數。

1823 年，七十一歲高齡的法國數學家勒讓德（Adrien Marie Legendre, 1752 - 1833）提出了「費馬最後定理」當 $n = 5$ 時的證明。 1828 年，年青的德國數學家狄利克雷（Peter Gustav Lejeune Dirichlet, 1805 -1859）亦獨立地

Pierre Fermat

Leonhard Euler

Carl Friedrich Gauss

Lejeune Dirichlet

證得同樣的結果。 其後，在 1832 年，狄利克雷更證明當 $n = 14$ 時，「費馬最後定理」成立。

1839 年，另一位法國人拉梅（Gabriel Lamé, 1795 -1870）就證到 $n = 7$。1847 年，拉梅更宣稱他已完成了「費馬最後定理」的證明。

拉梅將 $x^n + y^n$ 分解成 $(x + y)(x + \zeta y)(x + \zeta^2 y)\ldots(x + \zeta^{n-1} y)$，其中 $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$ ，即方程 $r^n = 1$ 的複數根。 如果 $x^n + y^n = z^n$，那麼拉梅認爲每一個 $(x + \zeta^k y)$ 都會是 n 次冪乘以一個複數單位，從而可導出矛盾，並能證明「費馬最後定理」成立。 不過，拉梅的證明很快便證實爲無效，這是因爲拉梅所構作的複數，並不一定滿足「唯一分解定理」。

甚麼是「唯一分解定理」呢？ 在一般的整數中，每一個合成數都祇可能被分解成一種「質因數連乘式」。 但在某些「複整數」中，情況就未必相同。例如：$6 = 2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$，而在 $a + b\sqrt{-5}$ 的複整數中，2、3、$(1 + \sqrt{-5})$ 和 $(1 - \sqrt{-5})$ 都是互不相同的質數。換句話說，形如 $a + b\sqrt{-5}$ 的複整數，並不符合「唯一分解定理」。

如果能夠滿足「唯一分解定理」，那麼當 $z^n = ab$ 時，我們就確信可以找到兩個互質的整數 u 和 v ，使得 $a = u^n$ 和 $b = v^n$ 了。 但如果未能滿足「唯一分解定理」，以上的推論就不成立了。例如：$6^2 = 2 \times 3 \times (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$，但右方的四個數，都並非是一個平方數，故此，當 $6^2 = ab$ 時，我們就不能肯定 a 和 b 是不是平方數了！ 這一點，亦正好是拉梅證明的一大漏洞！

爲了解決未能滿足「唯一分解定理」所帶來的問題，德國數學家庫默爾（Ernst Edward Kummer, 1810 -1893）就提出了「理想數」的想法。

已知 n 爲一個質數。 假設 $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$，即方程 $r^n = 1$ 的複數根，則稱

$$a_0 + a_1\zeta + a_2\zeta + \cdots + a_{n-1}\zeta^{n-1}$$

爲「分圓整數」，其中 a_i 爲整數。 並非每一個分圓整數集合都滿足「唯一分解定理」，但如果能夠加入一個額外的「數」，使到該分圓整數集合滿足「唯一分解定理」，則稱該數爲「理想數」。庫默爾發現，當 n 爲一些特殊的質數時，（他稱之爲「正規質數」，）就可以利用「理想數」來證明「費馬最後定理」在這情況下成立。

由此，庫默爾證明了當 $n < 100$ 時，「費馬最後定理」成立。

德國商人沃爾夫斯凱爾（Paul Friedrich Wolfskehl, 1856 - 1908）在他的遺囑上訂明，如果有人能夠在他死後一百年內證實「費馬最後定理」，則可以獲得十萬馬克的獎金。自此，「費馬最後定理」就吸引到世上不同人仕的注意，不論是數學家或者是業餘學者，都紛紛作出他們的「證明」。在 1909 至 1934 年間，「沃爾夫斯凱爾獎金」的評審委員會，就收到了成千上萬個「證明」，可惜的是當中並沒有一個能夠成立。自從經過了兩次世界大戰之後，該筆獎金的已大幅貶值，「費馬最後定理」的吸引力和熱潮，亦慢慢地降低了。

其實，研究「費馬最後定理」有甚麼好處呢？首先，就是可以滿足人類的求知慾。「費馬最後定理」是一道簡單易明的命題，但是它的證明卻並非一般人所能理解，這已經是一個非常之有趣的事情。其次，在證明該定理的過程之中，我們發現了不少新的數學現象，產生了不少新的數學工具，同時亦豐富了我們對數學，特別是數論的知識。有數學家更認爲，「費馬最後定理」就好像一隻會生金蛋的母雞，由它所衍生出來的數學理論，例如：「唯一分解定理」、「分圓整數」、「理想數」……等等，都是人類思想中最珍貴的產物。

(to be continued next issue)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *October 1, 1999.*

Problem 86. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}.$$

(Source: 1996 Vietnamese Math Olympiad)

Problem 87. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an *O*. Then, player II chooses another square and marks it with *X*. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning. (*Source: 1995 Israeli Math Olympiad*)

Problem 88. Find all positive integers n such that $3^{n-1}+5^{n-1}$ divides 3^n+5^n. (*Source: 1996 St. Petersburg City Math Olympiad*)

Problem 89. Let O and G be the circumcenter and centroid of triangle ABC, respectively. If R is the circumradius and r is the inradius of ABC, then show that $OG\le\sqrt{R(R-2r)}$. (*Source: 1996 Balkan Math Olympiad*)

Problem 90. There are n parking spaces (numbered 1 to n) along a one-way road down which n drivers $d_1,d_2,\ldots,d_n$ in that order are traveling. Each driver has a favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. (Two drivers may have the same favorite space.) If there is no free space after his favorite, he drives away. How many lists $a_1,a_2,\ldots,a_n$ of favorite parking spaces are there which permit all of the drivers to park? Here a_i is the favorite parking space number of d_i. (*Source: 1996 St. Petersburg City Math Olympiad*)

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Solutions

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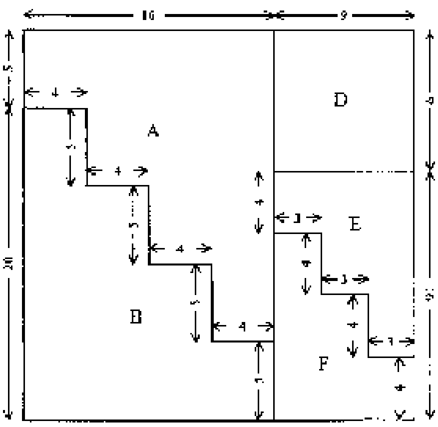
Problem 81. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area. (*Source: 1996 Irish Math Olympiad*)

Solution. SHAM Wang Kei (St. Paul's College, Form 4).

In the following diagram, A and B can be reassembled to form a 20×20 square and E and F can be reassembled to form a 12×12 square.



25×25

Other recommended solvers: **CHAN Man Wai** (St. Stephen's Girls' College, Form 4).

Problem 82. Show that if n is an integer greater than 1, then n^4+4^n cannot be a prime number. (*Source: 1977 Jozsef Kurschak Competition in Hungary*).

Solution. Gary NG Ka Wing (STFA Leung Kau Kui College, Form 6) and **NG Lai Ting** (True Light Girls' College, Form 6).

For even n, n^4+4^n is an even integer greater than 2, so it is not a prime. For odd $n>1$, write $n=2k-1$ for a positive integer $k>1$. Then $n^4+4^n=(n^2+2^n)^2-2^{n+1}n^2=(n^2+2^n-2^kn)(n^2+2^n+2^kn)$. Since the smaller factor $n^2+2^n-2^kn=(n-2^{k-1})^2+2^{2k-2}>1, n^4+4^n$ cannot be prime.

Other recommended solvers: **FAN Wai Tong** (St. Mark's School, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 6), **SHAM Wang Kei** (St. Paul's College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4) and **TAM Siu Lung** (Queen Elizabeth School, Form 6).

Problem 83. Given an alphabet with three letters *a, b, c*, find the number of words of *n* letters which contain an even number of *a*'s. (*Source: 1996 Italian Math Olympiad*).

Solution I. CHAO Khek Lun Harold (St. Paul's College, Form 4) **and Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6).

For a nonnegative even integer $2k\le n$, the number of n letter words with $2k$ a's is $C_{2k}^n2^{n-2k}$. The answer is the sum of these numbers, which can be simplified to $((2+1)^n+(2-1)^n)/2$ using binomial expansion.

Solution II. TAM Siu Lung (Queen Elizabeth School, Form 6).

Let S_n be the number of n letter words with even number of a's and T_n be the number of n letter words with odd number of a's. Then $S_n+T_n=3^n$. Among the S_n words, there are T_{n-1} words ended in a and $2S_{n-1}$ words ended in b or c. So we get $S_n=T_{n-1}+2S_{n-1}$. Similarly $T_n=S_{n-1}+2T_{n-1}$. Subtracting these, we get $S_n - T_n=S_{n-1}-T_{n-1}$. So $S_n-T_n=S_1-T_1=2-1=1$. Therefore, $S_n=(3^n+1)/2$.

Problem 84. Let M and N be the midpoints of sides AB and AC of ΔABC, respectively. Draw an arbitrary line through A. Let Q and R be the feet of the perpendiculars from B and C to this line, respectively. Find the locus of the intersection P of the lines QM and RN as the line rotates about A.

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 4).

Let S be the midpoint of side BC. From midpoint theorem, it follows $\angle MSN=\angle BAC$. Since M is the midpoint of the hypotenuse of right triangle AQM, we get $\angle BAQ=\angle AQM$. Similarly, $\angle CAR=\angle ARN$.

If the line intersects side BC, then either $\angle MPN=\angle QPR$ or $\angle MPN+\angle QPR=180^\circ$. In the former case, $\angle MPN=180^\circ-\angle PQR-\angle PRQ = 180^\circ-\angle AQM$ -

$\angle ARN = 180^\circ - \angle BAC$. So $\angle MPN$ + $\angle MSN = 180^\circ$. Then, *M, N, S, P* are concyclic. In the later case, $\angle MPN$ = $\angle PQR + \angle PRQ$ = $\angle AQM + \angle ARN$ = $\angle BAC$ = $\angle MSN$. So again *M, N, S, P* are concyclic. Similarly, if the line does not intersect side *BC*, there are 2 cases both lead to *M, N, S, P* concyclic. So the locus is on the circumcircle of *M, N, S*. Conversely, for every point *P* on this circle, draw line *MP* and locate *Q* on line *MP* so that *QM* = *AM*. The line *AQ* is the desired line and *QM, RN* will intersect at *P*.

Comments: The circle through *M, N, S* is the nine point circle of ΔABC. As there are 4 cases to deal with, it may be better to use coordinate geometry.

Other commended solvers: **FAN Wai Tong** (St. Mark's School, Form 6) and **TAM Siu Lung** (Queen Elizabeth School, Form 6).

Problem 85. Starting at $(1,1)$, a stone is moved in the coordinate plane according to the following rules:

(a) Form any point (*a, b*), the stone can be moved to (2*a, b*) or (*a,* 2*b*).

(b) From any point (*a, b*), the stone can be moved to $(a-b, b)$ if $a > b$, or to $(a, b-a)$ if $a < b$.

For which positive integers *x, y*, can the stone be moved to (*x, y*)? (*Source: 1996 German Math Olympiad*)

Solution. Let gcd(*x, y*) be the greatest common divisor (or highest common factor) of *x* and *y*. After rule (a), the gcd either remained the same or doubled. After rule (b), the gcd remain the same. So if (*x, y*) can be reached from (*a, b*), then gcd (*x, y*) = 2^n gcd(*a, b*) for a nonnegative integer *n*. If *a* = *b* = 1, then gcd(*x, y*) = 2^n.

Conversely, suppose gcd(*x, y*) = 2^n. Of those points (*a, b*) from which (*x, y*) can be reached, choose one that minimizes the sum *a* + *b*. If *a* or *b* is even, then (*x, y*) can be reached from (*a*/2, *b*) or (*a, b*/2) with a smaller sum. So *a* and *b* are odd. If *a* > *b* (or *a* < *b*), then (*x, y*) can be reached from ((*a* + *b*)/2, *b*) (or (*a*, (*a* + *b*)/2)) with a smaller sum. So *a* = *b*. Since 2^n = gcd(*x, y*) is divisible by *a* = gcd(*a, b*) and *a* is odd, so *a* = *b* = 1. Then (*x, y*) can be reached from (1, 1).

Olympiad Corner

(continued from page 1)

Problem 4. Determine all pairs (*a, b*) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Problem 5. Let *S* be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of *S* on its circumference, $n-1$ points in its interior and $n-1$ in its exterior. Prove that the number of good circles has the same parity as *n*.

Equation $x^4 + y^4 = z^4$

Recall the following theorem, see *Mathematical Excalibur*, Vol. 1, No. 2, pp. 2, 4 available at the web site

www.math.ust.hk/mathematical_excalibur/

Theorem. If *u, v* are *relatively prime* positive (i.e. *u, v* have no common prime divisor), $u > v$ and one is odd, the other even, then $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ give a *primitive* solution of $a^2 + b^2 = c^2$ (i.e. a solution where *a, b, c* are relatively prime). Conversely, every primitive solution is of this form, with a possible permutation of *a* and *b*.

Using this theorem, Fermat was able to show $x^4 + y^4 = z^4$ has no positive integral solutions. We will give the details below.

It is enough to show the equation $x^4 + y^4 = w^2$ has no positive integral solutions. Suppose $x^4 + y^4 = w^2$ has positive integral solutions. Let $x = a$, $y = b$, $w = c$ be a positive integral solution with *c* taken to be the least among all such solution. Now *a, b, c* are relatively prime for otherwise we can factor a common prime divisor and reduce *c* to get contradiction. Since $(a^2)^2 + (b^2)^2 = c^2$, by the theorem, there are relatively prime positive integers *u, v* (one is odd, the other even) such that $a^2 = u^2 - v^2$, $b^2 = 2uv$, $c = u^2 + v^2$. Here *u* is odd and *v* is even for otherwise $a^2 \equiv -1$ (mod 4), which is impossible. Now $a^2 + v^2 = u^2$ and *a, u, v* are relatively prime. By the theorem again, there are relatively prime positive integers *s, t* such that $a = s^2 - t^2$, $v = 2st$, $u = s^2 + t^2$. Now $b^2 = 2uv = 4st(s^2 + t^2)$. Since $s^2, t^2, s^2 + t^2$ are relatively prime, we must have $s = e^2$, $t = f^2, s^2 + t^2 = g^2$ for some positive integers *e, f, g*. Then $e^4 + f^4 = g^2$ with $g \le g^2 = s^2 + t^2 = u \le u^2 < c$. This contradicts the choice *c* being least. Therefore, $x^4 + y^4 = w^2$ has no positive integral solutions.

IMO1999

This year the International Mathematical Olympiad will be held in Romania. Based on their performances in qualifying examinations, the following students are selected to be Hong Kong team members:

Chan Ho Leung (Diocesan Boys' School, Form 7)
Chan Kin Hang (Bishop Hall Jubilee School, Form 5)
Chan Tsz Hong (Diocesan Boys' School, Form 7)
Law Ka Ho (Queen Elizabeth School, Form 6)
Ng Ka Wing (STFA Leung Kau Kui College, Form 6)
Wong Chun Wai (Choi Hung Estate Catholic Secondary School, Form 6)

Both Chan Kin Hang and Law Ka Ho were Hong Kong team members last year. This year the team leader is Dr. Tam Ping Kwan (Chinese University of Hong Kong) and the deputy leader will be Miss Luk Mee Lin (La Salle College).

Corrections

In the last issue of the *Mathematical Excalibur*, the definition of *power* given in the article *Power of Points Respect to Circles* should state "The *power* of a point *P* with respect to a circle is the number $d^2 - r^2$ as mentioned above." In particular, the power is positive when the point is outside the circle. The power is 0 when the point is on the circle. The power is negative when the point is inside the circle.

Mathematical Excalibur

Volume 4, Number 5 *October 1999 - December 1999*

Olympiad Corner

40th International Mathematical Olympiad, July 1999:

Time allowed: 4.5 Hours
Each problem is worth 7 points.

Problem 1. Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Problem 2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers x_1, x_2, …, $x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Problem 3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 15, 1999.

For individual subscription for the next five issues for the 99-00 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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費馬最後定理（二）

梁子傑
香港道教聯合會青松中學

在「數論」的研究之中，有一門分枝不可不提，它就是「橢圓曲線」(Elliptic Curve) $y^2 = x^3 + ax^2 + bx + c$（見 page 2 附錄）。

「橢圓曲線」並非橢圓形，它是計算橢圓周長時的一件「副產品」。但「橢圓曲線」本身卻有著一些非常有趣的數學性質，吸引著數學家的注視。

提到「橢圓曲線」，又不可不提「谷山 – 志村猜想」了。

1954 年，志村五郎 (Goro Shimura) 在東京大學結識了比他太一歲的谷山豊（Yutaka Taniyama, 1927 - 1958），之後，就開始了二人對「模形式」(modular form) 的研究。「模形式」，起源於法國數學家龐加萊（Henry Poincaré, 1854 - 1912）對「自守函數」的研究。所謂「自守函數」，可以說是「週期函數」的推廣，而「模形式」則可以理解為在複平面上的「週期函數」。

1955 年，谷山開始提出他的驚人猜想。三年後，谷山突然自殺身亡。其後，志村繼續谷山的研究，總結出以下的一個想法：「每條橢圓曲線，都可以對應一個模形式。」之後，人們就稱這猜想為「谷山 – 志村猜想」。

起初，大多數數學家都不相信這個猜想，但經過十多年的反覆檢算後，又沒有理據可以將它推翻。到了 70 年代，相信「谷山 – 志村猜想」的人越來越多，甚至以假定「谷山 – 志村猜想」成立的前提下進行他們的論證。

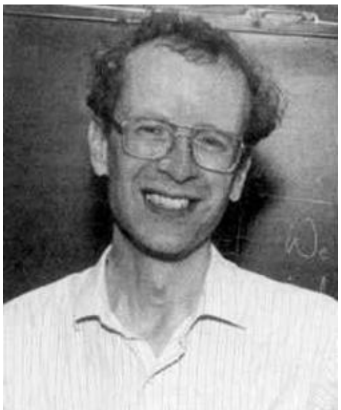
1984 年秋，德國數學家弗賴（Gerhand Frey），在一次數學會議上，提出了以下的觀點：

首先，假設「費馬最後定理」不成立。即能夠發現正整數 A、B、C 和 N，使得 $A^N + B^N = C^N$。於是利用這些數字構作橢圓曲線：$y^2 = x(x - A^N)(x + B^N)$。弗賴發現這條曲線有很多非常特別的性質，特別到不可能對應於任何一個「模形式」！換句話說，弗賴認為：如果「費馬最後定理」不成立，那麼「谷山 – 志村猜想」也是錯的！但倒轉來說，如果「谷山 – 志村猜想」成立，那麼「費馬最後定理」就必定成立！因此，弗賴其實是指出了一條證明「費馬最後定理」的新路徑：這就是去證明「谷山 – 志村猜想」！

可惜的是，弗賴在 1984 年的研究，並未能成功地證實他的觀點。不過，美國數學家里貝特（Kenneth Ribet），經過多次嘗試後，終於在 1986 年證實了有關的問題。

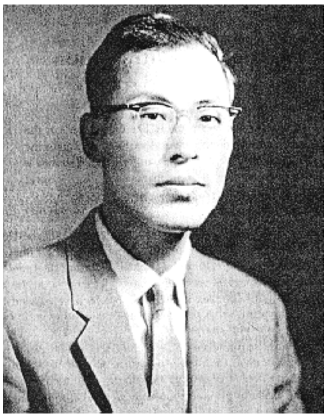
似乎，要證明「費馬最後定理」，現在祇需要證明「谷山 –志村猜想」就可以了。不過自從該猜想

Henri Poincaré



Andrew Wiles

Yutaka Taniyama



Goro Shimura

被提出以來，已經歷過差不多三十年的時間，數學家對這個證明，亦沒有多大的進展。不過，在這時候，英國數學家懷爾斯就開始他偉大而艱巨的工作。

懷爾斯（Andrew Wiles），出生於 1953 年。10 歲已立志要證明「費馬最後定理」。1975 年，開始在劍橋大學進行研究，專攻「橢圓曲線」和「岩澤理論」。在取得博士學位之後，就轉到美國的普林斯頓大學繼續工作。當他知道里貝特證實了弗賴的猜想後，就決定放棄當時手上的所有研究，專心於「谷山－志村猜想」的證明。由於他不想被人騷擾，他更決定要秘密地進行此項工作。

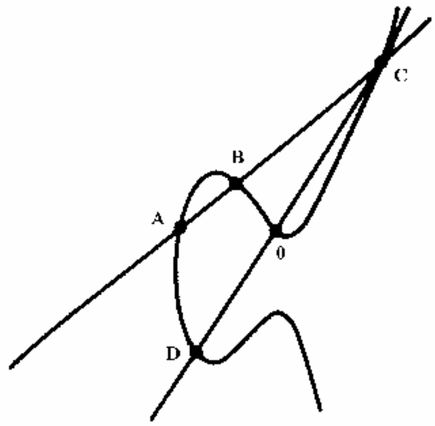
經過了七年的秘密工作後，懷爾斯認為他已證實了「谷山－志村猜想」，並且在 1993 年 6 月 23 日，在劍橋大學的牛頓研究所中，以「模形式、橢圓曲線、伽羅瓦表示論」為題，發表了他對「谷山－志村猜想」重要部份（即「費馬最後定理」）的證明。當日的演講非常成功，「費馬最後定理」經已被證實的消息，很快就傳遍世界。

不過，當懷爾斯將他長達二百頁的證明送給數論專家審閱時，卻發現當中出現漏洞。起初，懷爾斯以為很容易便可以將這個漏洞修補，但事與願違，到了 1993 年的年底，他承認他的證明出現問題，而且要一段時間才可解決。

到了 1994 年的 9 月，懷爾斯終於突破了證明中的障礙，成功地完成了一項人類史上的創舉，證明了「費馬最後定理」。1995 年 5 月，懷爾斯的證明，發表在雜誌《數學年鑑》之中。到了 1997 年 6 月 27 日，懷爾斯更獲得價值五萬美元的「沃爾夫斯凱爾獎金」，實現了他的童年夢想，正式地結束了這個長達 358 年的數學證明故事。

附錄：橢圓曲線

「橢圓曲線」是滿足方程 $y^2 = x^3 + ax^2 + bx + c$ 的點所組成的曲線，其中 a, b, c 為有理數使 $x^3 + ax^2 + bx + c$ 有不同的根。在曲線上定一個有理點 O。不難證明，當直線穿過兩個曲線上的有理點 A, B 後，該直線必定與曲線再相交於第三個有理點 C。由 C 和 O 再得一點 D 如下圖。我們可以將曲線上的有理點以 $A + B = D$ 為定義看成一個「群」(group)。由於以上性質可以用來解答很多相關的問題，故此「橢圓曲線」就成為數學研究的一個焦點。現時，「橢圓曲線」的理論，主要應用於現代編寫通訊密碼的技術方面。



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Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *December 4, 1999.*

Problem 91. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(This is the corrected version of problem 86.)

Problem 92. Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that $a_1 + a_2 + \ldots + a_n \ge n$ and $a_1^2 + a_2^2 + \ldots + a_n^2 \ge n^2$. Prove that $\max(a_1, a_2, \ldots, a_n) \ge 2$. (*Source: 1999 USA Math Olympiad*)

Problem 93. Two circles of radii R and r are tangent to line L at points A and B respectively and intersect each other at C and D. Prove that the radius of the circumcircle of triangle ABC does not depend on the length of segment AB. (*Source: 1995 Russian Math Olympiad*)

Problem 94. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a square.

Problem 95. Pieces are placed on an $n\times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (*Source: 1995 Iberoamerican Math Olympiad*)

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Solutions

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Problem 86. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1+\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(*Source: 1996 Vietnamese Math Olympiad*)

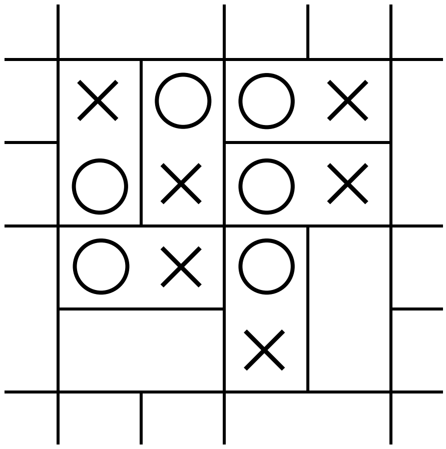
Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5), **FAN Wai Tong Louis** (St. Marks' School, Form 7), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7) and **NG Lai Ting** (True Light Girls' College, Form 7).

Clearly, x and y are nonzero. Dividing the second equation by the first equation, we then simplify to get $y = 24x/7$. So $x + y = 31x/7$. Substituting this into the first equation, we then simplifying, we get $x - (2/\sqrt{3})\sqrt{x} + 7/31 = 0$. Applying the quadratic formula to find $\sqrt{x}$, then squaring, we get $x = (41 \pm 2\sqrt{310})/93$. Then $y = 24x/7 = (328 \pm 16\sqrt{310})/217$, respectively. By direct checking, we see that both pairs (x, y) are solutions.

Other recommended solvers: **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 6), **CHAN Kwan Chuen** (HKSYC & IA Wong Tai Shan Memorial School, Form 4), **CHUI Man Kei** (STFA Leung Kau Kui College, Form 5), **HO Chung Yu** (HKU), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 4), **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **SUEN Yat Chung** (Carmel Divine Grace Foundation Secondary School, Form 6), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 5), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5), **WONG Chung Yin** (STFA Leung Kau Kui College), **WONG Tak Wai Alan** (University of Waterloo, Canada), **WU Man Kin Kenny** (STFA Leung Kau Kui College) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6).

Problem 87. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, player II chooses another square and marks it with X. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning. (*Source: 1995 Israeli Math Olympiad*).

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5).



Divide the board into 2×2 blocks. Then bisect each 2×2 block into two 1×2 tiles so that for every pair of blocks sharing a common edge, the bisecting segment in one will be horizontal and the other vertical. Since every five consecutive squares on the board contain a tile, after player I choose a square, player II could prevent player I from winning by choosing the other square in the tile.

Problem 88. Find all positive integers n such that $3^{n-1}+5^{n-1}$ divides 3^n+5^n. (*Source: 1996 St. Petersburg City Math Olympiad*).

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5), **HO Chung Yu** (HKU), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NG Lai Ting** (True Light Girls' College, Form 7), **SHUM Ho Keung** (PLK No.1 W.H. Cheung College, Form 6) and **TSE Ho Pak** (SKH Bishop Mok Sau Tseng Secondary School, Form 5).

For such an n, since

$$3(3^{n-1}+5^{n-1}) < 3^n+5^n < 5(3^{n-1}+5^{n-1}),$$

so $3^n+5^n = 4(3^{n-1}+5^{n-1})$. Cancelling, we get $5^{n-1} = 3^{n-1}$. This forces $n = 1$. Since 2 divides 8, $n = 1$ is the only solution.

Other recommended solvers: **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 6), **CHAN Kwan Chuen** (HKSYC & IA Wong Tai Shan Memorial School, Form 4), **CHAN Man Wai** (St. Stephen's Girls' College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **HON Chin Wing** (Pui Ching Middle School, Form 5), **LAW**

Siu Lun Jack (Ming Kei College, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 4), **NG Ka Chun** (Queen Elizabeth School), **NG Tin Chi** (TWGH Chang Ming Thien College, Form 7), **TAI Kwok Fung** (Carmel Divine Grace Foundation Secondary School, Form 6), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 5), **TSUI Ka Ho Willie** (Hoi Ping Chamber of Commerce Secondary School, Form 6), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5), **WONG Tak Wai Alan** (University of Waterloo, Canada), **YU Ka Lok** (Carmel Divine Grace Foundation Secondary School, Form 6) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6).

Problem 84. Let O and G be the circumcenter and centroid of triangle ABC, respectively. If R is the circumradius and r is the inradius of ABC, then show that $OG \le \sqrt{R(R-2r)}$. *(Source: 1996 Balkan Math Olympiad)*

Solution I. CHAO Khek Lun Harold (St. Paul's College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **NG Lai Ting** (True Light Girls' College, Form 7) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6)

Let line AG intersect side BC at A' and the circumcircle again at A''. Since $\cos BA'A + \cos CA'A = 0$, we can use the cosine law to get

$$A'A^2 = (2b^2 + 2c^2 - a^2)/4,$$

where a, b, c are the usual side lengths of the triangle. By the inter-secting chord theorem,

$$A'A \times A'A'' = A'B \times A'C = a^2/4.$$

Consider the chord through O and G interecting AA'' at G. By the intersecting chord theorem,

$$\begin{aligned}(R+OG)(R-OG) &= GA \times GA'' \\ &= (2A'A/3)(A'A/3 + A'A'') \\ &= (a^2+b^2+c^2)/9.\end{aligned}$$

Then

$$OG = \sqrt{R^2 - (a^2+b^2+c^2)/9}\,.$$

By the AM-GM inequality,

$$(a+b+c)(a^2+b^2+c^2) \ge$$
$$(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2}) = 9abc.$$

Now the area of the triangle is $(ab \sin C)/2 = abc/(4R)$ (by the extended sine law) on one hand and $(a + b + c)r/2$ on the other hand. So, $a + b + c = abc/(2rR)$. Using this, we simplify the inequality to get $(a^2+b^2+c^2)/9 \ge 2rR$. Then

$$\sqrt{R^2 - 2rR} \ge \sqrt{R^2 - (a^2+b^2+c^2)/9} = OG.$$

Solution II. NG Lai Ting (True Light Girls' College, Form 7)

Put the origin at the circumcenter. Let z_1, z_2, z_3 be the complex numbers corresponding to A, B, C, respectively on the complex plane. Then $OG^2 = \left|(z_1+z_2+z_3)/3\right|^2$. Using $|\omega|^2 = \omega\overline{\omega}$, we can check the right side equals $(3|z_1|^2 + 3|z_2|^2 + 3|z_3|^2 - |z_1 - z_2|^2 - |z_2 - z_3|^2 - |z_3 - z_1|^2)/9$. Since $|z_1| = |z_2| = |z_3| = R$ and $|z_1 - z_2| = c$, $|z_2 - z_3| = a$, $|z_3 - z_1| = b$, we get

$$OG^2 = (9R^2 - a^2 - b^2 - c^2)/9.$$

The rest is as in solution 1.

Problem 90. There are n parking spaces (numbered 1 to n) along a one-way road down which n drivers $d_1, d_2, \ldots, d_n$ in that order are traveling. Each driver has a favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. (Two drivers may have the same favorite space.) If there is no free space after his favorite, he drives away. How many lists $a_1, a_2, \ldots, a_n$ of favorite parking spaces are there which permit all of the drivers to park? Here a_i is the favorite parking space number of d_i. (*Source: 1996 St. Petersburg City Math Olympiad*).

Solution: Call a list of favorite parking spaces $a_1, a_2, \ldots, a_n$ which permits all drivers to park a *good* list. To each good list, associate the list $b_2, \ldots, b_n$, where b_i is the difference (mod n + 1) between the number a_i and the number of the space driver d_{i-1} took. Note from a_1 and $b_2, \ldots, b_n$, we can reconstruct $a_2, \ldots, a_n$. It follows that different good lists give rise to different lists of b_i's. Since there are n + 1 possible choices for each b_i, there are $(n+1)^{n-1}$ possible lists of $b_2, \ldots, b_n$. For each of these lists of the b_i's, imagine the n parking spaces are arranged in a circle with an extra parking space put at the end. Let d_1 park anywhere temporarily and put $d_i (i > 1)$ in the first available space after the space b_i away from the space taken by d_{i-1}. By shifting the position of d_1, we can ensure the extra parking space is not taken. This implies the corresponding list of $a_1, a_2, \ldots, a_n$ is good. So the number of good lists is $(n+1)^{n-1}$.

Comments: To begin the problem, one could first count the number of good lists in the cases n = 2 and n = 3. This will lead to the answer $(n+1)^{n-1}$. From the $n+1$ factor, it becomes natural to consider an extra parking space. The difficulty is to come up with the *one-to-one correspondence* between the good lists and the b_i lists. For this problem, only one incomplete solution with correct answer and right ideas was sent in by **CHAO Khek Lun Harold** (St. Paul's College, Form 5)

Olympiad Corner

continued from page 1)

Problem 3. (cont'd) N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N.

Problem 4. Determine all pairs (n, p) of positive integers such that p is a prime, $n \le 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Problem 5. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B, respectively. The lines MA and MB meet Γ_1 at C and D, respectively.

Prove that CD is tangent to Γ_2.

Problem 6. Determine all functions f: $\boldsymbol{R} \to \boldsymbol{R}$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all $x, y \in \boldsymbol{R}$.

Mathematical Excalibur

Volume 5, Number 1 *January 2000 – February 2000*

Olympiad Corner

8th Taiwan (ROC) Mathematical Olympiad, April 1999:

Time allowed: 4.5 Hours
Each problem is worth 7 points.

Problem 1. Determine all solutions (x, y, z) of positive integers such that

$$(x+1)^{y+1}+1=(x+2)^{z+1}.$$

Problem 2. Let $a_1, a_2, .., a_{1999}$ be a sequence of nonnegative integers such that for any integers i, j, with $i + j \leq 1999$,

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1.$$

Prove that there exists a real number x such that $a_n = [nx]$ for each n = 1, 2, …, 1999, where $[nx]$ denotes the largest integer less than or equal to nx.

Problem 3. There are 1999 people participating in an exhibition. Two of any 50 people do not know each other. Prove that there are at least 41 people, and each of them knows at most 1958 people.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is March 4, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Road to a Solution

Kin Y. Li

Due to family situation, I missed the trip to the 1999 IMO at Romania last summer. Fortunately, our Hong Kong team members were able to send me the problems by email. Of course, once I got the problems, I began to work on them. The first problem is the following.

Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry of S.

This was a nice problem. I spent sometime on it and got a solution. However, later when the team came back and I had a chance to look at the official solution, I found it a little beyond my expectation. Below I will present my solution and the official solution for comparison.

Here is the road I took to get a solution. To start the problem, I looked at the case of three points, say P_1, P_2, P_3, satisfying the condition. Clearly, the three points cannot be collinear (otherwise considering the perpendicular bisector of the segment joining two consecutive points on the line will yield a contradiction). Now by the condition, it follows that P_2 must be on the perpendicular bisector of segment $P_1\ P_3$. Hence, $P_1\ P_2 = P_2\ P_3$. By switching indices, P_3 should be on the perpendicular bisector of $P_2\ P_1$ and so $P_2\ P_3 = P_3\ P_1$. Thus, P_1, P_2, P_3 are the vertices of an equliateral triangle.

Next the case of four points required more observations. Again no three points are collinear. Also, from the condition, none of the point can be inside the triangle having the other three points as vertices. So the four points are the vertices of a convex quadrilateral. Then the sides have equal length as in the case of three points. Considering the perpendicular bisector of any side, by symmetry, the angles at the other two vertices must be the same. Hence all four angles are the same. Therefore, the four points are the vertices of a square.

After the cases of three and four points, it is quite natural to guess such sets are the vertices of regular polygons. The proof of the general case now follows from the reasonings of the two cases we looked at. First, no three points are collinear. Next, the smallest convex set enclosing the points must be a polygonal region with all sides having the same length and all angles the same. So the boundary of the region is a regular polygon. Finally, one last detail is required. In the case of four points, no point is inside the triangle formed by the other three points by inspection. However, for large number of points, inspection is not good enough. To see that none of the points is inside the polygonal region takes a little bit more work.

Again going back to the case of four points, it is natural to look at the situation when one of the point, say P, is inside the triangle formed by the other three points. Considering the perpendicular bisectors of three segments joining P to the other three points, we see that we can always get a contradiction.

Putting all these observations together, here is the solution I got:

Clearly, no three points of such a set is collinear (otherwise considering the perpendicular bisector of the two furthest points of S on that line, we will get a contradiction). Let H be the convex hull of such a set, which is the smallest convex set containing S. Since S is finite, the boundary of H is a polygon with the vertices P_1, $P_2, ..., P_n$ belonging to S. Let $P_i = P_j$ if $i \equiv j \pmod{n}$. For i = 1, 2, …, n, the condition on the set implies P_i is on the perpendicular bisector of $P_{i-1}P_{i+1}$. So $P_{i-1}P_i = P_iP_{i+1}$. Considering the perpendicular bisector of $P_{i-1}P_{i+2}$, we see that $\angle P_{i-1}P_iP_{i+1} = \angle P_iP_{i+1}P_{i+2}$. So the boundary of H is a regular polygon.

Next, there cannot be any point P of S inside the regular polygon. (To see this, assume such a P exists. Place it at the origin and the furthest point Q of S from P on the positive real axis. Since the origin P is in the interior of the convex polygon, not all the vertices can lie on or to the right of the y-axis. So there exists a vertex P_j to the left of the y-axis. Since the perpendicular bisector of PQ is an axis of symmetry, the mirror image of P_j will be a point in S further than Q from P, a contradiction.) So S is the set of vertices of some regular polygon. Conversely, such a set clearly has the required property.

Next we look at the official solution, which is shorter and goes as follows: Suppose $S = \{X_1, \ldots, X_n\}$ is such a set. Consider the *barycenter* of S, which is the point G such that

$$\vec{OG} = \frac{\vec{OX_1} + \cdots + \vec{OX_n}}{n}.$$

Note the barycenter does not depend on the origin. To see this, suppose we get a point G' using another origin O', i.e. $\vec{O'G'}$ is the average of $\vec{O'X_i}$ for $i = 1, \ldots, n$. Subtracting the two averages, we get $\vec{OG} - \vec{O'G'} = \vec{OO'}$. Adding $\vec{O'G'}$ to both sides, $\vec{OG} = \vec{OG'}$, so $G = G'$.

By the condition on S, after reflection with respect to the perpendicular bisector of every segment X_iX_j, the points of S are permuted only. So G is unchanged, which implies G is on every such perpendicular bisector. Hence, G is equidistant from all X_i's. Therefore, the X_i's are concyclic. For three consecutive points of S, say X_i, X_j, X_k, on the circle, considering the perpendicular bisector of segment X_iX_k, we have $X_iX_j = X_jX_k$. It follows that the points of S are the vertices of a regular polygon and the converse is clear.

Cavalieri's Principle

Kin Y. Li

Have you ever wondered why the volume of a sphere of radius r is given by the formula $\frac{4}{3}\pi r^3$? The r^3 factor can be easily accepted because volume is a three dimensional measurement. The π factor is probably because the sphere is round. Why then is there $\frac{4}{3}$ in the formula?

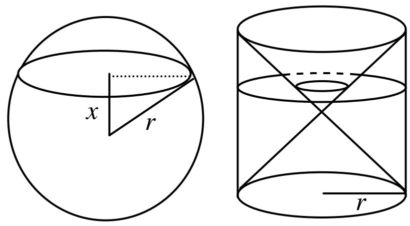
In school, most people told you it came from calculus. Then, how did people get the formula before calculus was invented? In particular, how did the early Egyptian or Greek geometers get it thousands of years ago?

Those who studied the history of mathematics will be able to tell us more of the discovery. Below we will look at one way of getting the formula, which may not be historically the first way, but it has another interesting application as we will see. First, let us introduce

Cavalieri's Principle: *Two objects having the same height and the same cross sectional area at each level must have the same volume.*

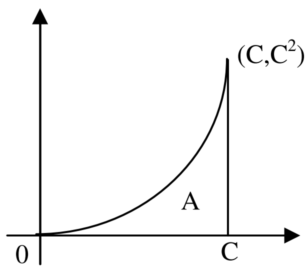
To understand this, imagine the two objects are very large, like pyramids that are built by piling bricks one level on top of another. By definition, the volume of the objects are the numbers of $1\times1\times1$ bricks used to build the objects. If at *each level* of the construction, the number of bricks used (which equals the cross sectional area numerically) is the same for the two objects, then the volume (which equals the total number of bricks used) would be the same for both objects.

To get the volume of a sphere, let us apply Cavalieri's principle to a solid sphere S of radius r and an object T made out from a solid right circular cylinder with height $2r$ and base radius r removing a pair of right circular cones with height r and base radius r having the center of the cylinder as the apex of each cone.



Both S and T have the same height $2r$. Now consider the cross sectional area of each at a level x units from the equatorial plane of S and T. The cross section for S is a circular disk of radius $\sqrt{r^2 - x^2}$ by Pythagoras' theorem, which has area $\pi(r^2 - x^2)$. The cross section for T is an annular ring of outer radius r and inner radius x, which has the same area $\pi r^2 - \pi x^2$. By Cavalieri's principle, S and T have the same volume. Since the volume of T is $\pi r^2(2r) - 2\times\frac{1}{3}\pi r^2 r = \frac{4}{3}\pi r^3$, so the volume of S is the same.

Cavalieri's principle is not only useful in getting the volume of special solids, but it can also be used to get the area of special regions in a plane! Consider the region A bounded by the graph of $y = x^2$, the x-axis and the line $x = c$ in the first quadrant.



The area of this region is less than the area of the triangle with vertices at (0, 0), $(c, 0)$, (c, c^2), which is $\frac{1}{2}c^3$. If you ask a little kid to guess the answer, you may get $\frac{1}{3}c^3$ since he knows $\frac{1}{3} < \frac{1}{2}$. For those who know calculus, the answer is easily seen to be correct. How can one explain this without calculus?

(continued on page 4)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *March 4, 2000.*

Problem 96. If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

Problem 97. A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (*Source: 1999 National Math Olympiad in Slovenia*)

Problem 98. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$. (*Source: 1998-99 Iranian Math Olympiad*)

Problem 99. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (*Source: 1996 Spanish Math Olympiad*)

Problem 100. The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (*Source: 1999 Czech and Slovak Math Olympiad*)

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Solutions

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Problem 91. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(*This is the corrected version of problem 86.*)

Solution. (CHENG Kei Tsi, LEE Kar Wai, TANG Yat Fai) (La Salle College, Form 5), **CHEUNG Yui Ho Yves** (University of Toronto), **HON Chin Wing** (Pui Ching Middle School, Form 5) **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LAU Chung Ming Vincent** (STFA Leung Kau Kui College, Form 5), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **Kevin LEE** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **MAK Hoi Kwan Calvin** (Form 4), **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4), **TANG King Fun** (Valtorta College, Form 5), **WONG Chi Man** (Valtorta College, Form 5) and **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5). (All solutions received were essentially the same.) Clearly, if (x, y) is a solution, then $x, y > 0$ and

$$1+\frac{1}{x+y}=\frac{2}{\sqrt{3x}}$$

$$1-\frac{1}{x+y}=\frac{4\sqrt{2}}{\sqrt{7y}}\,.$$

Taking the difference of the squares of both equations, we get

$$\frac{4}{x+y}=\frac{4}{3x}-\frac{32}{7y}\,.$$

Simplifying this, we get $0 = 7y^2 - 38xy - 24x^2 = (7y+4x)(y-6x)$. Since $x, y > 0$, $y = 6x$. Substituting this into the first given equation, we get $\sqrt{3x}\left(1+\frac{1}{7x}\right)=2$, which simplifies to $7\sqrt{3x}-14\sqrt{x}+\sqrt{3} = 0$. By the quadratic formula, $\sqrt{x}=(7\pm2\sqrt{7})/(7\sqrt{3})$. Then $x=(11\pm 4\sqrt{7})/21$ and $y = 6x = (22 \pm 8\sqrt{7})/7$. Direct checking shows these are solutions.

Comments: An alternative way to get the answers is to substitute $u = \sqrt{x}$, $v = \sqrt{y}$, $z = u + iv$, then the given equations become the real and imaginary parts of the complex equation $z+\frac{1}{z}=c$, where $c = \frac{2}{\sqrt{3}}+i\frac{4\sqrt{2}}{\sqrt{7}}$. Multiplying by z, we can apply the quadratic formula to get $u + iv$, then squaring u, v, we can get x, y.

Problem 92. Let $a_1, a_2, ..., a_n\ (n > 3)$ be real numbers such that $a_1 + a_2 + \cdots + a_n \ge n$ and $a_1^2 + a_2^2 + \cdots + a_n^2 \ge n^2$. Prove that $\max\ (a_1, a_2, ..., a_n) \ge 2$. (*Source: 1999 USA Math Olympiad*).

Solution. FAN Wai Tong Louis (St. Mark's School, Form 7).

Suppose $\max\ (a_1, a_2, ..., a_n) < 2$. By relabeling the indices, we may assume $2 > a_1 \ge a_2 \ge \cdots \ge a_n$. Let j be the largest index such that $a_j \ge 0$. For $i > j$, let $b_i = -a_i > 0$. Then

$$2j - n > (a_1 + \cdots + a_j) - n \ge b_{j+1} + \cdots + b_n.$$

So $(2j-n)^2 > b_{j+1}^2 + \cdots + b_n^2$. Then

$$4j + (2j-n)^2 > a_1^2 + \cdots + a_n^2 \ge n^2,$$

which implies $j > n - 1$. Therefore, $j = n$ and all $a_i \ge 0$. This yields $4n > a_1^2 + \cdots + a_n^2 \ge n^2$, which gives the contradiction that $3 \ge n$.

Other recommended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **WONG Wing Hong** (La Salle College, Form 2).

Problem 93. Two circles of radii R and r are tangent to line L at points A and B respectively and intersect each other at C and D. Prove that the radius of the circumcircle of triangle ABC does not depend on the length of segment AB. (*Source: 1995 Russian Math Olympiad*).

Solution. CHAO Khek Lun (St. Paul's College, Form 5).

Let O, O' be the centers of the circles of radius R, r, respectively. Let $\alpha = \angle CAB = \angle AOC/2$ and $\beta = \angle CBA = \angle BO'C/2$. Then $AC = 2R\sin\alpha$ and $BC = 2r\sin\beta$. The distance from C to AB is $AC\sin\alpha = BC\sin\beta$, which implies $\sin\alpha / \sin\beta = \sqrt{r/R}$. The circumradius of triangle

ABC is

$$\frac{AC}{2\sin\beta} = \frac{R\sin\alpha}{\sin\beta} = \sqrt{Rr}\ ,$$

which does not depend on the length of *AB*.

Other recommended solvers: **CHAN Chi Fung** (Carmel Divine Grace Foundation Secondary School, Form 6), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Bartholomew** (Queen Elizabeth School), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4).

Problem 94. Determine all pairs (*m, n*) of positive integers for which $2^m + 3^n$ is a square.

Solution. NGAN Chung Wai Hubert (St. Paul's Co-educational College, Form 7) and **YEUNG Kai Sing** (La Salle College, Form 3).

Let $2^m + 3^n = a^2$. Then a is odd and $a^2 = 2^m + 3^n \equiv (-1)^m$ (mod 3). Since squares are 0 or 1 (mod 3), m is even. Next $(-1)^n \equiv 2^m + 3^n = a^2 \equiv 1$ (mod 4) implies *n* is even, say $n = 2k$, $k \geq 1$. Then $2^m = (a+3^k)(a-3^k)$. So $a + 3^k = 2^r$, $a - 3^k = 2^s$ for integers $r > s \geq 0$ with $r + s = m$. Then $2 \cdot 3^k = 2^r - 2^s$ implies $s = 1$, so $2^{r-1} - 1 = 3^k$. Now $r + 1 = m$ implies *r* is odd. So

$$\left(2^{(r-1)/2}+1\right)\left(2^{(r-1)/2}-1\right) = 3^k .$$

Since the difference of the factors is 2, not both are divisible by 3. Then the factor $2^{(r-1)/2} - 1 = 1$. Therefore, $r = 3$, $k = 1$, $(m, n) = (4, 2)$, which is easily checked to be a solution.

Other recommended solvers: **CHAO Khek Lun** (St. Paul's College, Form 5), **CHENG Kei Tsi** (La Salle College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Batholomew** (Queen Elizabeth School), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NG Ting Chi** (TWGH Chang Ming Thien College, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4).

Problem 95. Pieces are placed on an $n \times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (*Source: 1995 Iberoamerican Olympiad*).

Solution. LEUNG Wai Ying (Queen Elizabeth School, Form 5).

Assign coordinates to the squares so (*x*, *y*) represents the square on the *x*-th column from the west and *y*-th row from the south. Suppose *k* pieces are enough to attack all squares. Then at least *n* – *k* columns, say columns $x_1, ..., x_{n-k}$, and *n* – *k* rows, say $y_1, ..., y_{n-k}$, do not contain any of the *k* pieces. Consider the 2(*n* - *k*) – 1 squares (x_1, y_1), (x_1, y_2), …, (x_1, y_{n-k}), (x_2, y_1), (x_3, y_1), …, (x_{n-k}, y_1). As they are on different diagonals and must be attacked diagonally by the *k* pieces, we have $k \geq 2(n-k) - 1$. Solving for *k*, we get $k \geq (2n-1)/3$. Now let *k* be the least integer such that $k \geq (2n-1)/3$. We will show *k* is the answer. The case *n* = 1 is clear. Next if *n* = 3*a* + 2 for a nonnegative integer *a*, then place *k* = 2*a* + 1 pieces at (1, *n*), (2, *n* – 2), (3, *n* – 4), …, (*a* + 1, *n* – 2*a*), (*a* + 2, *n* – 1), (*a* + 3, *n* – 3), (*a* + 4, *n* – 5), …, (2*a* + 1, *n* – 2*a* + 1). So squares with $x \leq 2a + 1$ or $y \geq n - 2a$ are under attacked horizontally or vertically. The other squares, with $2a + 2 \leq x \leq n$ and $1 \leq y \leq n - 2a - 1$, have $2a + 3 \leq x + y \leq 2n - 2a - 1$. Now the sums *x* + *y* of the *k* pieces range from *n* – *a* + 1 = 2*a* + 3 to *n* + *a* + 1 = 2*n* – 2*a* – 1. So the *k* pieces also attack the other squares diagonally.

Next, if *n* = 3*a* + 3, then *k* = 2*a* + 2 and we can use the 2*a* + 1 pieces above and add a piece at the southeast corner to attack all squares. Finally, if *n* = 3*a* + 4, then *k* = 2*a* + 3 and again use the 2*a* + 2 pieces in the last case and add another piece at the southeast corner.

Other recommended solvers: **(LEE Kar Wai Alvin, CHENG Kei Tsi Daniel, LI Chi Pang Bill, TANG Yat Fai Roger)** (La Salle College, Form 5), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 4. Let P^* denote all the odd primes less than 10000. Determine all possible primes $p \in P^*$ such that for each subset *S* of P^*, say *S* = { p_1, p_2, …, p_k }, with $k \geq 2$, whenever $p \notin S$, there must be some *q* in P^*, but not in *S*, such that *q* + 1 is a divisor of $(p_1+1)\ (p_2+1)\ \ldots\ (p_k+1)$.

Problem 5. The altitudes through the vertices *A, B, C* of an acute-angled triangle *ABC* meet the opposite sides at *D, E, F*, respectively, and *AB* > *AC*. The line *EF* meets *BC* at *P*, and the line through *D* parallel to *EF* meets the lines *AC* and *AB* at *Q* and *R*, respectively. *N* is a point on the side *BC* such that $\angle NQP + \angle NRP < 180^\circ$. Prove that *BN* > *CN*.

Problem 6. There are 8 different symbols designed on *n* different T-shirts, where $n \geq 2$. It is known that each shirt contains at least one symbol, and for any two shirts, the symbols on them are not all the same. Suppose that for any *k* symbols, $1 \leq k \leq 7$, the number of shirts containing at least one of the *k* symbols is even. Find the value on *n*.

Cavalieri

(continued from page 2)

To get the answer, we will apply Cavalieri's principle. Consider a solid right cylinder with height 1 and base region *A*. Numerically, the volume of this solid equals the area of the region *A*. Now rotate the solid so that the $1 \times c^2$ rectangular face becomes the base. As we expect the answer to be $\frac{1}{3}c^3$, we compare this rotated solid with a solid right pyramid with height *c* and square base of side *c*.

Both solids have height *c*. At a level *x* units below the top, the cross section of the rotated solid is a $1 \times x^2$ rectangle. The cross section of the right pyramid is a square of side *x*. So both solids have the same cross sectional areas at all levels. Therefore, the area of *A* equals numerically to the volume of the pyramid, which is $\frac{1}{3}c^3$.

Mathematical Excalibur

Volume 5, Number 2 *March 2000 – April 2000*

Olympiad Corner

28th United States of America Mathematical Olympiad, April 1999:

Time allowed: 6 Hours

Problem 1. Some checkers placed on an $n\times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2-2)/3$ checkers have been placed on the board.

Problem 2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$|AB-CD|+|AD-BC|\ge 2|AC-BD|$.

Problem 3. Let $p>2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$\{ra/p\}+\{rb/p\}+\{rc/p\}+\{rd/p\}=2$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is May 20, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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漫談質數

梁子傑
香港道教聯合會青松中學

我們從小學開始就已經認識甚麼是質數。一個大於 1 的整數，如果祇能被 1 或自己整除，則我們稱該數為「質數」。另外，我們叫 1 做「單位」，而其他的數字做「合成數」。例如：2、3、5、7 … 等等，就是質數，4、6、8、9 … 等等就是合成數。但是除了這個基本的定義之外，一般教科書中，就很少提到質數的其他性質了。而本文就為大家介紹一些與質數有關的人和事。

有人相信，人類在遠古時期，就經已發現質數。不過最先用文字紀錄質數性質的人，就應該是古希臘時代的偉大數學家歐幾里得 (Euclid) 了。

歐幾里得，約生於公元前 330 年，約死於公元前 275 年。他是古代亞歷山大里亞學派的奠基者。他的著作《幾何原本》，集合了平面幾何、比例論、數論、無理量論和立體幾何之大成，一致公認為數學史上的一本鉅著。

《幾何原本》全書共分十三卷，一共包含 465 個命題，當中的第七、八、九卷，主要討論整數的性質，後人又稱這學問為「數論」。第九卷的命題 20 和質數有關，它是這樣寫的：「預先任意給定幾個質數，則有比它們更多的質數。」

歐幾里得原文的證明並不易懂，但改用現代的數學符號，他的證明大致如下：

首先，假如 a、b、c … k 是一些質數。那麼 $abc\cdots k+1$ 或者是質數，或者不是。如果它是質數，那麼就加添了一個新的質數。如果它不是質數，那麼這個數就有一個質因子 p。如果 p 是 a、b、c … k 其中的一個數，由於它整除 $abc\cdots k$，於是它就能整除 1。但這是不可能的，因為 1 不能被其他數整除。因此 p 就是一個新的質數。總結以上兩個情況，我們總獲得一個新的質數。命題得證。

命題 20 提供了一個製造質數的方法，而且可以無窮無盡地製造下去。由此可知，命題 20 實際上是證明了質數有無窮多個。

到了十七世紀初，法國數學家默森 (Mersenne)（1588 – 1648）提出了一條計算質數的「公式」，相當有趣。

因為 $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$，所以如果 x^n-1 是質數，$x-1$ 必定要等於 1。由此得 $x=2$。另外，假如 $n=ab$ 並且 $a\le b$，又令 $x=2^a$，則 $2^n-1=(2^a)^b=x^b-1=(x-1)(x^{b-1}+x^{b-2}+\cdots+x+1)$。所以，如果 2^n-1 是質數，那麼 $x-1$ 必定又要等於 1。由此得 $2^a=2$，即 $a=1$，n 必定是質數。

綜合上述結果，默森提出了一條計算質數的公式，它就是 2^p-1，其中 p 為質數。例如：$2^2-1=3$，$2^3-1=7$，$2^5-1=31$ 等等。但默森的公式祇是計算質數時的「必要」條件，並不是一個「充分」條件；即是說，在某些情況下，由 2^p-1 計算出來的結果，未必一定是質數。例如：$2^{11}-1=2047=23\times 89$，這就不是質數了。因此由默森公式計算出來的數，其實也需要進一步的驗算，才可以知道它是否真正是一個質數。

由於現代的電腦主要用二進數來進行運算，而這又正好和默森公式配合，所以在今天，當人類找尋更大的質數時，往往仍會用上默森的方法。跟據互聯網上的資料，（網址為:

www.utm.edu/research/primes/largest.html），現時發現的最大質數為 $2^{6972593}-1$，它是由三位數學家在 1999 年 6 月 1 日發現的。

默森的好朋友費馬 (Fermat) (1601–1665) 亦提出過一條類似的質數公式。

設 $n=ab$ 並且 b 是一個奇數。令 $x=2^a$，則 $2^n+1=(2^a)^b+1=x^b+1=(x+1)(x^{b-1}-x^{b-2}+\cdots-x+1)$。注意：祇有當 b 為奇數時，上式才成立。很明顯，2^n+1 並非一個質數。故此，如果 2^n+1 是質數，那麼 n 必定不能包含奇因子，即 n 必定是 2 的乘冪。換句話說，費馬的質數公式為 $2^{2^n}+1$。

不難驗證，$2^{2^0}+1=3$，$2^{2^1}+1=5$，$2^{2^2}+1=17$，$2^{2^3}+1=257$，$2^{2^4}+1=65537$，它們全都是質數。問題是：跟著以後的數字，又是否質數呢？由於以後的數值增長得非常快，就連費馬本人，也解答不到這個問題了。

最先回答上述問題的人，是十八世紀瑞士大數學家歐拉 (Euler)（1707 – 1783）。歐拉出生於一個宗教家庭，17 歲已獲得碩士學位，一生都從事數學研究，縱使晚年雙目失明，亦不斷工作，可算是世上最多產的數學家。歐拉指出，$2^{2^5}+1$ 並非質數。他的證明如下：

記 $a=2^7$ 和 $b=5$。那麼 $a-b^3=3$ 而 $1+ab-b^4=1+(a-b^3)b=1+3b=2^4$。所以

$$\begin{aligned}&2^{32}+1=(2a)^4+1\\=\ &2^4a^4+1=(1+ab-b^4)a^4+1\\=\ &(1+ab)a^4+(1-a^4b^4)\\=\ &(1+ab)(a^4+(1-ab)(1+a^2b^2)),\end{aligned}$$

即 $1+ab=641$ 可整除 $2^{32}+1$，$2^{32}+1$ 並不是質數！

事實上，到了今天，祇要用一部電子計算機就可以知道：$2^{32}+1=4294967297=641\times6700417$。同時，跟據電腦的計算，當 n 大於 4 之後，由費馬公式計算出來的數字，再沒有發現另一個是質數了！不過，我們同時亦沒有一個數學方法來證明，費馬質數就祇有上述的五個數字。

自從歐拉證實 $2^{2^5}+1$ 並非質數之後，人們對費馬公式的興趣也隨之大減。不過到了 1796 年，當年青的數學家高斯發表了他的研究結果後，費馬質數又一再令人關注了。

高斯 (Gauss)（1777 – 1855），德國人。一個數學天才。3 歲已能指出父親帳簿中的錯誤。22 歲以前，已經成功地證明了多個重要而困難的數學定理。由於他的天份，後世人都稱他為「數學王子」。

高斯在 19 歲的時候發現，一個正質數多邊形可以用尺規作圖的充分和必要條件是，該多邊形的邊數必定是一個費馬質數！換句話說，祇有正三邊形（即正三角形）、正五邊形、正十七邊形、正 257 邊形和正 63357 邊形可以用尺規構作出來，其他的正質數多邊形就不可以了。（除非我們再發現另一個費馬質數。）高斯同時更提出了一個繪畫正十七邊形方案，打破了自古希臘時代流傳下來，最多祇可構作正五邊形的紀錄。

提到和質數有關的故事，就不可不提「哥德巴赫猜想」了。

哥德巴赫 (Goldbach) 是歐拉的朋友。1742 年，哥德巴赫向歐拉表示他發現每一個不小於 6 的偶數，都可以表示為兩個質數之和，例如：$8=3+5$、$20=7+13$、$100=17+83$ … 等。哥德巴赫問歐拉這是否一個一般性的現象。

歐拉表示他相信這是一個事實，但他無法作出一個證明。自此，人們就稱這個現象為「哥德巴赫猜想」。

自從「哥德巴赫猜想」被提出後，經過了整個十九世紀，對這方面研究的進展都很緩慢。直到 1920 年，挪威數學家布朗 (Brun) 證實一個偶數可以寫成兩個數字之和，其中每一個數字都最多祇有 9 個質因數。這可以算是一個重大的突破。

1948 年，匈牙利的瑞尼 (Renyi) 證明了一個偶數必定可以寫成一個質數加上一個有上限個因子所組成的合成數。1962 年，中國的潘承洞證明了一個偶數必定可以寫成一個質數加上一個由 5 個因子所組成的合成數。後來，有人簡稱這結果為 $(1+5)$。

1963 年，中國的王元和潘承洞分別證明了 $(1+4)$。1965 年，蘇聯的維諾格拉道夫 (Vinogradov) 證實了 $(1+3)$。1966 年，中國的陳景潤就證明了 $(1+2)$。這亦是世上現時對「哥德巴赫猜想」證明的最佳結果。

陳景潤（1933 – 1996），福建省福州人。出生於貧窮的家庭，由於戰爭的關係，自幼就在非常惡劣的環境下學習。1957 年獲得華羅庚的提拔，進入北京科學院當研究員。在「文化大革命」的十年中，陳景潤受到了批判和不公正的待遇，使他的工作和健康都大受傷害。1980 年，他當選為中國科學院學部委員。1984 年證實患上了「帕金遜症」，直至 1996 年 3 月 19 日，終於不治去世。

其實除了對「哥德巴赫猜想」的證明有貢獻外，陳景潤的另一個成就，就是對「孿生質數猜想」證明的貢獻。在質數世界中，我們不難發現有時有兩個質數，它們的距離非常接近，它們的差祇有 2，例如：3 和 5、5 和 7、11 和 13 … 10016957 和 10016959 … 等等。所謂「孿生質數猜想」，就是認為這些質數會有無窮多對。而在 1973 年，陳景潤就證得：「存在無窮多個質數 p，使得 $p+2$ 為不超過兩個質數之積。」

其實在質數的世界之中，還有很多更精彩更有趣的現象，但由於篇幅和個人能力的關係，未能一一盡錄。以下有一些書籍，內容豐富，值得對本文內容有興趣的人士參考。

參考書目

《數學和數學家的故事》
作者：李學數　出版社：廣角鏡

《天才之旅》
譯者：林傑斌　出版社：牛頓出版公司

《哥德巴赫猜想》
作者：陳景潤　出版社：九章出版社

《素數》
作者：王元　　出版社：九章出版社

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *May 20, 2000.*

Problem 101. A triple of numbers (a_1, a_2, a_3) = (3, 4, 12) is given. We now perform the following operation: choose two numbers a_i and a_j, $(i \neq j)$, and exchange them by $0.8\,a_i$ - $0.8\,a_j$ and $0.8\,a_i + 0.6\,a_j$. Is it possible to obtain after several steps the (unordered) triple (2, 8, 10)? *(Source: 1999 National Math Competition in Croatia)*

Problem 102. Let a be a positive real number and $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k \text{ , for all } n \geq 1.$$

Show that there exists a positive integer n such that x_n > 1999! (*Source: 1999 Romanian Third Selection Examination*)

Problem 103. Two circles intersect in points A and B. A line l that contains the point A intersects the circles again in the points C, D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN = 90^\circ$. (*Source: 1999 Romanian Fourth Selection Examination*)

Problem 104. Find all positive integers n such that 2^n-1 is a multiple of 3 and (2^n-1)/3 is a divisor of $4m^2 + 1$ for some integer m. (*Source: 1999 Korean Mathematical Olympiad*)

Problem 105. A rectangular parallelopiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelopiped is a cube. (*Source: 1999 National Math Olympiad in Slovenia*)

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Solutions

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Problem 96. If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

Solution. **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Consider the points (x, y) on the co-ordinate plane, where x = 1, 2, …, 7 and y = 1, 2, 3. In row 1, at least 4 of the 7 points are of the same color, say color A. In each of row 2 or 3, if 2 or more of the points directly above the A-colored points in row 1 are also A-colored, then there will be a rectangle with A-colored vertices. Otherwise, at least 3 of the points in each of row 2 and 3 are B-colored and they are directly above four A-colored points in row 1. Then there will be a rectangle with B-colored vertices.

Other recommended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHEUNG Chi Leung** (Carmel Divine Grace Foundation Secondary School, Form 6), **FAN Wai Tong** (St. Mark's School, Form 7), **LAM Shek Ming Sherman** (La Salle College), **LEE Kar Wai Alvin**, **LI Chi Pang Bill**, **TANG Yat Fai Roger** (La Salle College, Form 5), **LEE Kevin** (La Salle College, Form 4), **LEUNG Wai Ying**, **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5), **NG Wing Ip** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7), **WONG Wing Hong** (La Salle College, Form 2) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

Problem 97. A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat up to 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (*Source: 1999 National Math Olympaid in Slovenia*).

Solution. **TSE Ho Pak** (SKH Bishop Mok Sau Tseng Secondary School, Form 6).

Let the number of boys and girls be x and y, respectively. Then $7x + 3y \leq 59$ and $6x + 2y \geq 49$. Subtracting these, we get $x + y \leq 10$. Then $6x + 2(10 - x) \geq 49$ implies $x \geq 8$. Also, $7x + 3y \leq 59$ implies $x \leq 8$. So $x = 8$. To satisfy the inequalities then y must be 1.

Other recommended solvers: **AU Cheuk Yin Eddy** (Ming Kei College, Form 7), **CHAN Chin Fei** (STFA Leung Kau Kui College,), **CHAN Hiu Fai** (STFA Leung Kau Kui College, Form 6), **CHAN Man Wai** (St. Stephen's Girls' College, Form 5), **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHUNG Ngai Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **CHUNG Wun Tung Jasper** (Ming Kei College, Form 6), **FAN Wai Tong** (St. Mark's School, Form 7), **HONG Chin Wing** (Pui Ching Middle School, Form 5), **LAM Shek Ming Sherman** (La Salle College), **LEE Kar Wai Alvin, LI Chin Pang Bill, TANG Yat Fai Roger** (La Salle College, Form 5), **LEE Kevin** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 5), **LYN Kwong To** (Wah Yan College, Form 6), **MOK Ming Fai** (Carmel Divine Grace Foundation Secondary School, Form 6), **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **POON Wing Sze Jessica** (STFA Leung Kau Kui College), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho** (STFA Leung Kau Kui College), **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7), **WONG So Ting** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Wing Hong** (La Salle College, Form 2) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

Problem 98. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$. (*Source: 1998-99 Iranian Math Olympiad*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

Since $\angle CAQ = \angle CBQ = \angle DEP$ and $\angle AQC = 180^\circ - \angle ABD = \angle EPD$, so $\Delta AQC \sim \Delta EPD$. By Ptolemy's theorem, $BP\times ED = BD\times EP + BE\times DP$. So

$$BP = BD\times\frac{EP}{ED} + BE\times\frac{DP}{ED} =$$

$$AC\times\frac{AQ}{AC} + AC\times\frac{CQ}{AC} = AQ + CQ.$$

Other recommended solvers: **AU Cheuk Yin Eddy** (Ming Kei College, Form 7), **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **LAM Shek Ming Sherman** (La Salle College), **LEE Kevin** (La Salle College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

Problem 99. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (*Source: 1996 Spanish Math Olympiad*)

Solution. **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

If some agent watches less than 7 other agents, then he will miss at least 9 agents. The agent himself and these 9 agents will form a group violating the cycle condition. So every agent watches at least 7 other agents. Similarly, every agent is watched by at least 7 agents. (Then each agent can watch at most 15 –7 = 8 agents and is watched by at most 8 agents)

Define two agents to be "connected" if one watches the other. From above, we know that each agent is connected with at least 14 other agents. So each is "disconnected" to at most 1 agent. Since disconnectedness comes in pairs, among 11 agents, at least one, say *X*, will not disconnected to any other agents. Removing *X* among the 11 agents, the other 10 will form a cycle, say

$$X_1, X_2, ..., X_{10}, X_{11} = X_1.$$

Going around the cycle, there must be 2 agents X_i, X_{i+1} in the cycle such that X_i also watches *X* and X_{i+1} is watched by *X*. Then *X* can be inserted to the cycle between these 2 agents.

Other commended solvers: **CHAN Hiu Fai Philip, NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6) and **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Problem 100. The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (*Source: 1999 Czech and Slovak Math Olympiad*)

Solution. **FAN Wai Tong** (St. Mark's School, Form 7) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7)

Let $p_1 < p_2 < \cdots < p_n$ be distinct primes such that $p_1 + p_2 + \cdots + p_n = 27n$. Now $p_1 \neq 2$ (for otherwise $p_1 + p_2 + \cdots + p_n$ - $27n$ will be odd no matter *n* is even or odd). Since the primes less than 27 are 2, 3, 5, 7, 11, 13, 17, 19, 23, so $p_n = 27n$ - $(p_1 + \cdots + p_{n-1}) = 27 + (27 - p_1) + \cdots + (27 - p_{n-1}) \leq$ 27+(27-2)+(27-3) $+\cdots+$ (27 –23) = 145. Since p_n is prime, $p_n \leq 139$. Since the arithmetic mean of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 139 is 27. The answer to the problem is 139.

Other recommended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHEUNG Ka Chung**, **LAM Shek Ming Sherman**, **LEE Kar Wai Alvin**, **TANG Yat Fai Roger**, **WONG Wing Hong**, **YEUNG Kai Sing Kelvin** (La Salle College), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), and **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. ***(cont'd)***
for any integer *r* not divisible by *p*. Prove that at least two of the numbers *a*+*b*, *a*+*c*, *a*+*d*, *b*+*c*, *b*+*d*, *c*+*d* are divisible by *p*. (Note: $\{x\} = x - [x]$ denotes the fractional part of *x*.)

Problem 4. Let $a_1, a_2, ..., a_n (n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, ..., a_n) \geq 2$.

Problem 5. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Problem 6. Let *ABCD* be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle *BCD* meets *CD* at *E*. Let *F* be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle *ACF* meet line *CD* at *C* and *G*. Prove that the triangle *AFG* is isosceles.

Interesting Theorems About Primes

Below we will list some interesting theorem concerning prime numbers.

Theorem (due to Fermat in about 1640) *A prime number is the sum of two perfect squares if and only if it is 2 or of the form 4n + 1. A positive integer is the sum of two perfect squares if and only if in the prime factorization of the integer, primes of the form 4n + 3 have even exponents.*

Dirichlet's Theorem on Primes in Progressions (1837) *For every pair of relatively prime integers a and d, there are infinitely many prime numbers in the arithmetic progression* a, $a+d$, $a+2d$, $a+3d$, …, *(In particular, there are infinitely many prime numbers of the form 4n + 1, of the form 6n + 5, etc.)*

Theorem *There is a constant C such that if* p_1, p_2, …, p_n *are all the prime numbers less than x, then*

$$\ln(\ln x) - 1 < \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}$$
$$< \ln(\ln x) + C\ln(\ln(\ln x)).$$

In particular, if p_1, p_2, p_3, … *are all the prime numbers, then*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots = \infty.$$

(The second statement was obtained by Euler in about 1735. The first statement was proved by Chebysev in 1851.)

Chebysev's Theorem (1852) *If* $x > 1$, *then there exists at least one prime number between x and 2x.* (This was known as *Bertrand's postulate* because J. Bertrand verified this for *x* less than six million in 1845.)

Prime Number Theorem (due to J. Hadamard and Ch. de la Vallée Poussin independently in 1896) *Let* $\pi(x)$ *be the number of prime numbers not exceeding x, then*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

If p_n *is the n-th prime number, then*

$$\lim_{x\to\infty} \frac{p_n}{n\ln n} = 1.$$

(This was conjectured by Gauss in 1793 when he was about 15 years old.)

Brun's Theorem on Twin Primes (1919) *The series of reciprocals of the twin primes either is a finite sum or forms a convergent infinite series, i.e.*

$$\left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \cdots < \infty.$$

As a general reference to these results, we recommend the book *Fundamentals of Number Theory* by William J. Le Veque, published by Dover.

Mathematical Excalibur

Volume 5, Number 3 May 2000 – Septl 2000

Olympiad Corner

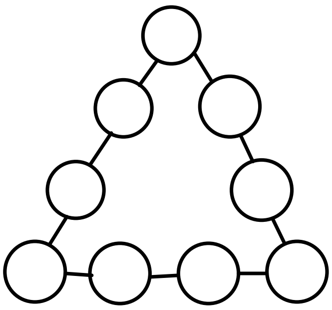
XII Asia Pacific Math Olympiad, March 2000:

Time allowed: 4 Hours

Problem 1. Compute the sum

$$S=\sum_{i=0}^{101}\frac{x_i^3}{1-3x_i+3x_i^2}\ \text{ for }\ x_i=\frac{i}{101}.$$

Problem 2. Given the following triangular arrangement of circles:



Each of the numbers 1,2,...,9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

(continued on page 4)

Acknowledgment: Thanks to Janet Wong, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is October 10, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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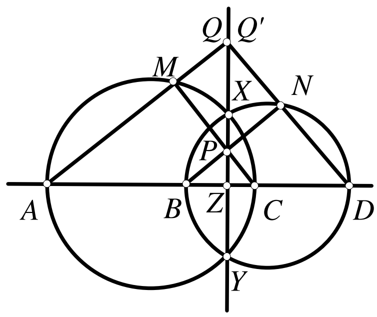
Coordinate Geometry

Kin Y. Li

When we do a geometry problem, we should first look at the given facts and the conclusion. If all these involve intersection points, midpoints, feet of perpendiculars, parallel lines, then there is a good chance we can solve the problem by coordinate geometry. However, if they involve two or more circles, angle bisectors and areas of triangles, then sometimes it is still possible to solve the problem by choosing a good place to put the origin and the x-axis. Below we will give some examples. *It is important to stay away from messy computations !*

Example 1. (1995 IMO) Let *A, B, C* and *D* be four distinct points on a line, in that order. The circles with diameters *AC* and *BD* intersect at the points *X* and *Y*. The line *XY* meets *BC* at the point *Z*. Let *P* be a point on the line *XY* different from *Z*. The line *CP* intersects the circle with diameter *AC* at the points *C* and *M*, and the line *BP* intersects the circle with diameter *BD* at the points *B* and *N*. Prove that the lines *AM, DN*, and *XY* are concurrent.

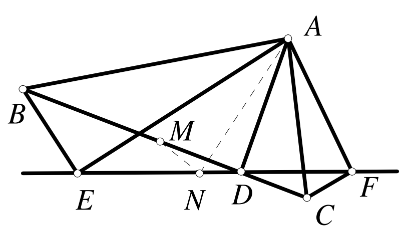


(*Remarks*. Quite obvious we should set the origin at *Z*. Although the figure is not symmetric with respect to line *XY*, there are pairs such as *M, N* and *A, D* and *B, C* that are symmetric in *roles*! So we work on the left half of the figure, the computations will be similar for the right half.)

Solution. (*Due to Mok Tze Tao, 1995 Hong Kong Team Member*) Set the origin at *Z* and the *x*-axis on line *AD*. Let the coordinates of the circumcenters of triangles *AMC* and *BND* be $(x_1, 0)$ and $(x_2, 0)$, and the circumradii be r_1 and r_2, respectively. Then the coordinates of A and *C* are $(x_1-r_1,\ 0)$ and $(x_1+r_1,\ 0)$, respectively. Let the coordinates of *P* be $(0, y_0)$. Since $AM \perp CP$ and the slope of *CP* is $-y_0/(x_1+r_1)$, the equation of *AM* works out to be $(x_1+r_1)x-y_0y=x_1^2-r_1^2$. Let *Q* be the intersection of *AM* with *XY*, then *Q* has coordinates $(0,(r_1^2-x_1^2)/y_0)$.

Similarly, let *Q'* be the intersection of *DN* with *XY*, then *Q'* has coordinates $(0,(r_2^2-x_2^2)/y_0)$. Since $r_1^2-x_1^2=ZX^2=r_2^2-x_2^2$, so $Q=Q'$.

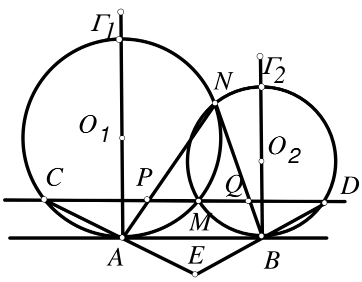
Example 2. (1998 APMO) Let *ABC* be a triangle and *D* the foot of the altitude from *A*. Let *E* and *F* be on a line passing through *D* such that *AE* is perpendicular to *BE, AF* is perpendicular to *CF*, and *E* and *F* are different from *D*. Let *M* and *N* be the midpoints of the line segments *BC* and *EF*, respectively. Prove that *AN* is perpendicular to *NM*.



(*Remarks*. We can set the origin at *D* and the *x*-axis on line *BC*. Then computing the coordinates of *E* and *F* will be a bit messy. A better choice is to set the line through *D,E.F* horizontal.)

Solution. (*Due to Cheung Pok Man, 1998 Hong Kong Team Member*) Set the origin at A and the x-axis parallel to line EF. Let the coordinates of D, E, F be $(d, b), (e, b), (f, b)$, respectively. The case $b=0$ leads to $D=E$, which is not allowed. So we may assume $b\neq 0$. Since $BE \perp AE$ and the slope of AE is b/e, so the equation of line BE works out to be $ex+by=e^2+b^2$. Similarly, the equations of lines CF and BC are $fx+by=f^2+b^2$ and $dx+by=d^2+b^2$, respectively. Solving the equations for BE and BC, we find B has coordinates $(d+e, b-(de/b))$. Similarly, C has coordinates $(d+f, b-(df/b))$. Then M has coordinates $(d+(e+f)/2, b-(de+df)/(2b))$ and N has coordinates $((e+f)/2, b)$. So the slope of AN is $2b/(e+f)$ and the slope of MN is $-(e+f)/(2b)$. Therefore, $AN \perp MN$.

Example 3. (2000 IMO) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP=EQ$.



(*Remarks.* Here if we set the x-axis on the line through the centers of the circles, then the equation of the line AB will be complicated. So it is better to have line AB on the x-axis.)

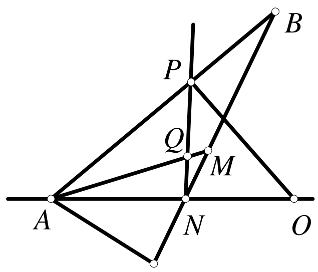
Solution. Set the origin at A and the x-axis on line AB. Let B, M have coordinates $(b,0)$, (s,t), respectively. Let the centers O_1, O_2 of Γ_1, Γ_2 be at $(0, r_1)$, (b, r_2), respectively. Then C, D have coordinates $(-s, t)$, $(2b-s,t)$, respectively. Since AB, CD are parallel, $CD=2b=2AB$ implies A, B are midpoints of CE, DE, respectively. So E is at $(s, -t)$. We see $EM \perp CD$.

To get $EP=EQ$, it is now left to show M is the midpoint of segment PQ. Since $O_1O_2 \perp MN$ and the slope of O_1O_2 is $(r_2-r_1)/b$, the equation of line MN is $bx+(r_2-r_1)y=bs+(r_2-r_1)t$. (This line should pass through the midpoint of segment AB.) Since $O_2M=r_2$ and $O_1M=r_1$, we get

$$(b-s)^2+(r_2-t)^2=r_2^2 \quad \text{and}$$
$$s^2+(r_1-t)^2=r_1^2.$$

Subtracting these equations, we get $b^2/2=bs+(r_2-r_1)t$, which implies $(b/2, 0)$ is on line MN. Since PQ, AB are parallel and line MN intersects AB at its midpoint, then M must be the midpoint of segment PQ. Together with $EM \perp PQ$, we get $EP=EQ$.

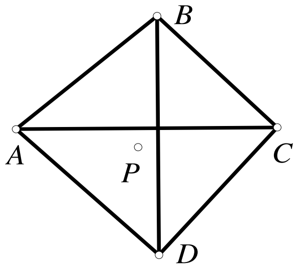
Example 4. (2000 APMO) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC.



(*Remarks.* Here the equation of the angle bisector is a bit tricky to obtain unless it is the x-axis. In that case, the two sides of the angle is symmetric with respect to the x-axis.)

Solution. *(Due to Wong Chun Wai, 2000 Hong Kong Team Member)* Set the origin at N and the x-axis on line NO. Let the equation of line AB be $y=ax+b$, then the equation of lines AC and PO are $y=-ax-b$ and $y=(-1/a)x+b$, respectively. Let the equation of BC be $y=cx$. Then B has coordinates $(b/(c-a), bc/(c-a))$, C has coordinates $(-b/(c+a), -bc(c+a))$, M has coordinates $(ab/(c^2-a^2), abc/(c^2-a^2))$, A has coordinates $(-b/a, 0)$, O has coordinates $(ab, 0)$ and Q has coordinates $(0, ab/c)$. Then BC has slope c and QO has slope $-1/c$. Therefore, $QO \perp BC$.

Example 5. (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.



(*Remarks*. The area of a triangle can be computed by taking the half length of the cross product. A natural candidate for the origin is P and having the diagonals parallel to the axes will be helpful.)

Solution. (*Due to Leung Wing Chung, 1998 Hong Kong Team Member*) Set the origin at P and the x-axis parallel to line AC. Then the equations of lines AC and BD are $y=p$ and $x=q$, respectively. Let $AP=BP=r$ and $CP=DP=s$. Then the coordinates of A, B, C, D are $(-\sqrt{r^2-p^2}, p)$, $(q, \sqrt{r^2-q^2})$, $(\sqrt{s^2-p^2}, p)$, $(q, -\sqrt{s^2-q^2})$, respectively. Using the determinant formula for finding the area of a triangle, we see that the areas of triangles ABP and CDP are equal if and only if

$$-\sqrt{r^2-p^2}\sqrt{r^2-q^2}-pq=-\sqrt{s^2-p^2}\sqrt{s^2-q^2}-pq.$$

Since $f(x) = -\sqrt{x^2-p^2}\ \sqrt{x^2-q^2}-pq$ is strictly decreasing when $x \geq |p|$ *and* $|q|$, equality of areas hold if and only if $r=s$, which is equivalent to A, B, C, D concyclic (since P being on the perpendicular bisectors of AB, CD is the only possible place for the center).

After seeing these examples, we would like to remind the readers that there are pure geometric proofs to each of the problems. For examples (1) and (3), there are proofs that only take a few lines. We encourage the readers to discover these simple proofs.

Although in the opinions of many people, a pure geometric proof is better and more beautiful than a coordinate geometric proof, we should point out that sometimes the coordinate geometric proofs may be preferred when there are many cases. For example (2), the different possible orderings of the points D, E, F on the line can all happen as some pictures will show. The coordinate geometric proofs above cover all cases.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *October 10, 2000.*

Problem 106. Find all positive integer ordered pairs *(a,b)* such that

$$gcd(a,b)+lcm(a,b)=a+b+6,$$

where gcd stands for greastest common divisor (or highest common factor) and lcm stands for least common multiple.

Problem 107. For $a, b, c > 0$, if $abc=1$, then show that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Problem 108. Circles C_1 and C_2 with centers O_1 and O_2 (respectively) meet at points A, B. The radii O_1B and O_2B intersect C_1 and C_2 at F and E. The line parallel to EF through B meets C_1 and C_2 at M and N, respectively. Prove that $MN=AE+AF$. (Source: 17th Iranian Mathematical Olympiad)

Problem 109. Show that there exists an increasing sequence a_1, a_2, a_3, ... of positive integers such that for every nonnegative integer k,the sequence $k+a_1$, $k+a_2$, $k+a_3$ ... contains only finitely many prime numbers. (Source: 1997 Math Olympiad of Czech and Slovak Republics)

Problem 110. In a park, 10000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (Source: 1997 German Mathematical Olympiad)

Comments. You may think of the trees being placed at (x,y), where $x, y = 0, 1, 2, ..., 99$.

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Solutions

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Problem 101. A triple of numbers $(a_1, a_2, a_3)=(3, 4, 12)$ is given. We now perform the following operation: choose two numbers a_i and a_j, $(i \neq j)$, and exchange them by $0.6a_i–0.8a_j$ and $0.8a_i+0.6a_j$. Is it possible to obtain after several steps the (unordered) triple (2, 8, 10) ? (*Source:* 1999 National Math Competition in Croatia)

Solution. **FAN Wai Tong** (St. Mark's School, Form 7), **KO Man Ho** (Wah Yan College, Kowloon, Form 6) and **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 6).

Since $(0.6a_i–0.8a_j)^2 + (0.8a_i+0.6a_j)^2 = a_i^2+a_j^2$, the sum of the squares of the triple of numbers before and after an operation stays the same. Since $3^2+4^2+12^2 \neq 2^2+8^2+10^2$, so (2,8,10) cannot be obtained.

Problem 102. Let a be a positive real number and $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $x_1=a$ and

$$x_{n+1} \ge (n+2)x_n - \sum_{k=1}^{n-1} kx_k, \text{ for all } n \ge 1.$$

Show that there exists a positive integer n such that $x_n > 1999!$ (*Source: 1999 Romanian Third Selection Examination*)

Solution. **FAN Wai Tong** (St. Mark's School, Form 7).

We will prove by induction that $x_{j+1} \ge 3x_j$ for every positive integer j. The case $j=1$ is true by the given inequality. Assume the cases $j=1, ..., n-1$ are true. Then $x_n \ge 3x_{n-1} \ge 9x_{n-2} \ge ...$ and

$$\frac{x_{n+1}}{x_n} \ge (n+2) - \sum_{k=1}^{n-1}\frac{kx_k}{x_n}$$

$$\ge (n+2) - \sum_{k=1}^{n-1}\frac{n-1}{3^{n-k}}$$

$$\ge (n+2)-(n-1)(\frac{1}{3}+\frac{1}{9}+...)$$

$$=\frac{n+5}{2}$$

$$\ge 3.$$

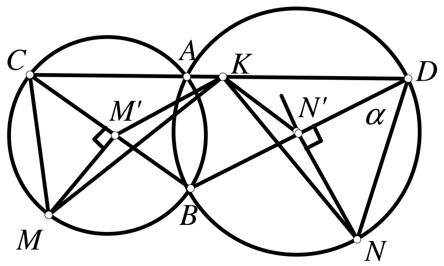
So the case $j = n$ is also true.

Since $a > 0$, we can take

$$n > 1+\log_3 (1999!/a).$$

Then $x_n \ge 3^{n-1}x_1 = 3^{n-1}a > 1999!$.

Problem 103. Two circles intersect in points A and B. A line l that contains the point A intersects again the circles in the points C, D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN=90°$. (*Source:* 1999 Romanian Fourth Selection Examination)



Solution. **FAN Wai Tong** (St. Mark's School, Form 7)

Let M' and N' be the midpoints of chords BC and BD respectively. From the midpoint theorem, we see that $BM'KN'$ is a parallelogram. Now

$$\angle KN'N = \angle KN'B + 90^\circ$$
$$= \angle KM'B + 90^\circ$$
$$= \angle KM'M.$$

Let $\alpha = \angle NDB = \angle NAB$. Then

$$\frac{KN'}{N'N} = \frac{M'B}{N'D\tan\alpha} = \frac{\frac{1}{2}BC}{\frac{1}{2}BD\tan\alpha}.$$

Now

$$\angle MCB = \angle MCB = \frac{1}{2}\angle CAB$$
$$=\frac{1}{2}(180^\circ - \angle DAB)$$
$$= 90^\circ - \angle NAB$$
$$= 90^\circ - \alpha.$$

So

$$\frac{MM'}{M'K}=\frac{CM'\cot\alpha}{BN'}=\frac{\frac{1}{2}BC\cot\alpha}{\frac{1}{2}BD}.$$

Then $KN'/N'N = MM'/M'K$. So triangles $MM'K, KN'N$ are similar. Then $\angle M'KM = \angle N'NK$ and

$$\begin{aligned}\angle MKN &= \angle M'KN' - \angle M'KM - \angle N'KN\\ &= \angle KN'D - (\angle N'NK + \angle N'KN)\\ &= 90^{\circ}\end{aligned}$$

Other commended solvers: **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

Problem 104. Find all positive integers n such that 2^n-1 is a multiple of 3 and $(2^n-1)/3$ is a divisor of $4m^2+1$ for some integer m. (*Source:* 1999 Korean Mathematical Olympiad)

Solution. (Official Solution)

(Some checkings should suggest n is a power of 2.) Now 2^n-1 is a multiple of 3 if and only if $(-1)^n \equiv 2^n \equiv$(mod 3), that is n is even. Suppose for some even n, $(2^n-1)/3$ is a divisor of $4m^2+1$ for some m. Assume n has an odd prime divisor d. Now $2^d-1\equiv 3$ (mod 4) implies one of its prime divisor p is of the form $4k+3$. Then p divides 2^d-1, which divides 2^n-1, which divides $4m^2+1$. Then p and $2m$ are relatively prime and so

$$1\equiv(2m)^{p-1}=(4m^2)^{2k+1}\equiv -1 \pmod p,$$

a contradiction. So n cannot have any odd prime divisor. Hence $n=2^j$ for some positive integer j.

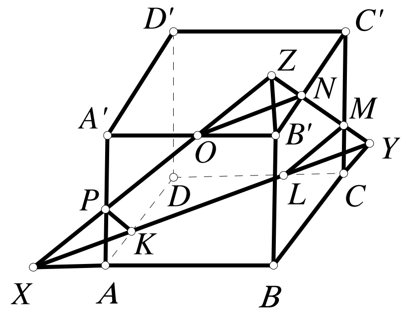
Conversely, suppose $n=2^j$. Let $F_i=2^{2^i}+1$. Using the factorization $2^{2b}-1=(2^b-1)\times(2^b+1)$ repeatedly on the numerator, we get

$$\frac{2^n-1}{3}=F_1F_2\cdots F_{j-1}.$$

Since F_i divides F_j-2 for $i<j$, the F_i's are pairwise relatively prime. By the Chinese remainder theorem, there is a positive integer x satisfying the simultaneous equations $x\equiv 0$ (mod 2) and $x\equiv 2^{2^{i-1}}$ (mod F_i) for $i=1, 2, \ldots, j-1$. Then $x=2m$ for some positive integer m and $4m^2+1= x^2+1\equiv 0$ (mod F_i) for $i=1, 2,\ldots j-1$. So $4m^2+1$ is divisible by $F_1F_2\ldots F_{j-1}=(2^n-1)/3$.

Problem 105. A rectangular parallelepiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelepiped is a cube. (*Source:* 1999 National Math Olympiad in Slovenia)

Solution. (Official Solution)



As in the figure, an equilateral triangle XYZ is formed by extending three alternate sides of the regular hexagon.

The right triangles XBZ and YBZ are congruent as they have a common side BZ and the hypotenuses have equal length. So $BX=BY$ and similarly $BX=BZ$. As the pyramids $XBYZ$ and $OB'NZ$ are similar and $ON=\frac{1}{3}XY$, it follows $B'Z=\frac{1}{3}BZ$. Thus we have $BB'=\frac{2}{3}BZ$ and similarly $AB=\frac{2}{3}BX$ and $CB=\frac{2}{3}BY$. Since $BX=BY=BZ$, we get $AB=BC=BB'$.

Other commended solvers: **FAN Wai Tong** (St. Mark's School, Form 7).

Olympiad Corner

(continued from page 1)

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.
Find all ways in which this can be done.

Problem 3. Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Proved that QO is perpendicular to BC.

Problem 4. Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}}<\frac{n!}{k!(n-k)!}<\frac{n^n}{k^k(n-k)^{n-k}}.$$

Problem 5. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0,1,\ldots, n$. A transposition of a_i with a_j is called *legal* if $i>0$, $a_i=0$ and $a_{i-1}+1=a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number of legal transpositions it becomes $(1,2,\ldots,n,0)$. For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular ?

2000 APMO and IMO

In April this year, Hong Kong IMO trainees participated in the XII Asia Pacific Mathematical Olympiad. The winners were

<u>Gold Award</u>

Fan Wai Tong (Form 7, St Mark's School)

<u>Silver Award</u>

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Chao Khek Lun (Form 5, St. Paul's College)

<u>Bronze Award</u>

Law Ka Ho (Form 7, Queen Elizabeth School)
Ng Ka Chun (Form 5, Queen Elizabeth School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)

<u>Honorable Mention</u>

Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Chau Suk Ling (Form 5, Queen Elizabeth School)
Choy Ting Pong (Form 7, Ming Kei College)

Based on the APMO and previous test results, the following trainees were selected to be the Hong Kong team members to the 2000 International Mathematical Olympiad, which was held in July in South Korea.

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Law Ka Ho (Form 7, Queen Elizabeth School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Fan Wai Tong (Form 7, St. Mark's School)

Mathematical Excalibur

Volume 5, Number 4 September 2000 – November 2000

Olympiad Corner

The 41st International Mathematical Olympiad, July 2000:

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points.

Problem 1. Two circles Γ_1 and Γ_2 intersect at *M* and *N*. Let ℓ be the common tangent to Γ_1 and Γ_2 so that *M* is closer to ℓ than *N* is. Let ℓ touches Γ_1 at *A* and Γ_2 at *B*. Let the line through *M* parallel to ℓ meets the circle Γ_1 again at *C* and the circle Γ_2 at *D*. Lines *CA* and *DB* meet at *E*; lines *AN* and *CD* meet at *P*; lines *BN* and *CD* meet at *Q*. Show that *EP* = *EQ*.

Problem 2. Let *a, b, c* be positive real numbers such that *abc* = 1. Prove that
$(a-1+1/b)(b-1+1/c)(c-1+1/a) \le 1$

Problem 3. Let $n \ge 2$ be a positive integer. Initially, there are *n* fleas on a horizontal line, not all at the same point. For a positive real number λ, define a move as follows:
Choose any two fleas, at points *A* and *B*, with *A* to the left of *B*; let the flea at *A* jump to the point *C* on the line to the line to the right of *B* with *BC*/*AB* = λ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

Jensen's Inequality

Kin Y. Li

In comparing two similar expressions, often they involve a common function. To see which expression is greater, the shape of the graph of the function on an interval is every important. A function *f* is said to be *convex* on an interval *I* if for any two points (x_1 , $f(x_1)$) and (x_2 , $f(x_2)$) on the graph, the segment joining these two points lie on or above the graph of the function over [x_1 , x_2]. That is,

$$f((1-t)x_1+tx_2) \le (1-t)\,f(x_1)+tf(x_2)$$

for every *t* in [0, 1]. If *f* is continuous on *I*, then it is equivalent to have

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$$

for every x_1, x_2 in *I*. If furthermore *f* is differentiable, then it is equivalent to have a nondecreasing derivative. Also, *f* is *strictly convex on I* if *f* is convex on *I* and equality holds in the inequalities above only when $x_1 = x_2$. We say a function *g* is *concave on an interval I* if the function –*g* is convex on *I*. Similarly, *g* is strictly concave on *I* if –*g* is strictly convex on *I*.

The following are examples of strictly convex functions on intervals:

$$x^p \text{ on } [0,\infty) \text{ for } p>1,$$

$$x^p \text{ on } (0,\infty) \text{ for } p<0,$$

$$a^x \text{ on } (-\infty,\infty) \text{ for } a>1,$$

$$\tan x \text{ on } [0, \tfrac{\pi}{2}).$$

The following are examples of strictly concave functions on intervals:

$$x^p \text{ on } [0,\infty) \text{ for } 0<p<1,$$

$$\log_a x \text{ on } (0,\infty) \text{ for } a>1,$$

$$\cos x \text{ on } [-\pi/2, \pi/2],$$

$$\sin x \text{ on } [0, \pi].$$

The most important inequalities con-cerning these functions are the following.

Jensen's Inequality. *If f is convex on an interval I and* $x_1, x_2, \ldots, x_n$ *are in I, then*

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}.$$

For strictly convex functions, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Generalized Jensen's Inequality. *Let f be continuous and convex on an interval I. If* $x_1, \ldots, x_n$ *are in I and* $0 < t_1, t_2, \ldots, t_n < 1$ *with* $t_1+t_2+\cdots+t_n = 1$, *then*

$$f(t_1x_1+t_2x_2+\cdots+t_nx_n) \le t_1f(x_1)+t_2f(x_2)+\cdots+t_nf(x_n)$$

(with the same equality condition for strictly convex functions).

Jensen's inequality is proved by doing a forward induction to get the cases *n* = 2^k, then a backward induction to get case *n* – 1 from case *n* by taking x_n to be the arithmetic mean of x_1, x_2, ..., x_{n-1}. For the generalized Jensen's inequality, the case all t_i 's are rational is proved by taking common denominator and the other cases are obtained by using continuity of the function and the density of rational numbers.

There are similar inequalities for concave and strictly concave functions by reversing the inequality signs.

Example 1. For a triangle *ABC*, show that $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$ and determine when equality holds.

Solution. Since $f(x) = \sin x$ is strictly concave on [0, π], so

$$\begin{aligned}\sin A + \sin B + \sin C &= f(A)+f(B)+f(C)\\ &\le 3f\left(\frac{A+B+C}{3}\right)\\ &= 3\sin\left(\frac{A+B+C}{3}\right)\\ &= \frac{3\sqrt{3}}{2}.\end{aligned}$$

Equality holds if and only if $A = B = C = \pi/3$, i.e. ΔABC is equilateral.

Example 2. If $a, b, c > 0$ and

$$a + b + c = 1,$$

then find the minimum of

$$\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}.$$

Solution. Note $0 < a, b, c < 1$. Let $f(x) = \left(x+\frac{1}{x}\right)^{10}$ on $I = (0, 1)$, then f is strictly convex on I because its second derivative

$$90\left(x+\frac{1}{x}\right)^{8}\left(1-\frac{1}{x^2}\right)^{2}+10\left(x+\frac{1}{x}\right)^{9}\left(\frac{2}{x^3}\right)$$

is positive on I. By Jensen's inequality,

$$\frac{10^{10}}{3^9}=3f\left(\frac{a+b+c}{3}\right)$$
$$\le f(a)+f(b)+f(c)$$
$$=\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}.$$

So the minimum is $10^{10}/3^9$, attained when $a = b = c = 1/3$.

Example 3. Prove that AM-GM in-equality, which states that if a_1, $a_2, \ldots, a_n \ge 0$, then

$$\frac{a_1+a_2+\cdots+a_n}{n} \ge \sqrt[n]{a_1a_2\cdots a_n}.$$

Solution. If one of the a_i's is 0, then the right side is 0 and the inequality is clear. If $a_1, a_2, \ldots, a_n > 0$, then since $f(x) = \log x$ is strictly concave on $(0, \infty)$, by Jensen's inequality,

$$\log\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$
$$\ge \frac{\log a_1+\log a_2+\cdots+\log a_n}{n}$$
$$=\log\left(\sqrt[n]{a_1a_2\cdots a_n}\right).$$

Exponentiating both sides, we get the AM-GM inequality.

Remarks. If we use the generalized Jensen's inequality instead, we can get the weighted AM-GM inequality. It states that if $a_1, \ldots, a_n > 0$ and $0 < t_1, \ldots, t_n < 1$ satisfying $t_1 + \cdots + t_n = 1$, then $t_1 a_1 + \cdots + t_n a_n \ge a_1^{t_1} \cdots a_n^{t_n}$ with equality if and only if all a_i's are equal.

Example 4. Prove the power mean inequality, which states that for a_1, $a_2, \ldots, a_n > 0$ and $s < t$, if

$$S_r=\left(\frac{a_1^r+a_2^r+\cdots+a_n^r}{n}\right)^{1/r},$$

then $S_s \le S_t$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Remarks. S_1 is the arithmetic mean (AM) and S_{-1} is the harmonic mean (HM) and S_2 is the root-mean-square (RMS) of a_1, $a_2, \cdots, a_n$. Taking limits, it can be shown that $S_{+\infty}$ is the maximum (MAX), S_0 is the geometric mean (GM) and $S_{-\infty}$ is the minimum (MIN) of $a_1, a_2, \cdots, a_n$.

Solution. In the cases $0 < s < t$ or $s < 0 < t$, we can apply Jensen's inequality to $f(x) = x^{t/s}$. In the case $s < t < 0$, we let $b_i = 1/a_i$ and apply the case $0 < -t < -s$. The other cases can be obtained by taking limit of the cases proved.

Example 5. Show that for $x, y, z > 0$,

$$x^5+y^5+z^5$$
$$\le x^5\sqrt{\frac{x^2}{yz}}+y^5\sqrt{\frac{y^2}{zx}}+z^5\sqrt{\frac{z^2}{xy}}.$$

Solution. Let $a = \sqrt{x}$, $b = \sqrt{y}$, $c = \sqrt{z}$, then the inequality becomes

$$a^{10}+b^{10}+c^{10} \le \frac{a^{13}+b^{13}+c^{13}}{abc}.$$

By the power mean inequality,

$$a^{13}+b^{13}+c^{13}=3S_{13}^{13}$$
$$=3S_{13}^{10}\,S_{13}^{3} \ge 3S_{10}^{10}\,S_0^3$$
$$=\left(a^{10}+b^{10}+c^{10}\right)abc.$$

Example 6. Prove Hölder's inequality, which states that if $p, q > 1$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real (or complex) numbers, then

$$\sum_{i=1}^{n}|a_ib_i| \le \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|b_i|^q\right)^{\frac{1}{q}}.$$

(The case $p = q = 2$ is the Cauchy-Schwarz inequality.)

Solution. Let

$$A=|a_1|^p+\cdots+|a_n|^p.$$
$$B=|b_1|^p+\cdots+|b_n|^q.$$

If A or B is 0, then either all a_i's or all b_i's are 0, which will make both sides of the inequality 0.

So we need only consider the case $A \ne 0$ and $B \ne 0$. Let $t_1 = 1/p$ and $t_2 = 1/q$, then $0 < t_1, t_2 < 1$ and $t_1 + t_2 = 1$. Let $x_i = |a_i|^p/A$ and $y_i = |b_i|^q/B$, then

$$x_1+\cdots+x_n=1, \quad y_1+\cdots+y_n=1.$$

Since $f(x) = e^x$ is strictly convex on $(-\infty, \infty)$, by the generalized Jensen's inequality,

$$x_i^{1/p}y_i^{1/q}=f(t_1\ln x_i+t_2\ln y_i)$$
$$\le t_1f(\ln x_i)+t_2f(\ln y_i)=\frac{x_i}{p}+\frac{y_i}{q}.$$

Adding these for $i = 1, \ldots, n$, we get

$$\sum_{i=1}^{n}\frac{|a_i||b_i|}{A^{1/p}B^{1/q}}=\sum_{i=1}^{n}x_i^{1/p}y_i^{1/q}$$
$$\le \frac{1}{p}\sum_{i=1}^{n}x_i+\frac{1}{q}\sum_{i=1}^{n}y_i=1.$$

Therefore,

$$\sum_{i=1}^{n}|a_i||b_i| \le A^{1/p}B^{1/q}$$
$$=\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|b_i|^q\right)^{1/q}.$$

Example 7. If $a, b, c, d > 0$ and

$$c^2+d^2=(a^2+b^2)^3,$$

then show that

$$\frac{a^3}{c}+\frac{b^3}{d} \ge 1.$$

Solution 1. Let

$$x_1=\sqrt{a^3/c}, \quad x_2=\sqrt{b^3/d},$$
$$y_1=\sqrt{ac}, \quad y_2=\sqrt{bd}.$$

By the Cauchy-Schwarz inequality,

$$\left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd)$$
$$=\left(x_1^2+x_2^2\right)\left(y_1^2+y_2^2\right)$$
$$\ge (x_1y_1+x_2y_2)^2$$
$$=(a^2+b^2)^2$$
$$=\sqrt{(a^2+b^2)(c^2+d^2)}$$
$$\ge ac+bd.$$

Cancelling $ac + bd$ on both sides, we get the desired inequality.

Solution 2. Let

$$x=(a^3/c)^{2/3}, \quad y=(b^3/d)^{2/3}.$$

By the $p = 3$, $q = 3/2$ case of Hölder's inequality,

$$a^2+b^2$$
$$=(c^{2/3})x+(d^{2/3})y$$
$$\le (c^2+d^2)^{1/3}(x^{3/2}+y^{3/2})^{2/3}$$

Cancelling $a^2+b^2=(c^2+d^2)^{1/3}$ on both sides, we get $1 \le x^{3/2}+y^{3/2} = (a^3/c)+(b^3/d)$.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *December 10, 2000.*

Problem 111. Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? *(Source: 1997 Czech-Slovak Match)*

Problem 112. Find all positive integers (x, n) such that $x^n + 2^n + 1$ is divisor of $x^{n+1} + 2^{n+1} + 1$. (*Source: 1998 Romanian Math Olympiad*)

Problem 113. Let $a, b, c > 0$ and $abc \le 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \ge a + b + c.$$

(Hint: Consider the case $abc = 1$ first.)

Problem 114. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) An infinite chessboard is given, with n black squares and the remainder white. Let the collection of black squares be denoted by G_0. At each moment $t = 1, 2, 3, \ldots$, a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares G_1, G_2, G_3, … are so formed. Prove that G_n is empty.

Problem 115. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) Find the locus of the points P in the plane of an equilateral triangle ABC for which the triangle formed with lengths PA, PB and PC has constant area.

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Solutions

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Problem 106. Find all positive integer ordered pairs (a, b) such that

$$\gcd(a,b) + \operatorname{lcm}(a,b) = a + b + 6,$$

where gcd stands for greatest common divisor (or highest common factor) and lcm stands for least common multiple.

Solution. **CHAN An Jack** and **LAW Siu Lun Jack** (Mei Kei College, Form 6), **CHAN Chin Fei** (STFA Leung Kau Kui College), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **FUNG Wing Kiu Ricky** (La Salle College), **HUNG Chung Hei** (Pui Ching Middle School, Form 5), **KO Man Ho** (Wah Yan College, Kowloon, Form 7), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Ka Ho** (HKU, Year 1), **LEE Kevin** (La Salle College), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **MAK Hoi Kwan Calvin** (La Salle College), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School), **POON Wing Sze Jessica** (STFA Leung Kau Kui College, Form 7), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 6), **TONG Chin Fung** (SKH Lam Woo Memorial Secondary School, Form 6), **WONG Wing Hong** (La Salle College, Form 3) and **YEUNG Kai Shing** (La Salle College, Form 4).

Let $m = \gcd(a, b)$, then $a = mx$ and $b = my$ with $\gcd(x, y) = 1$. In that case, $\operatorname{lcm}(a, b) = mxy$. So the equation becomes $m + mxy = mx + my + 6$. This is equivalent to $m(x - 1)(y - 1) = 6$. Taking all possible positive integer factorizations of 6 and requiring $\gcd(x, y) = 1$, we have $(m, x, y) = (1, 2, 7)$, $(1, 7, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(3, 2, 3)$ and $(3, 3, 2)$. Then $(a, b) = (2, 7)$, $(7, 2)$, $(3, 4)$, $(4, 3)$, $(6, 9)$ and $(9, 6)$. Each of these is easily checked to be a solution.

Other recommended solvers: **CHAN Kin Hang Andy** (Bishop Hall Jubilee School, Form 7) and **CHENG Kei Tsi Daniel** (La Salle College, Form 6).

Problem 107. For $a, b, c > 0$, if $abc = 1$, then show that

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

Solution 1. **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 7), **LAW Ka Ho** (HKU, Year 1) and **TSUI Ka Ho Willie** (Hoi Ping Chamber of Commerce Secondary School, Form 7).

By the AM-GM inequality and the fact $abc = 1$, we get

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \ge 2\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right)$$

$$= \left(\sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right) + \left(\sqrt{\frac{ab}{c}} + \sqrt{\frac{bc}{a}}\right) + \left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}}\right) \ge 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \ge$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + 3\sqrt[6]{abc} = \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

Solution 2. **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **LAW Ka Ho** (HKU, Year 1) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Without loss of generality, assume $a \ge b \ge c$. Then $1/\sqrt{a} \le 1/\sqrt{b} \le 1/\sqrt{c}$. By the rearrangement inequality,

$$\frac{b}{\sqrt{a}} + \frac{c}{\sqrt{b}} + \frac{a}{\sqrt{c}} \ge \frac{a}{\sqrt{a}} + \frac{b}{\sqrt{b}} + \frac{c}{\sqrt{c}} = \sqrt{a} + \sqrt{b} + \sqrt{c}$$

Also, by the AM-GM inequality,

$$\frac{c}{\sqrt{a}} + \frac{a}{\sqrt{b}} + \frac{b}{\sqrt{c}} \ge 3.$$

Adding these two inequalities, we get the desired inequality.

Generalization: Professor Murray S. Klamkin (University of Alberta, Canada) sent in a solution, which proved a stronger inequality and later generalized it to n variables. He made the sub-stitutions $x_1 = \sqrt{a}$, $x_2 = \sqrt{b}$, $x_3 = \sqrt{c}$ to get rid of square roots and let $S_m = x_1^m + x_2^m + x_3^m$ so that the inequality became

$$\frac{x_2^2 + x_3^2}{x_1} + \frac{x_3^2 + x_1^2}{x_2} + \frac{x_1^2 + x_2^2}{x_3} \ge S_1 + 3.$$

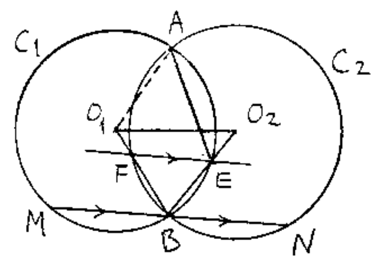
By the AM-GM inequality, $S_m \ge 3\sqrt[3]{x_1^m x_2^m x_3^m} = 3$. Since $S_2/3 \ge (S_1/3)^2 \ge S_1/3$ by the power mean inequality, we would get a stronger inequality by replacing $S_1 + 3$ by $2S_2$. Rearranging terms, this stronger inequality could be rewritten as $S_2(S_{-1} - 3) \ge S_1 - S_2$. Now the left side is nonnegative, but the right side is nonpositive. So the stronger inequality is true. If we replace 3 by n

and assume $x_1 \cdots x_n = 1$, then as above, we will get $S_m(S_{1-m} - n) \geq S_1 - S_m$ by the AM-GM and power mean inequalities. Expanding and regrouping terms, we get the stronger inequality in n variables, namely

$$\sum_{i=1}^{n} \frac{S_m - x_i^m}{x_i^{m-1}} \geq (n-1)S_m.$$

Other recommended solvers: **CHAN Chin Fei** (STFA Leung Kau Kui College), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 7), **LEE Kevin** (La Salle College, Form 5), **MAK Hoi Kwan Calvin** (La Salle College), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School) and **YEUNG Kai Shing** (La Salle College, Form 4).

Problem 108. Circles C_1 and C_2 with centers O_1 and O_2 (respectively) meet at points A, B. The radii O_1B and O_2B intersect C_1 and C_2 at F and E. The line parallel to EF through B meets C_1 and C_2 at M and N, respectively. Prove that $MN = AE + AF$. (*Source: 17th Iranian Mathematical Olympiad*)



Solution. **YEUNG Kai Shing** (La Salle College, Form 4).

As the case $F = E = B$ would make the problem nonsensible, the radius O_1B of C_1 can only intersect C_2, say at F. Then the radius O_2B of C_2 intersect C_1 at E. Since $\Delta\ EO_1B$ and $\Delta\ FO_2B$ are isosceles, $\angle EO_1F = 180^\circ - 2\angle FBE = \angle EO_2F$. Thus, E, O_2, O_1, F are concyclic. Then $\angle AEB = (360^\circ - \angle AO_1B)/2 = 180^\circ - \angle O_2O_1F = \angle O_2EF = \angle EBM$. So arc$AMB$ = arcMAE. Subtracting minor arcAM from both sides, we get minor arcMB = minor arcAE. So $MB = AE$. Similarly, $NB = AF$. Then $MN = MB + NB = AE + AF$.

Other recommended solvers: **Chan Kin Hang Andy** (Bishop Hall Jubilee School, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 6) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 109. Show that there exists an increasing sequence a_1, a_2, a_3, … of positive integers such that for every nonnegative integer k, the sequence $k + a_1$, $k + a_2$, $k + a_3$, … contains only finitely many prime numbers. (*Source: 1997 Math Olympiad of Czech and Slovak Republics*)

Solution. **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 7), **LAW Ka Ho** (HKU, Year 1) and **YEUNG Kai Shing** (La Salle College, Form 4).

Let $a_n = n! + 2$. Then for every non-negative integer k, if $n \geq k + 2$, then $k + a_n$ is divisible by $k + 2$ and is greater than $k + 2$, hence not prime.

Other commended solvers: **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **KO Man Ho** (Wah Yan College, Form 7), **LEE Kevin** (La Salle College, Form 5) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 110. In a park, 1000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (*Source: 1997 German Mathematical Olympiad*)

Solution. **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **Chau Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **FUNG Wing Kiu Ricky** (La Salle College), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Ka Ho** (HKU, Year 1), **LEE Kevin** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **LYN Kwong To** and **KO Man Ho** (Wah Yan College, Kowloon, Form 7), **POON Wing Sze Jessica** (STFA Leung Kau Kui College, Form 7) and **YEUNG Kai Shing** (La Salle College, Form 4).

In every 2×2 subsquare, only one tree can be cut. So a maximum of 2500 trees can be cut down. Now let the trees be at (x, y), where $x, y = 0, 1, 2, \ldots, 99$. If we cut down the 2500 trees at (x, y) with both x and y even, then the condition will be satisfied. To see this, consider the stumps at (x_1, y_1) and (x_2, y_2) with x_1, y_1, x_2, y_2 even. The cases $x_1 = x_2$ or $y_1 = y_2$ are clear. Otherwise, write $(y_2 - y_1)/(x_2 - x_1) = m/n$ in lowest term. Then either m or n is odd and so the tree at $(x_1 + m, y_1 + n)$ will be between (x_1, y_1) and (x_2, y_2).

Other recommended solvers: **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. ***(cont'd)*** Determine all values of λ such that, for any point M on the line and any initial position of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Problem 4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one a blue one, so that each contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Problem 5. Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime numbers, and $2^n + 1$ is divisible by n.

Problem 6. Let AH_1, BH_2, CH_3, be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the incircle of the triangle ABC.

Mathematical Excalibur

Volume 5, Number 5 — *November 2000 – December 2000*

Olympiad Corner

British Mathematical Olympiad, January 2000:

Time allowed: 3 hours 30 minutes

Problem 1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

Problem 2. Show that for every positive integer n,

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000.

Problem 3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that

$$AP + BP + CP$$

is minimized.

Problem 4. For each positive integer k, define the sequence $\{a_n\}$ by

$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n\, a_{n-1}$$

for each $n \geq 1$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *February 4, 2001.*

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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五點求圓錐曲線

梁子傑
香港道教聯合會青松中學

我們知道，圓錐曲線是一些所謂二次形的曲線，即一條圓錐曲線會滿足以下的一般二次方程：$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$，其中 A、B 及 C 不會同時等於 0。假設 $A \neq 0$，那麼我們可以將上式除以 A，並化簡成以下模式：

$$x^2 + bxy + cy^2 + dx + ey + f = 0。$$

以上的方程給了我們一個啟示：就是五點能夠定出一個圓錐曲線。因為如果我們知道了五個不同點的坐標，我們可以將它們分別代入上面的方程中，從而得到一個有 5 個未知數（即 b、c、d、e 和 f）和 5 條方程的方程組。祇要解出各未知數的答案，就可以知道該圓錐曲線的方程了。

不過，上述方法雖然明顯，但真正操作時又困難重重！這是由於有 5 個未知數的聯立方程卻不易解！而且我們在計算之初假設 x^2 的係數非零，但萬一這假設不成立，我們就要改設 B 或 C 非零，並需要重新計算一次了。

幸好，我們可以通過「圓錐曲線族」的想法來解此問題。方法見下例：

例： 求穿過 $A(1, 0)$, $B(3, 1)$, $C(0, 3)$, $D(-4, -1)$, $E(-2, -3)$ 五點的圓錐曲線方程。

解： 利用兩點式，先求出以下各直線的方程：

AB：$\dfrac{y-0}{x-1} = \dfrac{1-0}{3-1}$，即 $x - 2y - 1 = 0$

CD：$\dfrac{y-3}{x-0} = \dfrac{-1-3}{-4-0}$，即 $x - y + 3 = 0$

AC：$\dfrac{y-0}{x-1} = \dfrac{3-0}{0-1}$，即 $3x + y - 3 = 0$

BD：$\dfrac{y-1}{x-3} = \dfrac{-1-1}{-4-3}$，即 $2x - 7y + 1 = 0$

然後將 AB 和 CD 的方程「相乘」，得一條圓錐曲線的方程：

$(x - 2y - 1)(x - y + 3) = 0$，即 $x^2 - 3xy + 2y^2 + 2x - 5y - 3 = 0$。

注意： 雖然上述的方程是一條二次形「曲線」，但實際上它是由兩條直線所組成的。同時，亦請大家留意，該曲線同時穿過 A、B、C 和 D 四點。

類似地，我們又將 AC 和 BD「相乘」，得：

$(3x + y - 3)(2x - 7y + 1) = 0$，即 $6x^2 - 19xy - 7y^2 - 3x + 22y - 3 = 0$。

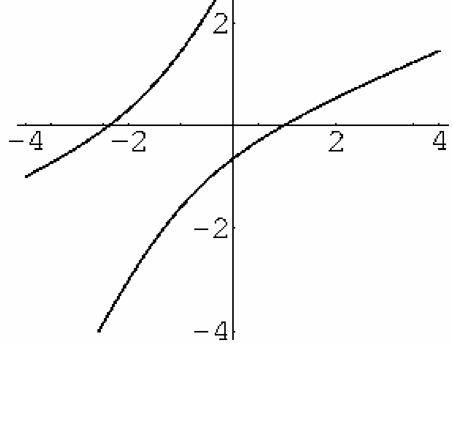
考慮圓錐曲線族：

$x^2 - 3xy + 2y^2 + 2x - 5y - 3 + k(6x^2 - 19xy - 7y^2 - 3x + 22y - 3) = 0$。很明顯，無論 k 取任何數值，這圓錐曲線族都會同樣穿過 A、B、C 和 D 四點。

最後，將 E 點的坐標代入曲線族中，得：$12 + k(-216) = 0$，即 $k = 1/18$，由此得所求的圓錐曲線方程為

$18(x^2 - 3xy + 2y^2 + 2x - 5y - 3) + (6x^2 - 19xy - 7y^2 - 3x + 22y - 3) = 0$，即

$24x^2 - 73xy + 29y^2 + 33x - 68y - 57 = 0$。



Majorization Inequality

Kin Y. Li

The majorization inequality is a generalization of Jensen's inequality. While Jensen's inequality provides one extremum (either maximum or minimum) to a convex (or concave) expression, the majorization inequality can provide both in some cases as the examples below will show. In order to state this inequality, we first introduce the concept of majorization for ordered set of numbers. If

$$x_1 \ge x_2 \ge \cdots \ge x_n,$$
$$y_1 \ge y_2 \ge \cdots \ge y_n,$$
$$x_1 \ge y_1, \quad x_1 + x_2 \ge y_1 + y_2, \quad \ldots,$$
$$x_1 + \cdots + x_{n-1} \ge y_1 + \cdots + y_{n-1}$$

and

$$x_1 + \cdots + x_n = y_1 + \cdots + y_n,$$

then we say $(x_1, x_2, \ldots, x_n)$ *majorizes* $(y_1, y_2, \ldots, y_n)$ and write

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n).$$

Now we are ready to state the inequality.

Majorization Inequality. *If the function f is convex on the interval I* = [*a, b*] *and*

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)$$

for $x_i, y_i \in I$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n)$$
$$\ge f(y_1) + f(y_2) + \cdots + f(y_n).$$

For strictly convex functions, equality holds if and only if $x_i = y_i$ *for i = 1, 2, …, n. The statements for concave functions can be obtained by reversing inequality signs.*

Next we will show that the majorization inequality implies Jensen's inequality. This follows from the observation that if $x_1 \ge x_2 \ge \cdots \ge x_n$, then $(x_1, x_2, \ldots, x_n) \succ$ (*x, x, …, x*), where *x* is the arithmetic mean of x_1, x_2, …, x_n. (Thus, applying the majorization inequality, we get Jensen's inequality.) For *k* = 1, 2, …, *n* - 1, we have to show $x_1 + \cdots + x_k \ge kx$. Since

$$(n-k)(x_1 + \cdots + x_k)$$
$$\ge (n-k)kx_k \ge k(n-k)x_{k+1}$$
$$\ge k(x_{k+1} + \cdots x_n).$$

Adding $k(x_1 + \cdots + x_k)$ to the two extremes, we get

$$n(x_1 + \cdots + x_k) \ge k(x_1 + \cdots + x_n) = knx.$$

Therefore, $x_1 + \cdots + x_k \ge kx.$

Example 1. For acute triangle ABC, show that

$$1 \le \cos A + \cos B + \cos C \le \frac{3}{2}$$

and determine when equality holds.

Solution. Without loss of generality, assume $A \ge B \ge C$. Then $A \ge \pi/3$ and $C \le \pi/3$. Since $\pi/2 \ge A \ge \pi/3$ and

$$\pi \ge A + B (= \pi - C) \ge 2\pi/3,$$

we have $(\pi/2, \pi/2, 0) \succ (A, B, C) \succ (\pi/3, \pi/3, \pi/3)$. Since *f(x)* = cos *x* is strictly concave on *I* = [0, $\pi/2$], by the majorization inequality,

$$1 = f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + f(0)$$
$$\le f(A) + f(B) + f(C)$$
$$= \cos A + \cos B + \cos C$$
$$\le f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = \frac{3}{2}.$$

For the first inequality, equality cannot hold (as two of the angles cannot both be right angles). For the second inequality, equality holds if and only if the triangle is equilateral.

Remarks. This example illustrates the equilateral triangles and the degenerate case of two right angles are extreme cases for convex (or concave) sums.

Example 2. Prove that if *a, b* ≥ 0, then

$$\sqrt[3]{a + \sqrt[3]{a}} + \sqrt[3]{b + \sqrt[3]{b}} \le \sqrt[3]{a + \sqrt[3]{b}} + \sqrt[3]{b + \sqrt[3]{a}}.$$

(*Source: Math Horizons, Nov. 1995, Problem 36 of Problem Section, proposed by E.M. Kaye*)

Solution. Without loss of generality, we may assume $b \ge a \ge 0$. Among the numbers

$$x_1 = b + \sqrt[3]{b}, \quad x_2 = b + \sqrt[3]{a},$$
$$x_3 = a + \sqrt[3]{b}, \quad x_4 = a + \sqrt[3]{a},$$

x_1 is the maximum and x_4 is the minimum. Since $x_1 + x_4 = x_2 + x_3$, we get $(x_1, x_4) \succ (x_2, x_3)$ or (x_3, x_2) (depends on which of x_2 or x_3 is larger). Since *f(x)* = $\sqrt[3]{x}$ is concave on the interval [0, ∞), so by the majorization inequality,

$$f(x_4) + f(x_1) \le f(x_3) + f(x_2),$$

which is the desired inequality.

Example 3. Find the maximum of $a^{12} + b^{12} + c^{12}$ if $-1 \le a, b, c \le 1$ and $a + b + c = -1/2$.

Solution. Note the continuous function *f(x)* = x^{12} is convex on [–1, 1] since $f''(x) = 132x^{10} \ge 0$ on (–1, 1). If 1 ≥ *a* ≥ *b* ≥ *c* ≥ –1 and

$$a + b + c = -\frac{1}{2},$$

then we get (1, –1/2, –1) $\succ$ (*a, b, c*). This is because 1 ≥ *a* and

$$\frac{1}{2} = 1 - \frac{1}{2} \ge -c - \frac{1}{2} = a + b.$$

So by the majorization inequality,

$$a^{12} + b^{12} + c^{12}$$
$$= f(a) + f(a) + f(c)$$
$$\le f(1) + f\left(-\frac{1}{2}\right) + f(-1)$$
$$= 2 + \frac{1}{2^{12}}.$$

The maximum value 2 + (1/ 2^{12}) is attained when *a* = 1, *b* = –1/2 and *c* = –1.

Remarks. The example above is a simplification of a problem in the 1997 Chinese Mathematical Olympiad.

Example 4. (1999 IMO) Let *n* be a fixed integer, with *n* ≥ 2.

(a) Determine the least constant *C* such that the inequality

$$\sum_{1 \le i < j \le n} x_i x_j \left(x_i^2 + x_j^2\right) \le C \left(\sum_{1 \le i \le n} x_i\right)^4$$

holds for all real numbers $x_1, x_2, \ldots, x_n \ge 0$.

(b) For this constant *C*, determine when equality holds.

Solution. Consider the case *n* = 2 first. Let $x_1 = m + h$ and $x_2 = m - h$, then $m = (x_1 + x_2)/2$, $h = (x_1 - x_2)/2$ and

$$x_1 x_2 \left(x_1^2 + x_2^2\right) = 2\left(m^4 - h^4\right)$$
$$\le 2m^4 = \frac{1}{8}(x_1 + x_2)^4$$

with equality if and only if *h* = 0, i.e. $x_1 = x_2$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *February 4, 2001.*

Problem 116. Show that the interior of a convex quadrilateral with area A and perimeter P contains a circle of radius A/P.

Problem 117. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

Problem 118. Let R be the real numbers. Find all functions $f: R \to R$ such that for all real numbers x and y,

$$f(xf(y) + x) = xy + f(x).$$

Problem 119. A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear. (*Source: 1997 Chinese Senior High Math Competitiion*)

Problem 120. Twenty-eight integers are chosen from the interval [104, 208]. Show that there exist two of them having a common prime divisor.

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Solutions

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Problem 111. Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? *(Source: 1997 Czech-Slovak Match)*

Solution 1. **LEE Kai Seng** (HKUST).

Take a smallest ball B with center at O and radius r. Any other ball touching B at x contains a smaller ball of radius r also touching B at x. Since these smaller balls are contained in the ball with center O and radius $3r$, which has a volume 27 times the volume of B, there are at most 26 of these other balls touching B.

Solution 2. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Consider a smallest ball S with center O and radius r. Let S_i and S_j (with centers O_i and O_j and radii r_i and r_j, respectively) be two other balls touching S at P_i and P_j, respectively. Since r_i, $r_j \ge$ r, we have $O_i\,O_j \ge r_i + r_j \ge r + r_i = O\,O_i$ and similarly $O_i\,O_j \ge O\,O_j$. So $O_i\,O_j$ is the longest side of $\Delta\,O\,O_i\,O_j$. Hence $\angle P_iOP_j = \angle O_iOO_j \ge 60^\circ$.

For ball S_i, consider the solid cone with vertex at O obtained by rotating a 30° angle about OP_i as axis. Let A_i be the part of this cone on the surface of S. Since $\angle P_iOP_j \ge 60^\circ$, the interiors of A_i and A_j do not intersect. Since the surface area of each A_i is greater than $\pi(r\sin 30^\circ)^2 = \pi r^2/4$, which is 1/16 of the surface area of S, S can touch at most 15 other balls. So the answer to the question is no.

Other recommended solvers: **CHENG Kei Tsi** (La Salle College, Form 6).

Problem 112. Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$. *(Source: 1998 Romanian Math Olympiad)*

Solution. **CHENG Kei Tsi** (La Salle College, Form 6), **LEE Kevin** (La Salle College, Form 5) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

For $x = 1$, $2(1^n + 2^n + 1) > 1^{n+1} + 2^{n+1} + 1 > 1^n + 2^n + 1$. For $x = 2$, $2(2^n + 2^n + 1) > 2^{n+1} + 2^{n+1} + 1 > 2^n + 2^n + 1$. For $x = 3$, $3(3^n + 2^n + 1) > 3^{n+1} + 2^{n+1} + 1 > 2(3^n + 2^n + 1)$. So there are no solutions with $x = 1, 2, 3$.

For $x \ge 4$, if $n \ge 2$, then we get $x(x^n + 2^n + 1) > x^{n+1} + 2^{n+1} + 1$. Now

$$\begin{aligned} &x^{n+1} + 2^{n+1} + 1 \\ &= (x-1)(x^n + 2^n + 1) \\ &\quad + x^n - (2^n + 1)\,x + 3\cdot 2^n + 2 \\ &> (x-1)(x^n + 2^n + 1) \end{aligned}$$

because for $n = 2$, $x^n - (2^n + 1)x + 2^{n+1} = x^2 - 5x + 8 > 0$ and for $n \ge 3$, $x^n - (2^n + 1)x \ge x(4^{n-1} - 2^n - 1) > 0$. Hence only $n = 1$ and $x \ge 4$ are possible. In that case, $x^n + 2^n + 1 = x + 3$ is a divisor of $x^{n+1} + 2^{n+1} + 1 = x^2 + 5 = (x-3)(x+3) + 14$ if and only if $x + 3$ is a divisor of 14. Since $x + 3 \ge 7$, $x = 4$ or 11. So the solutions are $(x, y) = (4, 1)$ and $(11, 1)$.

Problem 113. Let $a, b, c > 0$ and $abc \le 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \ge a + b + c.$$

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Since $abc \le 1$, we get $1/(bc) \ge a$, $1/(ac) \ge b$ and $1/(ab) \ge c$. By the AM-GM inequality,

$$\frac{2a}{c} + \frac{c}{b} = \frac{a}{c} + \frac{a}{c} + \frac{c}{b} \ge 3\sqrt[3]{\frac{a^2}{bc}} \ge 3a.$$

Similarly, $2b/a + a/c \ge 3b$ and $2c/b + b/a \ge 3c$. Adding these and dividing by 3, we get the desired inequality.

Alternatively, let $x = \sqrt[9]{a^4b/c^2}$, $y = \sqrt[9]{c^4a/b^2}$ and $z = \sqrt[9]{b^4c/a^2}$. We have $a = x^2y$, $b = z^2x$, $c = y^2z$ and $xyz = \sqrt[3]{abc} \le 1$. Using this and the re-arrangement inequality, we get

$$\begin{aligned} &\frac{a}{c} + \frac{b}{a} + \frac{c}{b} = \frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx} \\ &\ge xyz\left(\frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx}\right) = x^3 + y^3 + z^3 \\ &\ge x^2y + y^2z + z^2x = a + b + c. \end{aligned}$$

Problem 114. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) An infinite chessboard is given, with n black squares and the remainder white. Let the collection of black squares be denoted by G_0. At each moment $t = 1, 2, 3, \ldots$, a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares G_1, G_2, G_3, … are so formed. Prove that G_n is empty.

Solution. **LEE Kai Seng** (HKUST).

Call a rectangle (made up of squares on the chess board) *desirable* if with respect to its left-lower vertex as origin, every square in the first quadrant outside the rectangle is white. The most crucial fact is that knowing only the colouring of the squares in a desirable rectangle, we can determine their colourings at all later moments. Note that the smallest rectangle enclosing all black squares is a desirable rectangle. We will prove by induction that all squares of a desirable rectangle with at most *n* black squares will become white by *t* = *n*. The case *n* = 1 is clear. Suppose the cases *n* < *N* are true. Let *R* be a desirable rectangle with *N* black squares. Let R_0 be the smallest rectangle in *R* containing all *N* black squares, then R_0 is also desirable. Being smallest, the leftmost column and the bottom row of R_0 must contain some black squares. Now the rectangle obtained by deleting the left column of R_0 and the rectangle obtained by deleting the bottom row of R_0 are desirable and contain at most *n* - 1 black squares. So by *t* = *n* - 1, all their squares will become white. Finally the left bottom corner square of R_0 will be white by *t* = *n*.

Comments: This solution is essentially the same as the proposer's solution.

Other commended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 115. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) Find the locus of the points *P* in the plane of an equilateral triangle *ABC* for which the triangle formed with *PA, PB* and *PC* has constant area.

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Without loss of generality, assume *PA* ≥ *PB, PC*. Consider *P* outside the circumcircle of ΔABC first. If *PA* is between *PB* and *PC*, then rotate ΔPAC about *A* by 60° so that *C* goes to *B* and *P* goes to *P'*. Then $\Delta APP'$ is equilateral and the sides of $\Delta PBP'$ have length *PA, PB, PC*.

Let *O* be the circumcenter of ΔABC, *R* be the circumradius and $x = AB = AC = \sqrt{3}AO = \sqrt{3}R$. The area of $\Delta PBP'$ is the sum of the areas of $\Delta PAP'$, ΔPAB, $\Delta P'AB$ (or ΔPAC), which is

$$\frac{\sqrt{3}}{4}PA^2 + \frac{1}{2}PA\cdot x\sin\angle PAB$$
$$+\frac{1}{2}PA\cdot x\sin PAC.$$

Now

$$\sin\angle PAB + \sin\angle PAC$$
$$= 2\sin 150^\circ \cos(\angle PAB - 150^\circ)$$
$$= -\cos(\angle PAB + 30^\circ)$$
$$= -\cos\angle PAO = \frac{PO^2 - PA^2 - R^2}{2PA\cdot R}.$$

Using these and simplifying, we get the area of $\Delta PBP'$ is $\sqrt{3}(PO^2 - R^2)/4$.

If *PC* is between *PA* and *PB*, then rotate ΔPAC about *C* by 60° so that *A* goes to *B* and *P* goes to *P'*. Similarly, the sides of $\Delta PBP'$ have length *PA, PB, PC* and the area is the same. The case *PB* is between *PA* and *PC* is also similar.

For the case *P* is inside the circumcircle of ΔABC, the area of the triangle with sidelengths *PA, PB, PC* can similarly computed to be $\sqrt{3}(R^2 - PO^2)/4$. Therefore, the locus of *P* is the circle(s) with center *O* and radius $\sqrt{R^2 \pm 4c/\sqrt{3}}$, where *c* is the constant area.

Comments: The proposer's solution only differed from the above solution in the details of computing areas.

Olympiad Corner

(continued from page 1)

Problem 4. ***(cont'd)***

Determine all values of *k* for which 2000 is a term of the sequence.

Problem 5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy as a trio, and one of Bashful and Sneezy as a pair. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow White agreed to take part as well. In how many ways could the four teams then be formed?

Majorization Inequality

(continued from page 2)

For the case *n* > 2, let $a_i = x_i/(x_1+\cdots+x_n)$ for *i* = 1, …, *n*, then $a_1+\cdots+a_n = 1$. In terms of a_i's, the inequality to be proved becomes

$$\sum_{1\le i<j\le n} a_i a_j\left(a_i^2 + a_j^2\right) \le C.$$

The left side can be expanded and regrouped to give

$$\sum_{i=1}^{n} a_i^3\left(a_1+\cdots+a_{i-1}+a_{i+1}+\cdots a_n\right)$$
$$= a_1^3(1-a_1)+\cdots+a_n^3(1-a_n).$$

Now $f(x) = x^3(1-x) = x^3 - x^4 =$ is strictly convex on $\left[0,\frac{1}{2}\right]$ because the second derivative is positive on $\left(0,\frac{1}{2}\right)$.

Since the inequality is symmetric in the a_i's, we may assume $a_1 \ge a_2 \ge \cdots \ge a_n$.

If $a_1 \le \frac{1}{2}$, then since

$$\left(\frac{1}{2},\frac{1}{2},0,...,0\right) \succ \left(a_1, a_2, ..., a_n\right),$$

by the majorization inequality,

$$f(a_1)+f(a_2)+\cdots+f(a_n)$$
$$\le f\left(\frac{1}{2}\right)+f\left(\frac{1}{2}\right)+f(0)+\cdots+f(0)=\frac{1}{8}.$$

If $a_1 > \frac{1}{2}$, then $1-a_1, a_2, ..., a_n$ are in [0, $\frac{1}{2}$]. Since

$$(1-a_1, 0, ..., 0) \succ (a_2, ..., a_n),$$

by the majorization inequality and case *n* = 2, we have

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$
$$\le f(a_1)+f(1-a_1)+f(0)+\cdots+f(0)$$
$$= f(a_1)+f(1-a_1)\le\frac{1}{8}.$$

Equality holds if and only if two of the variables are equal and the other $n-2$ variables all equal 0.

Mathematical Excalibur

Volume 6, Number 1 January 2001 – March 2001

Olympiad Corner

17th Balkan Mathematical Olympiad, 3-9 May 2000:

Time allowed: 4 hours 30 minutes

Problem 1. Find all the functions f : $\mathbf{R} \rightarrow \mathbf{R}$ with the property:

$$f(xf(x)+f(y))=(f(x))^2+y,$$

for any real numbers x and y.

Problem 2. Let ABC be a nonisosceles acute triangle and E be an interior point of the median AD, $D \in (BC)$. The point F is the orthogonal projection of the point E on the straight line BC. Let M be an interior point of the segment EF, N and P be the orthogonal projections of the point M on the straight lines AC and AB, respectively. Prove that the two straight lines containing the bisectrices of the angles PMN and PEN have no common point.

Problem 3. Find the maximum number of rectangles of the dimensions $1\times 10\sqrt{2}$, which is possible to cut off from a rectangle of the dimensions 50×90, by using cuts parallel to the edges of the initial rectangle.

(continued on page 2)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *April 15, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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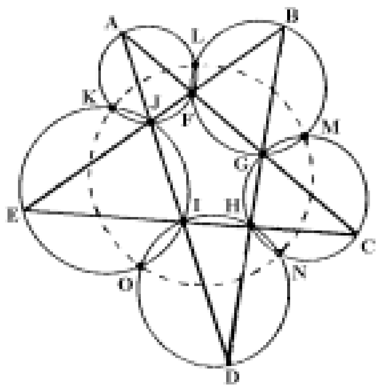
Concyclic Problems

Kin Y. Li

Near Christmas last year, I came across two beautiful geometry problems. I was informed of the first problem by a reporter, who was covering President Jiang Zemin's visit to Macau. While talking to students and teachers, the President posed the following problem.

For any pentagram ABCDE obtained by extending the sides of a pentagon FGHIJ, prove that neighboring pairs of the circumcircles of ΔAJF, BFG, CGH, DHI, EIJ intersect at 5 concyclic points K, L, M, N, O as in the figure.



The second problem came a week later. I read it in the Problems Section of the November issue of the *American Mathematical Monthly*. It was proposed by Floor van Lamoen, Goes, The Netherlands. Here is the problem.

A triangle is divided by its three medians into 6 smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

To get the readers appreciating these problems, here I will say, *stop reading, try to work out these problems and come back to compare your solutions with those given below!*

Here is a guided tour of the solutions. The first step in enjoying geometry problems is to draw accurate pictures with compass and ruler!

Now we look at ways of getting solutions to these problems. Both are concyclic problems with more than 4 points. Generally, to do this, we show the points are concyclic four at a time. For example, in the first problem, if we can show *K, L, M, N* are concyclic, then by similar reasons, *L, M, N, O* will also be concyclic so that all five points lie on the circle passing through *L, M, N*.

There are two common ways of showing 4 points are concyclic. One way is to show the sum of two opposite angles of the quadrilateral with the 4 points as vertices is 180°. Another way is to use the converse of the intersecting chord theorem, which asserts that if lines WX and YZ intersect at P and $PW\cdot PX$ = $PY\cdot PZ$, then W, X, Y, Z are concyclic. (The equation implies ΔPWY, PZX are similar. Then $\angle PWY = \angle PZX$ and the conclusion follows.)

For the first problem, as the points *K, L, M, N, O* are on the circumcirles, checking the sum of opposite angles equal 180° is likely to be easier as we can use the theorem about angles on the same segment to move the angles. To show *K, L, M, N* are concyclic, we consider showing $\angle LMN + \angle LKN = 180^\circ$. Since the sides of $\angle LMN$ are in two circumcircles, it may be wise to break it into two angles *LMG* and *GMN*. Then the strategy is to *change* these to other angles *closer* to $\angle LKN$.

Now $\angle LMG = 180^\circ - \angle LFG = \angle LFA$ = $\angle LKA$. (So far, we are on track. We *bounced* $\angle LMG$ to $\angle LKA$, which shares a side with $\angle LKN$.) Next, $\angle GMN$ = $\angle GCN$ = $\angle ACN$. Putting these together, we have

$$\begin{aligned}&\angle LMN+\angle LKN\\&=\angle LKA+\angle ACN+\angle LKN\\&=\angle AKN+\angle ACN.\end{aligned}$$

Now if we can only show *A, K, N, C* are concyclic, then we will get 180° for the displayed equations above and we will finish. However, life is not that easy. This turned out to be the hard part. If you draw the circle through *A, C, N*, then you see it goes through *K* as expected and surprisingly, it also goes through another point, *I*. With this discovery, there is new hope. Consider the arc through *B, I, O*. On the two sides of this arc, you can see there are *corresponding point pairs* (*A, C*), (*K, N*), (*J, H*), (*F, G*). So to show *A, K, N, C* are concyclic, we can first try to show *N* is on the circle through *A, C, I,* then in that argument, if we interchange *A* with *C, K,* with *N* and so on, we should also get *K* is on the circle through *C, A, I*. Then *A, K, N, C* (and *I*) will be concyclic and we will finish.

Wishful thinking like this sometimes works! Here are the details:

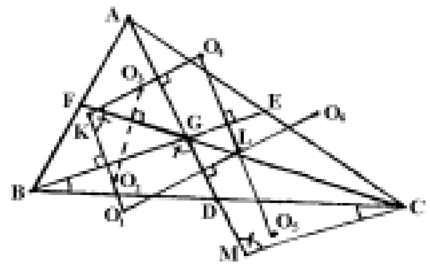
$$\angle ACN = \angle GCN = 180^\circ - \angle GHN$$

$$= \angle NHD = \angle NID = 180^\circ - \angle AIN .$$

So *N* is on the circle *A, C, I*. *Interchanging letters*, we get *similarly K* is on circle *C, A, I*. So *A, K, N, C* (and *I*) are concyclic. Therefore, *K, L, M, N, O* are indeed concyclic.

(*History*. My friend C.J. Lam did a search on the electronic database JSTOR and came across an article titled *A Chain of Circles Associated with the 5-Line* by J.W. Clawson published in the *American Mathematical Monthly*, volume 61, number 3 (March 1954), pages 161-166. There the problem was attributed to the nineteenth century geometer Miquel, who published the result in Liouville's Journal de Mathematiques, volume 3 (1838), pages 485-487. In that paper, Miquel proved his famous theorem that for four pairwise intersecting lines, taking three of the lines at a time and forming the circles through the three intersecting points, the four circles will always meet at a common point, which nowadays are referred to as the *Miquel point*. The first problem was then deduced as a corollary of this Miquel theorem.)

For the second problem, as the 6 circumcenters of the smaller triangles are not on any circles that we can see immediately, so we may try to use the converse of the intersecting chord theorem. For a triangle *ABC*, let *G, D, E, F* be the centroid, the midpoints of sides *BC, CA, AB*, respectively. Let O_1, O_2, O_3, O_4, O_5, O_6 be the circumcenters of triangles *DBG, BFG, FAG, AEG, ECG, CDG*, respectively.



Well, should we draw the 6 circumcircles? It would make the picture complicated. The circles do not seem to be helpful at this *early* stage. We give up on drawing the circles, but the circumcenters are important. So at least we should locate them. To locate the circumcenter of ΔFAG, for example, which two sides do we draw perpendicular bisectors? Sides *AG* and *FG* are the choices because they are also the sides of the other small triangles, so we can save some work later. Trying this out, we discover these perpendicular bisectors produce many parallel lines and parallelograms!

Since circumcenters are on perpendicular bisectors of chords, lines $O_3\,O_4$, $O_6\,O_1$ are perpendicular bisectors of *AG, GD*, respectively. So they are perpendicular to line *AD* and are $\frac{1}{2}$ *AD* units apart. Similarly, the two lines $O_1\,O_2$, $O_4\,O_5$ are perpendicular to line *BE* and are $\frac{1}{2}$ *BE* units apart. Aiming in showing O_1, O_2, O_3, O_4 are concyclic by the converse of the intersecting chord theorem, let *K* be the intersection of lines $O_1\,O_2$, $O_3\,O_4$ and *L* be the intersection of the lines $O_4\,O_5$, $O_6\,O_1$. Since the area of the parallelogram KO_4LO_1 is

$$\frac{1}{2}AD\cdot KO_4 = \frac{1}{2}BE\cdot KO_1,$$

we get $KO_1/KO_4 = AD/BE$.

Now that we get ratio of KO_1 and KO_4, we should examine KO_2 and KO_3. Trying to understand ΔKO_2O_3, we first find its angles. Since $KO_2 \perp BG$, $O_2O_3 \perp FG$ and $KO_3 \perp AG$, we see that $\angle KO_2O_3 = \angle BGF$ and $\angle KO_3O_2 = \angle FGA$. Then $\angle O_2KO_3 = \angle DGB$. At this point, you can see the angles of ΔKO_2O_3 equal the three angles with vertices at *G* on the left side of segment *AD*.

Now we try to put these three angles together in another way to form another triangle. Let *M* be the point on line *AG* such that *MC* is parallel to *BG*. Since $\angle MCG = \angle BGF$, $\angle MGC = \angle FGA$ (and $\angle GMC = \angle BGD$,) we see ΔKO_2O_3, *MCG* are similar.

The sides of ΔMCG are easy to compute in term of *AD, BE, CF*. As *AD* and *BE* occurred in the ratio of KO_1 and KO_4, this is just what we need! Observe that ΔMCD, *GBD* are congruent since $\angle MCD = \angle GBD$ (by *MC* parallel to *GB*), *CD* = *BD* and $\angle MDC = \angle GDB$. So

$$MG = 2GD = \frac{2}{3}AD,$$

$$MC = GB = \frac{2}{3}BE$$

(and $CG = \frac{2}{3}CF$. Incidentally, this means the three medians of a triangle can be put together to form a triangle! Actually, this is well-known and was the reason we considered ΔMCG.) We have $KO_3/KO_2 = MG/MC = AD/BE = KO_1/KO_4$.

So $KO_1\cdot KO_2 = KO_3\cdot KO_4$, which implies O_1, O_2, O_3, O_4 are concyclic. Similarly, we see that O_2, O_3, O_4, O_5 concyclic (using the parallelogram formed by the lines O_1O_2, O_4O_5, O_2O_3, O_5O_6 instead) and O_3, O_4, O_5, O_6 are concyclic.

Olympiad Corner

(continued from page 1)

Problem 4. We say that a positive integer *r* is a *power*, if it has the form $r = t^s$ where *t* and *s* are integers, $t \ge 2$, $s \ge 2$. Show that for any positive integer *n* there exists a set *A* of positive integers, which satisfies the conditions:

1. *A* has *n* elements;
2. any element of *A* is a power;
3. for any r_1, r_2, …, $r_k\,(2 \le k \le n)$ from *A* the number $\dfrac{r_1 + r_2 + \cdots + r_k}{k}$ is a power.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***April 15, 2001***.

Problem 121. Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.
(Two integers are *relative prime* if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus 37 = 22 + 15 represents the number 37 in the desired way.) (*Source: Second Bay Area Mathematical Olympaid*)

Problem 122. Prove that the product of the lengths of the three angle bisectors of a triangle is less than the product of the lengths of the three sides. (*Source: 1957 Shanghai Junior High School Math Competition*)

Problem 123. Show that every convex quadrilateral with area 1 can be covered by some triangle of area at most 2. (*Source: 1989 Wuhu City Math Competition*)

Problem 124. Find the least integer n such that among every n distinct numbers a_1, a_2, …, a_n, chosen from [1, 1000], there always exist a_i, a_j such that

$$0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}\ .$$

(*Source: 1990 Chinese Team Training Test*)

Problem 125. Prove that

$$\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + \cdots + \tan^2 89^\circ$$

is an integer.

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Solutions

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Problem 116. Show that the interior of a convex quadrilateral with area A and perimeter P contains a circle of radius A/P.

Solution 1. **CHAO Khek Lun** (St. Paul's College, Form 6).

Draw four rectangles on the sides of the quadrilateral and each has height A/P pointing inward. The sum of the areas of the rectangles is A. Since at least one interior angle of the quadrilateral is less than 180°, at least two of the rectangles will overlap. So the union of the four rectangular regions does not cover the interior of the quadrilateral. For any point in the interior of the quadrilateral not covered by the rectangles, the distance between the point and any side of the quadrilateral is greater than A/P. So we can draw a desired circle with that point as center.

Solution 2. **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Let $BCDE$ be a quadrilateral with area A and perimeter P. One of the diagonal, say BD is inside the quadrilateral. Then either ΔBCD or ΔBED will have an area greater than or equal to $A/2$. Suppose this is ΔBCD. Then $BCDE$ contains the incircle of ΔBCD, which has a radius of

$$\begin{aligned} &\frac{2[BCD]}{BC+CD+DB} \\ &> \frac{2[BCD]}{BC+CD+DE+EB} \\ &\geq \frac{A}{P}, \end{aligned}$$

where the brackets denote area. Hence, it contains a circle of radius A/P.
Comment: Both solutions do not need the convexity assumption.

Problem 117. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

Solution. **CHAO Khek Lun** (St. Paul's College, Form 6) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Suppose the sides are a, b, c, d with $a < b < c < d$. Since $d < a + b + c < 3d$ and d divides $a + b + c$, we have $a + b + c = 2d$. Now each of a, b, c divides $a + b + c + d = 3d$. Let $x = 3d/a$, $y = 3d/b$ and $z = 3d/c$. Then $a < b < c < d$ implies $x > y > z > 3$. So $z \geq 4$, $y \geq 5$, $x \geq 6$. Then

$$2d = a+b+c \leq \frac{3d}{6} + \frac{3d}{5} + \frac{3d}{4} < 2d,$$

a contradiction. Therefore, two of the sides are equal.

Problem 118. Let R be the real numbers. Find all functions $f: R \to R$ such that for all real numbers x and y,

$$f(xf(y) + x) = xy + f(x).$$

Solution 1. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Putting $x = 1$, $y = -1 - f(1)$ and letting $a = f(y) + 1$, we get

$$f(a) = f(f(y) + 1) = y + f(1) = -1.$$

Putting $y = a$ and letting $b = f(0)$, we get

$$b = f(xf(a) + x) = ax + f(x),$$

so $f(x) = -ax + b$. Putting this into the equation, we have

$$a^2xy - abx - ax + b = xy - ax + b.$$

Equating coefficients, we get $a = \pm 1$ and $b = 0$, so $f(x) = x$ or $f(x) = -x$. We can easily check both are solutions.

Solution 2. **LEE Kai Seng** (HKUST).

Setting $x = 1$, we get

$$f(f(y)+1) = y + f(1).$$

For every real number a, let $y = a - f(1)$, then $f(f(y) + 1) = a$ and f is surjective. In particular, there is b such that $f(b) = -1$. Also, if $f(c) = f(d)$, then

$$\begin{aligned} c + f(1) &= f(f(c)+1) \\ &= f(f(d)+1) \\ &= d + f(1). \end{aligned}$$

So $c = d$ and f is injective. Taking $x = 1$, $y = 0$, we get $f(f(0) + 1) = f(1)$. Since f is injective, we get $f(0) = 0$.
For $x \neq 0$, let $y = -f(x)/x$, then

$$f(xf(y) + x) = 0 = f(0).$$

By injectivity, we get $xf(y) + x = 0$. Then

$$f(-f(x)/x) = f(y) = -1 = f(b)$$

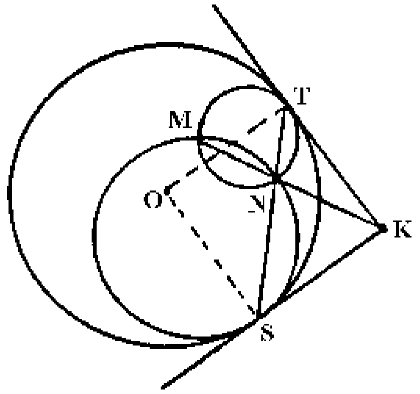
and so $-f(x)/x = b$ for every $x \neq 0$. That is, $f(x) = -bx$. Putting this into the given equation, we find $f(x) = x$ or $f(x) = -x$, which are checked to be solutions.

Other commended solvers: **CHAO Khek Lun** (St. Paul's College, Form 6) and **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6).

Problem 119. A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N

closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear. (*Source: 1997 Chinese Senior High Math Competition*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).



Consider the tangent lines at S and at T. (Suppose they are parallel, then S, O, T will be collinear so that M and N will be equidistant from ST, contradicting N is closer to ST.) Let the tangent lines meet at K, then $\angle OSK = 90^\circ = \angle OTK$ implies O, S, K, T lie on a circle with diameter OK. Also, $KS^2 = KT^2$ implies K is on the *radical axis MN* of the two inside circles. So M, N, K are collinear.

If S, N, T are collinear, then $\angle SMT$ = $\angle SMN + \angle TMN$ = $\angle NSK + \angle KTN$ = $180^\circ - \angle SKT$. So M, S, K, T, O are concyclic. Then $\angle OMN$ = $\angle OMK$ = $\angle OSK$ = 90°.

Conversely, if $OM \perp MN$, then $\angle OMK$ = $90^\circ = \angle OSK$ implies M, S, K, T, O are concyclic. Then

$$\begin{aligned}\angle SKT &= 180^\circ - \angle SMT \\ &= 180^\circ - \angle SMN - \angle TMN \\ &= 180^\circ - \angle NSK - \angle KTN.\end{aligned}$$

Thus, $\angle TNS = 360^\circ - \angle NSK - \angle SKT$ - $\angle KTN = 180^\circ$. Therefore, S, N, T are collinear.

Comments: For the meaning of *radical axis*, we refer the readers to pages 2 and 4 of *Math Excalibur, vol. 4, no. 3* and the corrections on page 4 of *Math Excalibur, vol. 4, no. 4*.

Other commended solvers: **CHAO Khek Lun** (St. Paul's College, Form 6).

Problem 120. Twenty-eight integers are chosen from the interval [104, 208]. Show that there exist two of them having a common prime divisor.

Solution 1. **CHAO Khek Lun** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6) and **CHUNG Tat Chi** (Queen Elizabeth School, Form 4).

Applying the inclusion-exclusion principle, we see there are 82 integers on [104, 208] that are divisible by 2, 3, 5 or 7. There remain 23 other integers on the interval. If 28 integers are chosen from the interval, at least 28 – 23 = 5 are among the 82 integers that are divisible by 2, 3, 5 or 7. So there will exist two that are both divisible by 2, 3, 5 or 7.

Solution 2. **CHAN Yun Hung** (Carmel Divine Grace Foundation Secondary School, Form 4), **KWOK Sze Ming** (Queen Elizabeth School, Form 5), **LAM Shek Ming** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **WONG Tak Wai Alan** (University of Toronto) and **WONG Wing Hong** (La Salle College, Form 3).

There are 19 prime numbers on the interval. The remaining 86 integers on the interval are all divisible by at least one of the prime numbers 2, 3, 5, 7, 11 and 13 since 13 is the largest prime less than or equal to $\sqrt{208}$. So every number on the interval is a multiple of one of these 25 primes. Hence, among any 26 integers on the interval at least two will have a common prime divisor.

A Proof of the Majorization Inequality

Kin Y. Li

Quite a few readers would like to see a proof of the majorization inequality, which was discussed in the last issue of the *Mathematical Excalibur*. Below we will present a proof. We will first make one observation.

Lemma. Let $a < c < b$ and f be convex on an interval I with a, b, c on I. Then the following are true:

$$\frac{f(c)-f(a)}{c-a} \le \frac{f(b)-f(a)}{b-a}$$

and

$$\frac{f(b)-f(c)}{b-c} \le \frac{f(b)-f(a)}{b-a}.$$

Proof. Since $a < c < b$, we have $c = (1 - t)a + tb$ for some $t \in (0, 1)$. Solving for t, we get $t = (c - a)/(b - a)$. Since f is convex on I,

$$f(c) \le (1-t)f(a) + t f(b)$$

$$= \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b),$$

which is what we will get if we solve for $f(c)$ in the two inequalities in the statement of the lemma.

In brief the lemma asserts that the slopes of chords are increasing as the chords are moving to the right. Now we are ready to proof the majorization inequality.

Suppose

$$(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n).$$

Since $x_i \ge x_{i+1}$ and $y_i \ge y_{i+1}$ for i = 1, 2, …, n – 1, it follows from the lemma that the slopes

$$m_i = \frac{f(x_i)-f(y_i)}{x_i - y_i}$$

satisfy $m_i \ge m_{i+1}$ for 1 ≤ i ≤ n – 1. (For example, if $y_{i+1} \le y_i \le x_{i+1} \le x_i$, then applying the lemma twice, we get

$$\begin{aligned} m_{i+1} &= \frac{f(x_{i+1})-f(y_{i+1})}{x_{i+1}-y_{i+1}} \\ &\le \frac{f(x_{i+1})-f(y_i)}{x_{i+1}-y_i} \\ &\le \frac{f(x_i)-f(y_i)}{x_i-y_i} = m_i \end{aligned}$$

and similarly for the other ways y_{i+1}, y_i, x_{i+1}, x_i are distributed.)

For k = 1, 2, …, n, let

$$X_k = x_1 + x_2 + \cdots + x_k$$

and

$$Y_k = y_1 + y_2 + \cdots + y_k.$$

Since $X_k \ge Y_k$ for k = 1, 2, …, n – 1 and $X_n = Y_n$, we get

$$\sum_{k=1}^{n}(X_k - Y_k)(m_k - m_{k+1}) \ge 0,$$

where we set $m_{n+1} = 0$ for convenience. Expanding the sum, grouping the terms involving the same m_k 's and letting $X_0 = 0 = Y_0$, we get

$$\sum_{k=1}^{n}(X_k - X_{k-1} - Y_k + Y_{k-1})m_k \ge 0,$$

which is the same as

$$\sum_{k=1}^{n}(x_k - y_k)m_k \ge 0.$$

Since $(x_k - y_k)m_k = f(x_k) - f(y_k)$, we get

$$\sum_{k=1}^{n}(f(x_k) - f(y_k))m_k \ge 0.$$

Transferring the $f(y_k)$ terms to the right, we get the majorization inequality.

Mathematical Excalibur

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Olympiad Corner

The 2000 Canadian Mathematical Olympiad

Problem 1. At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners).

Problem 2. A *permutation* of the integers 1901, 1902, …, 2000 is a sequence a_1, a_2, …, a_{100} in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1,\ s_2 = a_1 + a_2,$$
$$s_3 = a_1 + a_2 + a_3, \ldots, s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

(continued on page 4)

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *June 30, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Base n Representations

Kin Y. Li

When we write down a number, it is understood that the number is written in base 10. We learn many interesting facts at a very young age. Some of these can be easily explained in terms of base 10 representation of a number. Here is an example.

Example 1. Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. How about divisibility by 11?

Solution. Let $M = d_m 10^m + \cdots + d_1 10 + d_0$, where d_i = 0, 1, 2, …, 9. The binomial theorem tells us $10^k = (9+1)^k = 9N_k + 1$. So

$$M = d_m(9N_m + 1) + \cdots + d_1(9+1) + d_0$$
$$= 9(d_m N_m + \cdots + d_1) + (d_m + \cdots + d_1 + d_0).$$

Therefore, M is a multiple of 9 if and only if $d_m + \cdots + d_1 + d_0$ is a multiple of 9.

Similarly, we have $10^k = 11N'_k + (-1)^k$. So M is divisible by 11 if and only if $(-1)^m d_m + \cdots - d_1 + d_0$ is divisible by 11.

Remarks. In fact, we can also see that the remainder when M is divided by 9 is the same as the remainder when the sum of the digits of M is divided by 9. Recall the notation $a \equiv b \pmod{c}$ means a and b have the same remainder when divided by c. So we have $M \equiv d_m + \cdots + d_1 + d_0 \pmod 9$.

The following is an IMO problem that can be solved using the above remarks.

Example 2. (*1975 IMO*) Let A be the sum of the decimal digits of 4444^{4444}, and B be the sum of the decimal digits of A. Find the sum of the decimal digits of B.

Solution. Since $4444^{4444} < (10^5)^{4444} = 10^{22220}$, so $A < 22220 \times 9 = 199980$. Then $B < 1 + 9 \times 5 = 46$ and the sum of the decimal digits of B is at most 3+9=12. Now $4444 \equiv 7 \pmod 9$ and $7^3 = 343 \equiv 1 \pmod 9$ imply $4444^3 \equiv 1 \pmod 9$. Then $4444^{4444} = (4444^3)^{1481} 4444 \equiv 7 \pmod 9$. By the remarks above, *A, B* and the sum of the decimal digits of B also have remainder 7 when divided by 9. So the sum of the decimal digits of B being at most 12 must be 7.

Although base 10 representations are common, numbers expressed in other bases are sometimes useful in solving problems, for example, base 2 is common. Here are a few examples using other bases.

Example 3. (*A Magic Trick*) A magician asks you to look at four cards. On the first card are the numbers 1, 3, 5, 7, 9, 11, 13, 15; on the second card are the numbers 2, 3, 6, 7, 10, 11, 14, 15; on the third card are the numbers 4, 5, 6, 7, 12, 13, 14, 15; on the fourth card are the numbers 8, 9, 10, 11, 12, 13, 14, 15. He then asks you to pick a number you saw in one of these cards and hand him all the cards that have that number on them. Instantly he knows the number. Why?

Solution. For n = 1, 2, 3, 4, the numbers on the n-th card have the common feature that their n-th digits from the end in base 2 representation are equal to 1. So you are handing the base 2 representation of your number to the magician. As the numbers are less than 2^4, he gets your number easily.

Remarks. A variation of this problem is the following. A positive integer less than 2^4 is picked at random. What is the least number of yes-no questions you can ask that always allow you to know the number? Four questions are enough as you can ask if each of the four digits of the number in base 2 is 1 or not. Three questions are not enough as there are 15 numbers and three questions can only provide $2^3 = 8$ different yes-no combinations.

Example 4. (*Bachet's Weight Problem*) Give a set of distinct integral weights that allowed you to measure any object having weight $n = 1, 2, 3, \ldots, 40$ on a balance. Can you do it with a set of no more than four distinct integral weights?

Solution. Since the numbers 1 to 40 in base 2 have at most 6 digits, we can do it with the set 1, 2, 4, 8, 16, 32. To get a set with fewer weights, we observe that we can put weights from this set on both sides of the balance! Consider the set of weights 1, 3, 9, 27. For example to determine an object with weight 2, we can put it with a weight of 1 on one side to balance a weight of 3 on the other side. Note the sum of 1, 3, 9, 27 is 40. For any integer n between 1 and 40, we can write it in base 3. If the digit 2 appears, change it to $3-1$ so that n can be written as a *unique* sum and difference of 1, 3, 9, 27. For example, $22 = 2\cdot 9 + 3 + 1 = (3-1)9 + 3 + 1 = 27 - 9 + 3 + 1$ suggests we put the weights of 22 with 9 on one side and the weights of 27, 3, 1 on the other side.

Example 5. (*1983 IMO*) Can you choose 1983 pairwise distinct nonnegative integers less than 10^5 such that no three are in arithmetic progression?

Solution. Start with 0, 1 and at each step add the *smallest* integer which is not in arithmetic progression with any two preceding terms. We get 0, 1, 3, 4, 9, 10, 12, 13, 27, 18, … . In base 3, this sequence is

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, \ldots$$

(Note this sequence is the nonnegative integers in base 2.) Since 1982 in base 2 is 11110111110, so switching this from base 3 to base 10, we get the 1983th term of the sequence is $87843 < 10^5$. To see this sequence works, suppose x, y, z with $x < y < z$ are three terms of the sequence in arithmetic progression. Consider the rightmost digit in base 3 where x differs from y, then that digit for z is a 2, a contradiction.

Example 6. Let $[r]$ be the greatest integer less than or equal to r. Solve the equation

$$[x]+[2x]+[4x]+[8x]$$
$$+[16x]+[32x]=12345.$$

Solution. If x is a solution, then since $r-1 < [r] \le r$, we have $63x - 6 < 12345 \le 63x$. It follows that $195 < x < 196$. Now write the number x in base 2 as $11000011.abcde\ldots$, where the digits $a, b, c, d, e, \ldots$ are 0 or 1. Substituting this into the equation, we will get $12285 + 31a + 15b + 7c + 3d + e = 12345$. Then $31a + 15b + 7c + 3d + e = 60$, which is impossible as the left side is at most $31 + 15 + 7 + 3 + 1 = 57$. Therefore, the equation has no solution.

Example 7. (*Proposed by Romania for 1985 IMO*) Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for n = 1, 2, 3, … (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

Solution. Write $\sqrt{2}$ in base 2 as $b_0.b_1b_2b_3\ldots$, where each $b_i = 0$ or 1. Since $\sqrt{2}$ is irrational, there are infinitely many $b_k = 1$. If $b_k = 1$, then in base 2, $2^{k-1}\sqrt{2} = b_0 \cdots b_{k-1}.b_k \cdots$. Let $m = [2^{k-1}\sqrt{2}]$, then

$$2^{k-1}\sqrt{2} - 1 < [2^{k-1}\sqrt{2}] = m < 2^{k-1}\sqrt{2} - \frac{1}{2}.$$

Multiplying by $\sqrt{2}$ and adding $\sqrt{2}$, we get $2^k < (m+1)\sqrt{2} < 2^k + \frac{\sqrt{2}}{2}$. Then $[(m+1)\sqrt{2}] = 2^k$.

Example 8. (*American Mathematical Monthly, Problem 2486*) Let p be an odd prime number. For any positive integer k, show that there exists a positive integer m such that the rightmost k digits of m^2, when expressed in the base p, are all 1's.

Solution. We prove by induction on k. For $k = 1$, take $m = 1$. Next, suppose m^2 in base p, ends in k 1's, i.e.

$$m^2 = 1 + p + \cdots + p^{k-1} + (ap^k + \cdots).$$

This implies m is not divisible by p. Let gcd stand for greatest common divisor (or highest common factor). Then gcd(m, p) = 1. Now

$$(m+cp^k)^2 = m^2 + 2mcp^k + c^2p^{2k}$$
$$= 1 + p + \cdots + p^{k-1} + (a+2mc)p^k + \cdots.$$

Since $\gcd(2m, p) = 1$, there is a positive integer c such that $(2m)c \equiv 1 - a \pmod{p}$. This implies $a + 2mc$ is of the form $1 + Np$ and so $(m+cp^k)^2$ will end in at least $(k+1)$ 1's as required.

Example 9. Determine which binomial coefficients $C_r^n = \frac{n!}{r!(n-r)!}$ are odd.

Solution. We remark that modulo arithmetic may be extended to polynomials with integer coefficients. For example, $(1+x)^2 = 1 + 2x + x^2 \equiv 1 + x^2$ (mod 2). If $n = a_m + \cdots + a_1$, where the a_i's are distinct powers of 2. We have $(1+x)^{2^k} \equiv 1 + x^{2^k}$ (mod 2) by induction on k and so

$$(1+x)^n \equiv (1+x^{a_m})\cdots(1+x^{a_1}) \pmod{2}.$$

The binomial coefficient C_r^n is odd if and only if the coefficient of x^r in $(1+x^{a_m})\cdots(1+x^{a_1})$ is 1, which is equivalent to r being 0 or a sum of one or more of the a_i's. For example, if $n = 21 = 16 + 4 + 1$, then C_r^n is odd for r = 0, 1, 4, 5, 16, 17, 20, 21 only.

Example 10. (*1996 USAMO*) Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

This is a difficult problem. Here we will try to lead the reader to a solution. For a problem that we cannot solve, we can try to change it to an easier problem. How about changing the problem to positive integers, instead of integers? At least we do not have to worry about negative integers. That is still not too obvious how to proceed. So can we change it to an even simpler problem? How about changing 2 to 10?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***June 30, 2001***.

Problem 126. Prove that every integer can be expressed in the form $x^2+y^2-5z^2$, where *x, y, z* are integers.

Problem 127. For positive real numbers *a, b, c* with $a+b+c=abc$, show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\le\frac{3}{2},$$

and determine when equality occurs. (*Source: 1998 South Korean Math Olympiad*)

Problem 128. Let *M* be a point on segment *AB*. Let *AMCD, BEHM* be squares on the same side of *AB*. Let the circumcircles of these squares intersect at *M* and *N*. Show that *B, N, C* are collinear and *H* is the orthocenter of ΔABC. (*Source: 1979 Henan Province Math Competition*)

Problem 129. If *f(x)* is a polynomial of degree 2*m*+1 with integral coefficients for which there are 2*m*+1 integers $k_1,k_2,\ldots,k_{2m+1}$ such that $f(k_i)=1$ for $i=1,2,\cdots,2m+1$, prove that *f(x)* is not the product of two nonconstant polynomials with integral coefficients.

Problem 130. Prove that for each positive integer *n*, there exists a circle in the *xy*-plane which contains exactly *n* lattice points in its interior, where a *lattice point* is a point with integral coordinates. (*Source: H. Steinhaus, Zadanie 498, Matematyka 10 (1957), p. 58*)

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Solutions

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Problem 121. Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.
(Two integers are *relative prime* if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus 37 = 22 + 15 represents the number 37 in the desired way.) (*Source: Second Bay Area Mathematical Olympaid*)

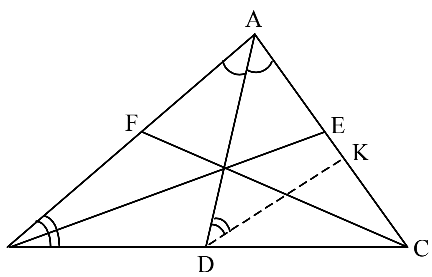
Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 5), **CHONG Fan Fei** (Queen's College, Form 4), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4), **LAW Siu Lun** (Ming Kei College, Form 6), **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College), **WONG Wing Hong** (La Salle College, Form 3) & **YEUNG Kai Sing** (La Salle College, Form 4).

For an integer $n\ge 7$, n is either of the form $2j+1$ ($j>2$) or $4k$($k>1$) or $4k+2$($k>1$). If $n=2j+1$, then j and $j+1$ are relatively prime and $n=j+(j+1)$. If $n=4k$, then $2k-1$ (>1) and $2k+1$ are relatively prime and $n=(2k-1)+(2k+1)$. If $n=4k+2$, then $2k-1$ and $2k+3$ are relatively prime and $n=(2k-1)+(2k+1)$.

Other commended solvers: **HON Chin Wing** (Pui Ching Middle School, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6) & **WONG Tak Wai Alan** (University of Toronto).

Problem 122. Prove that the product of the lengths of the three angle bisectors of a triangle is less than the product of the lengths of the three sides. (*Source: 1957 Shanghai Junior High School Math Competition*).

Solution. **YEUNG Kai Sing** (La Salle College, Form 4).

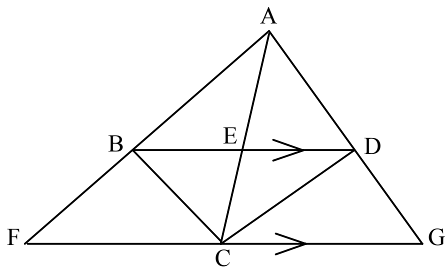


Let *AD, BE* and *CF* be the angle bisectors of ΔABC, where *D* is on *BC, E* is on *CA* and *F* is on *AB*. Since $\angle ADC=\angle ABD+\angle BAD>\angle ABD$, there is a point *K* on *CA* such that $\angle ADK=\angle ABD$. Then ΔABD is similar to ΔADK. So $AB/AD=AD/AK$. Then $AD^2=AB\cdot AK<AB\cdot CA$. Similarly, $BE^2<BC\cdot AB$ and $CF^2<CA\cdot BC$. Multiplying these in-equalities and taking square roots, we get $AD\cdot BE\cdot CF<AB\cdot BC\cdot CA$.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 5), **HON Chin Wing** (Pui Ching Middle School, Form 6) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 123. Show that every convex quadrilateral with area 1 can be covered by some triangle of area at most 2. (*Source: 1989 Wuhu City Math Competition*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).



Let *ABCD* be a convex quadrilateral with area 1. Let *AC* meet *BD* at *E*. Without loss of generality, suppose $AE\ge EC$. Construct ΔAFG, where lines *AB* and *AD* meet the line parallel to *BD* through *C* at *F* and *G* respectively. Then ΔABE is similar to ΔAFC. Now $AE\ge EC$ implies $AB\ge BF$. Let $[XY\cdots Z]$ denote the area of polygon $XY\cdots Z$, then $[ABC]\ge[FBC]$. Similarly, $[ADC]\ge[GDC]$. Since $[ABC]+[ADC]=[ABCD]=1$, so $[AFG]=[ABCD]+[FBC]+[GDC]\le 2[ABCD]=2$ and ΔAFG covers *ABCD*.

Problem 124. Find the least integer *n* such that among every n distinct numbers $a_1, a_2, \ldots, a_n$, chosen from [1,1000], there always exist a_i, a_j such that

$$0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}.$$

(*Source: 1990 Chinese Team Training Test*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

For $n \le 10$, let $a_i = i^3$ $(i = 1, 2, \cdots, n)$. Then the inequality cannot hold since $0 < i^3 - j^3$ implies $i - j \ge 1$ and so $i^3 - j^3 = (i-j)^3 + 3ij(i-j) \ge 1 + 3ij$. For $n = 11$, divide [1,1000] into intervals $[k^3+1, (k+1)^3]$ for k = 0, 1, …, 9. By pigeonhole principle, among any 11 distinct numbers $a_1, a_2, \cdots, a_{11}$ in [1, 1000], there always exist a_i, a_j, say $a_i > a_j$, in the same interval. Let $x = \sqrt[3]{a_i}$ and $y = \sqrt[3]{a_j}$, then $0 < x - y < 1$ and $0 < a_i - a_j = x^3 - y^3 = (x-y)^3 + 3xy(x-y) < 1 + 3xy = 1 + 3\sqrt[3]{a_i a_j}$.

Other commended solvers: **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6), **WONG Wing Hong** (La Salle College, Form 3) & **YEUNG Kai Sing** (La Salle College, Form 4).

Problem 125. Prove that

$$\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + \cdots + \tan^2 89^\circ$$

is an integer.

Solution. **CHAO Khek Lun** (St. Paul's College, Form 6).

For $\theta = 1^\circ, 3^\circ, 5^\circ, \ldots, 89^\circ$, we have $\cos\theta \ne 0$ and $\cos 90\theta = 0$. By de Moivre's theorem, $\cos 90\theta + i \sin 90\theta = (\cos\theta + i\sin\theta)^{90}$. Taking the real part of both sides, we get

$$0 = \sum_{k=0}^{45} (-1)^k C_{2k}^{90} \cos^{90-2k}\theta \sin^{2k}\theta.$$

Dividing by $\cos^{90}\theta$ on both sides and letting $x = \tan^2\theta$, we get

$$0 = \sum_{k=0}^{45} (-1)^k C_{2k}^{90} x^k.$$

So $\tan^2 1^\circ, \tan^2 3, \tan^2 5^\circ, \ldots, \tan^2 89^\circ$ are the 45 roots of this equation. Therefore, their sum is $C_{88}^{90} = 4005$.

Olympiad Corner

(continued from page 1)

How many of these permutations will have no terms of the sequence s_1, …, s_{100} divisible by three?

Problem 3. Let $A = (a_1, a_2, \cdots, a_{2000})$ be a sequence of integers each lying in the interval [-1000, 1000]. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

Problem 4. Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$
$$\angle ABD = 2\angle CDB$$

and $AB = CB$.

Prove that $AD = CD$.

Problem 5. Suppose that the real numbers $a_1, a_2, \cdots, a_{100}$ satisfy

$$a_1 \ge a_2 \ge \cdots \ge a_{100} \ge 0,$$
$$a_1 + a_2 \le 100$$

and $a_3 + a_4 + \cdots + a_{100} \le 100$.

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \cdots, a_{100}$ which achieve this maximum.

Base n Representations

(continued from page 2)

Now try an example, say n = 12345. We can write n in more than one ways in the form $a + 10b$. Remember we want a, b to be unique in the set X. Now for b in X, $10b$ will *shift* the digits of b to the left one space and fill the last digit with a 0. Now we can try writing n = 12345 = 10305 + 10(204). So if we take X to be the positive integers whose even position digits from the end are 0, then the problem will be solved for $n = a + 10b$. How about $n = a + 2b$? If the reader examines the reasoning in the case $a + 10b$, it is easy to see the success comes from separating the digits and observing that multiplying by 10 is a shifting operation in base 10. So for $a + 2b$, we take X to be the set of positive integers whose base 2 even position digits from the end are 0, then the problem is solved for positive integers.

How about the original problem with *integers*? It is tempting to let X be the set of positive or negative integers whose base 2 even position digits from the end are 0. It does not work as the example $1 + 2\cdot 1 = 3 = 5 + 2(-1)$ shows uniqueness fails. Now what other ways can we describe the set X we used in the last paragraph? Note it is also the set of positive integers whose base 4 representations have only digits 0 or 1. How can we take care of uniqueness and negative integers at the same time? One idea that comes close is the Bachet weights.

The brilliant idea in the official solution of the 1996 USAMO is do things in base (−4). That is, show every integer has a unique representation as $\sum_{i=0}^{k} c_i (-4)^i$, where each c_i = 0, 1, 2 or 3 and $c_k \ne 0$. Then let X be the set of integers whose base (−4) representations have only $c_i = 0$ or 1 will solve the problem.

To show that an integer n has a base (−4) representation, find an integer m such that $4^0 + 4^2 + \cdots + 4^{2m} \ge n$ and express

$$n + 3\ (4^1 + 4^3 + \cdots + 4^{2m-1})$$

in base 4 as $\sum_{i=0}^{2m} b_i 4^i$. Now set $c_{2i} = b_{2i}$ and $c_{2i-1} = 3 - b_{2i-1}$. Then $n = \sum_{i=0}^{2m} c_i (-4)^i$.

To show the uniqueness of base (−4) representation of n, suppose n has two distinct representations with digits c_i's and d_i's. Let j be the smallest integer such that $c_j \ne d_j$. Then

$$0 = n - n = \sum_{i=j}^{k} (c_i - d_i)(-4)^i$$

would have a nonzero remainder when divided by 4^{j+1}, a contradiction.

Mathematical Excalibur

Volume 6, Number 3 **June 2001 – October 2001**

Olympiad Corner

The 42nd International Mathematical Olympiad, Washington DC, USA, 8-9 July 2001

Problem 1. Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

Problem 3. Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

(continued on page 4)

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 10, 2001***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Pell's Equation (I)

Kin Y. Li

Let d be a positive integer that is not a square. The equation $x^2 - dy^2 = 1$ with variables x, y over integers is called ***Pell's equation***. It was Euler who attributed the equation to John Pell (1611-1685), although Brahmagupta (7th century), Bhaskara (12th century) and Fermat had studied the equation in details earlier.

A solution (x, y) of Pell's equation is called *positive* if both x and y are positive integers. Hence, positive solutions correspond to the lattice points in the first quadrant that lie on the hyperbola $x^2 - dy^2 = 1$. A positive solution (x_1, y_1) is called the *least positive solution* (or *fundamental solution*) if it satisfies $x_1 < x$ and $y_1 < y$ for every other positive solution (x, y). (As the hyperbola $x^2 - dy^2 = 1$ is strictly increasing in the first quadrant, the conditions for being least are the same as requiring $x_1 + y_1\sqrt{d} < x + y\sqrt{d}$.)

Theorem. *Pell's equation $x^2 - dy^2 = 1$ has infinitely many positive solutions. If (x_1, y_1) is the least positive solution, then for $n = 1, 2, 3, \ldots,$ define*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.$$

The pairs (x_n, y_n) are all the positive solutions of the Pell's equation. The x_n's and y_n's are strictly increasing to infinity and satisfy the recurrence relations $x_{n+2} = 2x_1x_{n+1} - x_n$ and $y_{n+2} = 2x_1y_{n+1} - y_n$.

We will comment on the proof. The least positive solution is obtained by writing $\sqrt{d}$ as a simple continued fraction. It turns out

$$\sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where $a_0 = [\sqrt{d}]$ and $a_1, a_2, \ldots$ is a periodic positive integer sequence. The continued fraction will be denoted by $\langle a_0, a_1, a_2, \ldots \rangle$. The *k-th convergent* of $\langle a_0, a_1, a_2, \ldots \rangle$ is the number $\frac{p_k}{q_k} = \langle a_0, a_1, a_2, \ldots, a_k \rangle$ with p_k, q_k relatively prime. Let $a_1, a_2, \ldots, a_m$ be the period for $\sqrt{d}$. The least positive solution of Pell's equation turns out to be

$$(x_1, y_1) = \begin{cases} (p_{m-1}, q_{m-1}) & \text{if } m \text{ is even} \\ (p_{2m-1}, q_{2m-1}) & \text{if } m \text{ is odd} \end{cases}.$$

For example, $\sqrt{3} = \langle 1, 1, 2, 1, 2, \ldots \rangle$ and so $m = 2$, then $\langle 1, 1 \rangle = \frac{2}{1}$. We check $2^2 - 3 \cdot 1^2 = 1$ and clearly, $(2, 1)$ is the least positive solution of $x^2 - 3y^2 = 1$. Next, $\sqrt{2} = \langle 1, 2, 2, \ldots \rangle$ and so $m = 1$, then $\langle 1, 2 \rangle = \frac{3}{2}$. We check $3^2 - 2 \cdot 2^2 = 1$ and again clearly $(3, 2)$ is the least positive solution of $x^2 - 2y^2 = 1$.

Next, if there is a positive solution (x, y) such that $x_n + y_n\sqrt{d} < x + y\sqrt{d} < x_{n+1} + y_{n+1}\sqrt{d}$, then consider $u + v\sqrt{d} = (x + y\sqrt{d})/(x_n + y_n\sqrt{d})$. We will get $u + v\sqrt{d} < x_1 + y_1\sqrt{d}$ and $u - v\sqrt{d} = (x - y\sqrt{d})/(x_n - y_n\sqrt{d})$ so that $u^2 - dv^2 = (u - v\sqrt{d})(u + v\sqrt{d}) = 1$, con-tradicting (x_1, y_1) being the least positive solution.

To obtain the recurrence relations, note that

$$(x_1+y_1\sqrt{d})^2 = x_1^2+dy_1^2+2x_1y_1\sqrt{d}$$
$$=2x_1^2-1+2x_1y_1\sqrt{d}$$
$$=2x_1(x_1+y_1\sqrt{d})-1.$$

So

$$x_{n+2}+y_{n+2}\sqrt{d}$$
$$=(x_1+y_1\sqrt{d})^2(x_1+y_1\sqrt{d})^n$$
$$=2x_1(x_1+y_1\sqrt{d})^{n+1}-(x_1+y_1\sqrt{d})^n$$
$$=2x_1x_{n+1}-x_n+(2x_1y_{n+1}-y_n)\sqrt{d}.$$

The related equation $x^2-dy^2=-1$ may not have a solution, for example, $x^2-3y^2=-1$ cannot hold as $x^2-3y^2\equiv x^2+y^2\neq -1$ (mod 4). However, if d is a prime and $d\equiv 1$ (mod 4), then a theorem of Lagrange asserts that it will have a solution. In general, if $x^2-dy^2=-1$ has a least positive solution (x_1,y_1), then all its positive solutions are pairs (x, y), where $x+y\sqrt{d}=(x_1+y_1\sqrt{d})^{2n-1}$ for some positive integer n.

In passing, we remark that some k-th convergent numbers are special. If the length m of the period for $\sqrt{d}$ is even, then $x^2-dy^2=1$ has $(x_n,y_n)=(p_{nm-1},q_{nm-1})$ as all its positive solutions, but $x^2-dy^2=-1$ has no integer solution. If m is odd, then $x^2-dy^2=1$ has (p_{jm-1},y_{jm-1}) with j even as all its positive solutions and $x^2-dy^2=-1$ has (p_{jm-1},q_{jm-1}) with j odd as all its positive solutions.

Example 1. Prove that there are infinitely many triples of consecutive integers each of which is a sum of two squares.

Solution. The first such triple is $8=2^2+2^2$, $9=3^2+0^2$, $10=3^2+1^2$, which suggests we consider triples x^2-1, x^2, x^2+1. Since $x^2-2y^2=1$ has infinitely many positive solutions (x, y), we see that $x^2-1=y^2+y^2$, $x^2=x^2+0^2$ and x^2+1 satisfy the requirement and there are infinitely many such triples.

Example 2. Find all triangles whose sides are consecutive integers and areas are also integers.

Solution. Let the sides be $z-1, z, z+1$. Then the semiperimeter $s=\frac{3z}{2}$ and the area is $A=\frac{z\sqrt{3(z^2-4)}}{4}$. If A is an integer, then z cannot be odd, say $z = 2x$, and $z^2-4=3\omega^2$. So $4x^2-4=3\omega^2$, which implies ω is even, say $\omega=2y$. Then $x^2-3y^2=1$, which has $(x_1,y_1)=$ (2, 1) as the least positive solution. So all positive solutions are (x_n,y_n), where $x_n+y_n\sqrt{3}=(2+\sqrt{3})^n$. Now $x_n-y_n\sqrt{3}=(2-\sqrt{3})^n$. Hence,

$$x_n=\frac{(2+\sqrt{3})^n+(2-\sqrt{3})^n}{2}$$

and

$$y_n=\frac{(2+\sqrt{3})^n-(2-\sqrt{3})^n}{2\sqrt{3}}.$$

The sides of the triangles are $2x_n-1$, $2x_n, 2x_n+1$ and the areas are $A=3x_ny_n$.

Example 3. Find all positive integers k, m such that $k<m$ and

$$1+2+\cdots+k=(k+1)+(k+2)+\cdots+m.$$

Solution. Adding $1+2+\cdots+k$ to both sides, we get $2k(k+1)=m(m+1)$, which can be rewritten as $(2m+1)^2-2(2k+1)^2=-1$. Now the equation $x^2-2y^2=-1$ has $(1,1)$ as its least positive solution. So its positive solutions are pairs $x_n+y_n\sqrt{2}=(1+\sqrt{2})^{2n-1}$. Then

$$x_n=\frac{(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}}{2}$$

and

$$y_n=\frac{(1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1}}{2\sqrt{2}}.$$

Since $x^2-2y^2=-1$ implies x is odd, so x is of the form $2m+1$. Then $y^2=2m^2+m+1$ implies y is odd, so y is of the form $2k+1$. Then $(k,m)=\left(\frac{y_n-1}{2},\ \frac{x_n-1}{2}\right)$ with $n = 2, 3, 4, \ldots$ are all the solutions.

Example 4. Prove that there are infinitely many positive integers n such that n^2+1 divides $n!$.

Solution. The equation $x^2-5y^2=-1$ has $(2,1)$ as the least positive solution. So it has infinitely many positive solutions. Consider those solutions with $y>5$. Then $5<y<2y\le x$ as $4y^2\le 5y^2-1=x^2$. So $2(x^2+1)=5\cdot y\cdot 2y$ divides $x!$, which is more than we want.

Example 5. For the sequence $a_n=\left[\sqrt{n^2+(n+1)^2}\right]$, prove that there are infinitely many n's such that $a_n-a_{n+1}>1$ and $a_{n+1}-a_n=1$.

Solution. First consider the case $n^2+(n+1)^2=y^2$, which can be rewritten as $(2n+1)^2-2y^2=-1$. As in example 3 above, $x^2-2y^2=-1$ has infinitely many positive solutions and each x is odd, say $x=2n+1$ for some n. For these n's, $a_n=y$ and $a_{n-1}=\left[\sqrt{(n-1)^2+n^2}\right]=\left[\sqrt{y^2-4n}\right]$. The equation $y^2=n^2+(n+1)^2$ implies $n>2$ and $a_{n-1}\le\sqrt{y^2-4n}<y-1=a_n-1$. So $a_n-a_{n-1}>1$ for these n's.

Also, for these n's, $a_{n+1}=\left[\sqrt{(n+1)^2+(n+2)^2}\right]=\left[\sqrt{y^2+4n+4}\right]$.

As $n<y<2n+1$, we easily get $y+1<\sqrt{y^2+4n+4}<y+2$. So $a_{n+1}-a_n=(y+1)-y=1$.

Example 6. (American Math Monthly E2606, proposed by R.S. Luthar) Show that there are infinitely many integers n such that $2n+1$ and $3n+1$ are perfect squares, and that such n must be multiples of 40.

Solution. Consider $2n+1=u^2$ and $3n+1=v^2$. On one hand, $u^2+v^2\equiv 2$ (mod 5) implies u^2, $v^2\equiv 1$ (mod 5), which means n is a multiple of 5.

On the other hand, we have $3u^2-2v^2=1$. Setting $u=x+2y$ and $v=x+3y$, the equation becomes $x^2-6y^2=1$. It has infinitely many positive solutions. Since $3u^2-2v^2=1$, u is odd, say $u=2k+1$. Then $n=2k^2+2k$ is even. Since $3n+1=v^2$, so v is odd, say $v=4m\pm 1$. Then $3n=16m^2\pm 8m$, which implies n is also a multiple of 8.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***November 10, 2001***.

Problem 131. Find the greatest common divisor (or highest common factor) of the numbers $n^n - n$ for $n = 3$, 5, 7, … .

Problem 132. Points *D, E, F* are chosen on sides *AB, BC, CA* of ΔABC, respectively, so that *DE* = *BE* and *FE* = *CE*. Prove that the center of the circumcircle of ΔADF lies on the angle bisector of $\angle DEF$. (*Source: 1989 USSR Math Olympiad*)

Problem 133. (a) Are there real numbers a and b such that $a+b$ is rational and $a^n + b^n$ is irrational for every integer $n \ge 2$? (b) Are there real numbers *a* and *b* such that $a+b$ is irrational and $a^n + b^n$ is rational for every integer $n \ge 2$? (*Source: 1989 USSR Math Olympiad*)

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (*Source: 2001 Bulgarian Winter Math Competition*)

Problem 135. Show that for $n \ge 2$, if $a_1, a_2, \ldots, a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \ge$$
$$(a_1^2 a_2+1)(a_2^2 a_3+1)\cdots(a_n^2 a_1+1).$$

(*Source: 7th Czech-Slovak-Polish Match*)

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Solutions

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Problem 126. Prove that every integer can be expressed in the form $x^2 + y^2 - 5z^2$, where *x, y, z* are integers.

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5), **IP Ivan** (St. Joseph's College, Form 6), **KOO Koopa** (Boston College, Sophomore), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LAU Wai Shun** (Tsuen Wan Public Ho Chuen Yiu Memorial College, Form 6), **LEE Kevin** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC IA Wong Tai Shan Memorial College), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **YEUNG Kai Sing** (La Salle College, Form 5) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

For n odd, say $n = 2k + 1$, we have $(2k)^2 + (k+1)^2 - 5k^2 = 2k + 1 = n$. For n even, say $n = 2k$, we have $(2k-1)^2 + (k-2)^2 - 5(k-1)^2 = 2k = n$.

Problem 127. For positive real numbers *a, b, c* with *a* + *b* + *c* = *abc*, show that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \le \frac{3}{2},$$

and determine when equality occurs. (*Source: 1998 South Korean Math Olympiad*)

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **KOO Koopa** (Boston College, Sophomore), **LEE Kevin** (La Salle College, Form 6) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Let $A = \tan^{-1} a$, $B = \tan^{-1} b$, $C = \tan^{-1} c$. Since *a, b, c* > 0, we have $0 < A, B, C < \frac{\pi}{2}$. Now *a* + *b* + *c* = *abc* is the same as tan *A* + tan *B* + tan *C* = tan *A* tan *B* tan *C*. Then

$$\tan C = \frac{-(\tan A + \tan B)}{1 - \tan A \tan B} = \tan(\pi - A - B)$$

which implies $A + B + C = \pi$. In terms of *A, B, C* the inequality to be proved is cos *A* + cos *B* + cos $C \le \frac{3}{2}$, which follows by applying Jensen's inequality to *f*(*x*) = cos *x* on $(0, \frac{\pi}{2})$.

Other commended solvers: **CHENG Man Chuen** (CUHK, Math Major, Year 1), **IP Ivan** (St. Joseph's College, Form 6), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **TSUI Ka Ho** (Hoi Ping Chamber of Commerce Secondary School, Form 7), **WONG Wing Hong** (La Salle College, Form 4) and **YEUNG Kai Sing** (La Salle College, Form 5).

Problem 128. Let *M* be a point on segment *AB*. Let *AMCD, BEHM* be squares on the same side of *AB*. Let the circumcircles of these squares intersect at *M* and *N*. Show that *B, N, C* are collinear and *H* is the orthocenter of ΔABC. (*Source: 1979 Henan Province Math Competition*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

Since $\angle BNM = \angle BHM = 45^\circ = \angle CDM = \angle CDM$, it follows *B, N, C* are collinear. Next, $CH \perp AB$. Also, $BH \perp ME$ and $ME \| AC$ imply $BH \perp AC$. So *H* is the orthocenter of ΔABC.

Other commended solvers: **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **IP Ivan** (St. Joseph's College, Form 6), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **LAM Shek Ming Sherman** (La Salle College, Form 6), **Lee Kevin** (La Salle College, Form 6), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **YEUNG Kai Sing** (La Salle College, Form 5).

Problem 129. If *f(x)* is a polynomial of degree 2*m* + 1 with integral coefficients for which there are 2*m* + 1 integers $k_1, k_2, \ldots, k_{2m+1}$ such that $f(k_i) = 1$ for *i* = 1, 2, …, 2*m* + 1, prove that *f(x)* is not the product of two nonconstant polynomials with integral coefficients.

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **IP Ivan** (St. Joseph's College, Form 6), **KOO Koopa** (Boston College, Sophomore), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LEE Kevin** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form

7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College**),** **YEUNG Kai Sing** (La Salle College, Form 5) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

Suppose f is the product of two non-constant polynomials with integral co-efficients, say $f = PQ$. Since $1 = f(k_i) = P(k_i)\,Q(k_i)$ and $P(k_i)$, $Q(k_i)$ are integers, so either both are 1 or both are –1. As there are $2m + 1$ k_i's, either $P(k_i) = Q(k_i) = 1$ for at least $m + 1$ k_i's or $P(k_i) = Q(k_i) = -1$ for at least $m + 1$ k_i's. Since deg f = $2m + 1$, one of deg P or deg Q is at most m. This forces P or Q to be a constant polynomial, a contradiction.

Other commended solvers: **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Problem 130. Prove that for each positive integer n, there exists a circle in the xy-plane which contains exactly n lattice points in its interior, where a *lattice point* is a point with integral coordinates. (*Source: H. Steinhaus, Zadanie 498, Matematyka 10 (1957), p. 58*)

Solution. **CHENG Man Chuen** (CUHK, Math Major, Year 1) and **IP Ivan** (St. Joseph's College, Form 6).

Let $P = \left(\sqrt{2}, \frac{1}{3}\right)$. Suppose lattice points $(x_0, y_0), (x_1, y_1)$ are the same distance from P. Then $\left(x_0 - \sqrt{2}\right)^2 + \left(y_0 - \frac{1}{3}\right)^2 = \left(x_1 - \sqrt{2}\right)^2 + \left(y_1 - \frac{1}{3}\right)^2$. Moving the x terms to the left, the y terms to the right and factoring, we get

$$(x_0 - x_1)\left(x_0 + x_1 - 2\sqrt{2}\right)$$

$$= (y_0 - y_1)\left(y_0 + y_1 - \frac{2}{3}\right).$$

As the right side is rational and $\sqrt{2}$ is irrational, we must have $x_0 = x_1$. Then the left side is 0, which forces $y_1 = y_0$ since $y_1 + y_0$ is integer. So the lattice points are the same.

Now consider the circle with center at P and radius r. As r increases from 0 to infinity, the number of lattice points inside the circle increase from 0 to infinity. As the last paragraph shows, the increase cannot jump by 2 or more. So the statement is true.

Other commended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **KOO Koopa** (Boston College, Sophomore), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **NG Ka Chun** (Queen Elizabeth School, Form 7) and **YEUNG Kai Sing** (La Salle College, Form 4).

Olympiad Corner

(continued from page 1)

Problem 4. Let n be an odd integer greater than 1, let $k_1, k_2, ..., k_n$ be given integers. For each of the n! permutations $a = (a_1, a_2, ..., a_n)$ of 1, 2, …, n, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that n! is a divisor of $S(b) – S(c)$.

Problem 5. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$ac + bd = (b + d + a – c)(b + d – a + c)$.

Prove that $ab + cd$ is not prime.

Pell's Equation (I)

(continued from page 2)

Example 7. Prove that the only positive integral solution of $5^a - 3^b = 2$ is $a = b = 1$.

Solution. Clearly, if a or b is 1, then the other one is 1, too. Suppose (a, b) is a solution with both $a, b > 1$. Considering (mod 4), we have $1 – (–1)^b \equiv 2$ (mod 4), which implies b is odd. Considering (mod 3), we have $(-1)^a \equiv 2$ (mod 3), which implies a is odd.

Setting $x = 3^b + 1$ and $y = 3^{(b-1)/2}\,5^{(a-1)/2}$, we get $15y^2 = 3^b 5^a = 3^b(3^b + 2) = (3^b + 1)^2 - 1 = x^2 - 1$. So (x, y) is a positive solution of $x^2 - 15y^2 = 1$. The least positive solution is $(4,1)$. Then $(x, y) = (x_n, y_n)$ for some positive integer n, where $x_n + y_n\sqrt{15} = (4 + \sqrt{15})^n$. After examining the first few y_n's, we observe that y_{3k} are the only terms that are divisible by 3. However, they also seem to be divisible by 7, hence cannot be of the form $3^c 5^d$.

To confirm this, we use the recurrence relations on y_n. Since $y_1 = 1$, $y_2 = 8$ and $y_{n+2} = 8y_{n+1} - y_n$, taking y_n (mod 3), we get the sequence 1, 2, 0, 1, 2, 0… and taking y_n (mod 7), we get 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ….

Therefore, no $y = y_n$ is of the form $3^c 5^d$ and $a, b > 1$ cannot be solution to $5^a - 3^b = 2$.

Example 8. Show that the equation $a^2 + b^3 = c^4$ has infinitely many solutions.

Solution. We will use the identity

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

which is a standard exercise of mathematical induction. From the identity, we get $\left(\frac{(n-1)n}{2}\right)^2 + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n > 1$. All we need to do now is to show there are infinitely many positive integers n such that $n(n + 1)/2 = k^2$ for some positive integers k. Then $(a, b, c) = ((n – 1)n/2, n, k)$ solves the problem.

Now $n(n + 1)/2 = k^2$ can be rewritten as $(2n+1)^2 - 2(2k)^2 = 1$. We know $x^2 - 2y^2 = 1$ has infinitely many positive solutions. For any such (x, y), clearly x is odd, say $x = 2m + 1$. They $y^2 = 2m^2 + 2m$ implies y is even. So any such (x, y) is of the form $(2n+1,\ 2k)$. Therefore, there are infinitely many such n.

Mathematical Excalibur

Volume 6, Number 4 October 2001 – December 2001

Olympiad Corner

The 18th Balkan Mathematical Olympiad, Belgrade, Yugoslavia, 5 May 2001

Problem 1. Let n be a positive integer. Show that if a and b are integers greater then 1 such that $2^n - 1 = ab$, then the number $ab - (a - b) - 1$ is of the form $k \cdot 2^{2m}$, where k is odd and m is a positive integer.

Problem 2. Prove that if a convex pentagon satisfies the following conditions:
(1) all interior angles are congruent; and
(2) the lengths of all sides are rational numbers,
then it is a regular pentagon.

Problem 3. Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

Problem 4. A cube of dimensions $3\times3\times3$ is divided into 27 congruent unit cubical cells.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***15 January 2001***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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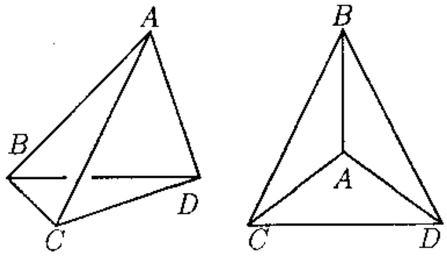
六個頂點的多面體

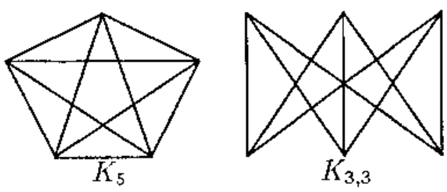
吳主居 **(Richard Travis NG)**

圖一左顯示一個四面體，用邊作輪廓，如將底面擴張，然後把其他的邊壓下去，可得到一個平面圖，各邊衹在頂點處相交，如圖一右所示。



（插圖一）

非平面圖不能是多面體的輪廓，最基本的非平面圖有兩個。第一個有五個頂點，兩兩相連，稱為 K_5，見圖二左。第二個有六個頂點，分為兩組，各有三個，同組的互不相連，不同組的則兩兩相連，稱為 $K_{3,3}$，見圖二右。



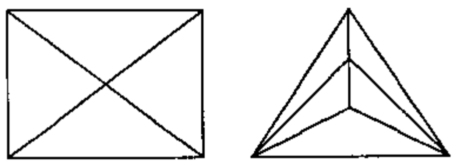
（插圖二）

一個頂點所在邊上的數量，稱為它的度數，一個代表多面體的平面圖，每個頂點的度數，都不能少於 3，所以任何多面體，都不少於四個頂點。假如它衹有四個頂點，它們的度數必定是 (3，3，3，3)，唯一的可能就是圖一的四面體。

假如一個多面體有五個頂點，看來它們的度數可能會是：

(3，3，3，3，3)，(3，3，3，3，4)，(3，3，3，4，4)，(3，3，4，4，4)，(3，4，4，4，4) 或 (4，4，4，4，4)。

不過很快便會發現，左面那三組是不可能的，因為各頂點度數之和，必定是邊數的雙倍，不可能是奇數。右面第一組是個四邊形為底的金字塔，見圖三左，第二組是個三角形為底的雙金字塔，見圖三右。最後一組是 K_5，不是平面圖，不能代表多面體。

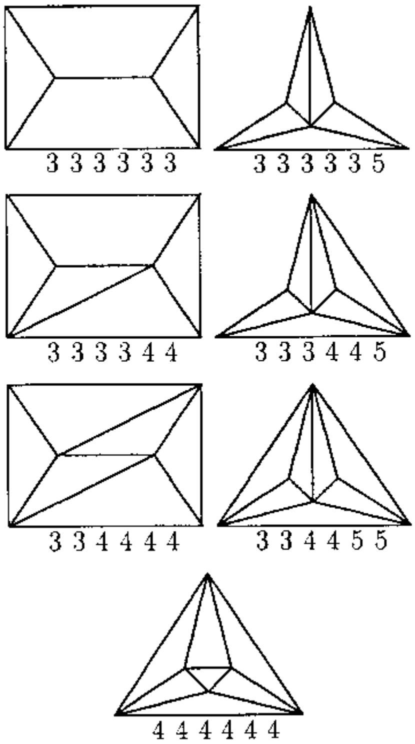


（插圖三）

六個頂點的多面體，有多少個呢？每個頂點的度數，都是 3，4 或 5，有下列可能:

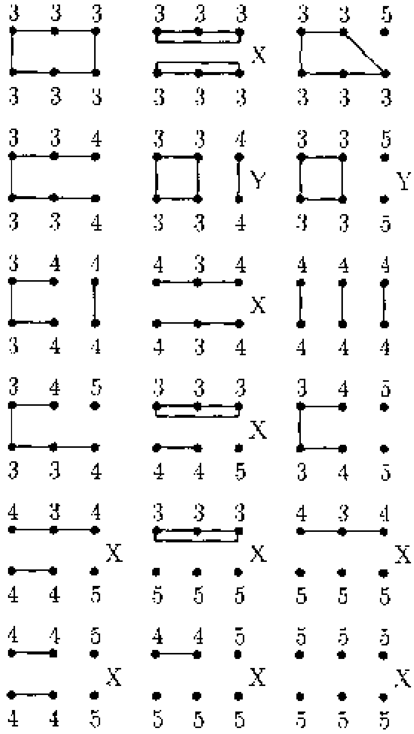
(3，3，3，3，3，3)，(3，3，3，3，3，5)，
(3，3，3，3，4，4)，(3，3，3，3，5，5)，
(3，3，3，4，4，5)，(3，3，4，4，4，4)，
(3，3，3，5，5，5)，(3，3，4，4，5，5)，
(3，4，4，4，4，5)，(4，4，4，4，4，4)，
(3，3，5，5，5，5)，(3，4，4，5，5，5)，
(4，4，4，4，5，5)，(3，5，5，5，5，5)，
(4，4，5，5，5，5)或(5，5，5，5，5，5)。

這十六組其中四組，有兩種表達方式，所以共有二十種情況，我們發現有七個不同的多面體，見圖四。



（插圖四）

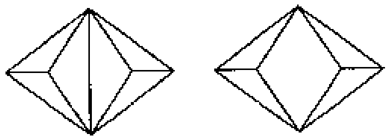
最後證明，就衹有這七種。我們先試劃這些圖，因為頂點太多兩兩相連，衹劃出缺了的邊比較容易，亦立即發現 (3，3，5，5，5，5) 和 (3，5，5，5，5，5) 這兩組是不能成立的，其餘十八種在圖五列出。



（插圖五）

沒做記號那七組，代表我們那七種多面體，用 X 做記號的，都含有 $K_{3,3}$ 在內，所以不可能是平面圖。用 Y 做記號的，雖然它們都是平面圖，但不能代表多面體。

先看圖六左的 (3，3，3，3，5，5)，它僅能代表兩個共邊的四面體，不是一個多面體。再看圖六右 (3，3，3，3，4，4) 的第二種情況，兩個度數為 4 的頂點互不相連，兩個四邊形的面，有兩個不相鄰的公共頂點，這也是不可能的。



（插圖六）

Remarks by Professor Andy Liu
(University of Alberta, Canada)

Polyhedra with Six Vertices is the work of Richard Travis Ng, currently a Grade 12 student at Archbishop MacDonald High School in Edmonton, Canada. The result is equivalent to that in John McClellan's *The Hexahedra Problem* (Recreational Mathematics Magazine, 4, 1961, 34-40), which counts the number of polyhedra with six faces. The problem is also featured in Martin Gardner's "New Mathematical Dviersions" (Mathematical Association of America, 1995, 224-225 and 233). However, the proof in this article is much simpler.

The 2001 Hong Kong IMO team with Professor Andrew Wiles at Washington, DC taken on July 13, 2001. From left to right, *Leung Wai Ying, Yu Hok Pun, Ko Man Ho, Professor Wiles, Cheng Kei Tsi, Chan Kin Hang, Chao Khek Lun*.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***15 January 2001.***

Problem 136. For a triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are rational, prove that $\cos A$, $\cos B$, $\cos C$ must also be rational.
If $\cos A$, $\cos B$, $\cos C$ are rational, must at least one of $\sin A$, $\sin B$, $\sin C$ be rational?

Problem 137. Prove that for every positive integer n,

$$(\sqrt{3}+\sqrt{2})^{1/n}+(\sqrt{3}-\sqrt{2})^{1/n}$$

is irrational.

Problem 138. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) If $a+b$ and $a-b$ are relatively prime integers, find the greatest common divisor (or the highest common factor) of $2a+(1+2a)(a^2-b^2)$ and $2a(a^2+2a-b^2)(a^2-b^2)$.

Problem 139. Let a line intersect a pair of concentric circles at points *A, B, C, D* in that order. Let E be on the outer circle and F be on the inner circle such that chords AE and BF are parallel. Let G and H be points on chords BF and AE that are the feet of perpendiculars from C to BF and from D to AE, respectively. Prove that $EH = FG$. (*Source: 1958 Shanghai City Math Competition*)

Problem 140. A convex pentagon has five equal sides. Prove that the interior of the five circles with the five sides as diameters do not cover the interior of the pentagon.

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Solutions

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Problem 131. Find the greatest common divisor (or highest common factor) of the numbers $n^n - n$ for $n = 3, 5, 7, \ldots$.

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **Jack LAU Wai Shun** (Tsuen Wan Public Ho Chuen Yiu Memorial College, Form 6), **LEE Tsun Man Clement** (St. Paul's College, Form 3), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **Boris YIM Shing Yik** (Wah Yan College, Kowloon) and **YUEN Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 6).

Since the smallest number is $3^3-3=24$, the greatest common divisor is at most 24. For $n=2k+1$,

$$n^n-n=n\left((n^2)^k-1\right)$$

$$=(n-1)n(n+1)(n^{2k-2}+\cdots+1).$$

Now one of $n-1,\ n,\ n+1$ is divisible by 3. Also, $(n-1)(n+1)=4k(k+1)$ is divisible by 8. So n^n-n is divisible by 24. Therefore, the greatest common divisor is 24.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **LAW Siu Lun** (CCC Ming Kei College, Form 7), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **Campion LOONG** (STFA Leung Kau Kui College, Form 6), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Ho Chung** (Queen's College, Form 3), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 7), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **Tak Wai Alan WONG** (University of Toronto, Canada), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Problem 132. Points *D, E, F* are chosen on sides *AB, BC, CA* of ΔABC, respectively, so that $DE = BE$ and $FE = CE$. Prove that the center of the circumcircle of ΔADF lies on the angle bisector of $\angle DEF$. (*Source: 1989 USSR Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **KWONG Tin Yan** (True Light Girls' College, Form 6), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Ho Chung** (Queen's College, Form 3), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

Let O be the circumcenter of ΔADF and α, β, γ be the measures of angles A, B, C of ΔABC . Then $\angle DOF = 2\alpha$ and $180^\circ - \angle DEF =$ $\angle BED + \angle CEF$ $= 360^\circ - 2\beta - 2\gamma =$ $2\alpha = \angle DOF$. So $ODEF$ is a cyclic quadrilateral. Since $OD = OF$, $\angle DEO$ $= \angle FEO$. So O is on the angle bisector of $\angle DEF$.

Other commended solvers: **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Problem 133. (a) Are there real numbers a and b such that $a+b$ is rational and a^n+b^n is irrational for every integer $n \ge 2$? (b) Are there real numbers a and b such that $a+b$ is irrational and a^n+b^n is rational for every integer $n \ge 2$? (*Source: 1989 USSR Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

(a) Let $a = \sqrt{2}+1$ and $b = -\sqrt{2}$. Then $a+b=1$ is rational. For an integer $n \geq 2$, from the binomial theorem, since binomial coefficients are positive integers, we get

$$(\sqrt{2}+1)^n = r_n\sqrt{2}+s_n,$$

where r_n, s_n are positive integers. For every positive integer k, we have $a^{2k} + b^{2k} = r_{2k}\sqrt{2} + s_{2k} + 2^k$ and $a^{2k+1} + b^{2k+1} = (r_{2k+1}-2^k)\sqrt{2} + s_{2k+1}$. Since

$$r_{2k+1} \geq 2^k + C_2^{2k+1}2^{k-1} > 2^k,$$

a^n+b^n is irrational for $n \geq 2$.

(b) Suppose such a and b exist. Then neither of them can be zero from cases $n = 2$ and 3. Now

$$(a^2+b^2)^2 = (a^4+b^4)+2a^2b^2$$

implies a^2b^2 is rational, but then

$$(a^2+b^2)(a^3+b^3)$$
$$=(a^5+b^5)+a^2b^2(a+b)$$

will imply $a+b$ is rational, which is a contradiction.

Other commended solvers: **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TSUI Chun Wa** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (*Source: 2001 Bulgarian Winter Math Competition*)

Solution. (*Official Solution)*

Peter may use the following strategy: he plans to write three 1's and 1998 0's, until Ivan begins to write a 1. Once Ivan writes his first 1, then Peter will switch to follow Ivan exactly from that point to the end.

If Peter succeeded to write three 1's and 1998 0's, then Ivan wrote only 0's and the number formed would be 21×4^{1998}. This is not the sum of two perfect squares since 21 is not the sum of two perfect squares.

If Ivan wrote a 1 at some point, then Peter's strategy would cause the number to have an even number of 0's on the right preceded by two 1's. Hence, the number would be of the form $(4n+3)4^m$. This kind of numbers are also not the sums of two perfect squares, otherwise we have integers x, y such that

$$x^2+y^2=(4n+3)4^m,$$

which implies x, y are both even if m is a positive integer. Keep on canceling 2 from both x and y. Then at the end, we will get $4n + 3$ as a sum of two perfect squares, which is impossible by checking the sum of odd and even perfect squares.

Other commended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Problem 135. Show that for $n \geq 2$, if $a_1, a_2, ..., a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \geq$$
$$(a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1).$$

(*Source: 7th Czech-Slovak-Polish Match*)

Solution 1. **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

First we shall prove that

$$\left(a_1^3+1\right)^2\left(a_2^3+1\right) \geq \left(a_1^2a_2+1\right)^3.$$

By expansion, this is the same as

$$a_1^6a_2^3+2a_1^3a_2^3+a_2^3+a_1^6+2a_1^3+1$$
$$\geq a_1^6a_2^3+3a_1^4a_2^2+3a_1^2a_2+1\,.$$

This follows by regrouping and factoring to get

$$a_1^3(a_1-a_2)^2(a_1+2a_2)$$
$$+\ (a_1-a_2)^2(2a_1+a_2) \geq 0$$

or from

$$a_2^3+2a_1^3 \geq 3\left(a_2^3a_1^3a_1^3\right)^{1/3} = 3a_1^2a_2,$$

$$2a_1^3a_2^3+a_1^6 \geq 3\left(a_1^{12}a_2^6\right)^{1/3} = 3a_1^4a_2^2,$$

by the AM-GM inequality. Similarly, we get

$$\left(a_i^3+1\right)^2\left(a_{i+1}^3+1\right) \geq \left(a_i^2a_{i+1}+1\right)^3$$

for i = 2, 3, …, n with $a_{n+1} = a_1$. Multiplying these inequalities and taking cube root, we get the desired inequality.

Solution 2. **Murray KLAMKIN** (University of Alberta, Canada) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Let $a_{n+1}=a_1$. For i = 1, 2, …, n, by Hölder's inequality, we have

$$(a_i^3+1)^{2/3}(a_{i+1}^3+1)^{1/3}$$
$$\geq (a_i^3)^{2/3}(a_{i+1}^3)^{1/3}+1.$$

Multiplying these n inequalities, we get the desired inequality.

Comments: For the statement and proof of Hölder's inequality, we refer the readers to vol. 5, no. 4, page 2 of *Math Excalibur*.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Olympiad Corner

(continued from page 1)

One of these cells is empty and the others are filled with unit cubes labeled in an arbitrary manner with numbers 1, 2, …, 26. An admissible move is the moving of a unit cube into an adjacent empty cell. Is there a finite sequence of admissible moves after which the unit cube labeled with k and the unit cube labeled with $27-k$ are interchanged, for each k = 1, 2, …, 13? (Two cells are said to be adjacent if they share a common face.)

Mathematical Excalibur

Volume 6, Number 5 *January 2002 – February 2002*

Olympiad Corner

The 10^{th} Winter Camp, Taipei, Taiwan, February 14, 2001.

Problem 1. Determine all integers a and b which satisfy that

$$a^{13}+b^{90}=b^{2001}.$$

Problem 2. Let $\langle a_n \rangle$ be sequence of real numbers satisfying the recurrence relation

$$a_1=k,\ \ a_{n+1}=\left[\sqrt{2}a_n\right],\ \ n=1,2,\ldots.$$

where $[x]$ denotes the largest number which is less or equal than x. Find all positive integers k for which three exist three consecutive terms a_{i-1}, a_i, a_{i+1} satisfy $2a_i = a_{i-1}+a_{i+1}$.

Problem 3. A real number r is said to be *attainable* if there is a triple of positive real numbers (a, b, c) such that a, b, c are not the lengths of any triangle and satisfy the inequality

$$rabc > a^2b+b^2c+c^2a.$$

(a) Determine whether or not $\frac{7}{2}$ is *attainable*.

(b) Find all positive integer n such that n is *attainable*.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 23, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Vector Geometry

Kin Y. Li

A *vector* $\overrightarrow{XY}$ is an object having a magnitude (the length XY) and a direction (from X to Y). Vectors are very useful in solving certain types of geometry problems. First, we will mention some basic concepts related to vectors. Two vectors are considered the same if and only if they have the same magnitudes and directions. A vector $\overrightarrow{OX}$ from the origin O to a point X is called a *position vector*. For convenience, often a position vector $\overrightarrow{OX}$ will simply be denoted by X, when the position of the origin is understood, so that the vector $\overrightarrow{XY}$ = $\overrightarrow{OY}-\overrightarrow{OX}$ will simply be $Y – X$. The length of the position vector $\overrightarrow{OX}$ = X will be denoted by $|X|$. We have the triangle inequality $|X+Y|\le|X|+|Y|$, with equality if and only if $X = tY$ for some $t \ge 0$. Also, $|cX|$ = $|c||X|$ for number c.

For a point P on the line XY, in terms of position vectors, $P = tX + (1 – t)Y$ for some real number t. If P is on the segment XY, then $t = PY/XY \in [0, 1]$.

Next, we will present some examples showing how vectors can be used to solve geometry problems.

Example 1. (*1980 Leningrad High School Math Olympiad*) Call a segment in a convex quadrilateral a *midline* if it joins the midpoints of opposite sides. Show that if the sum of the midlines of a quadrilateral is equal to its semiperimeter, then the quadrilateral is a parallelogram.

Solution. Let $ABCD$ be such a convex quadrilateral. Set the origin at A. The sum of the lengths of the midlines is

$$\frac{|B+C-D|+|D+C-B|}{2}$$

and the semiperimeter is

$$\frac{|B|+|C-D|+|D|+|C-B|}{2}.$$

So

$$|B+C-D|+|D+C-B|$$
$$=|B|+|C-D|+|D|+|C-B|$$

By triangle inequality, $|B|+|C-D|\ge$ $|B+C-D|$, with equality if and only if $B=t(C-D)$ (or $AB\|CD$). Similarly, $|D|+|C-B|\ge|D+C-B|$, with equality if and only if $AD\|BC$. For the equation to be true, both triangle inequalities must be equalities. In that case, $ABCD$ is a parallelogram.

Example 2. (*Crux Problem 2333*) D and E are points on sides AC and AB of triangle ABC, respectively. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED, respectively, such that $BF:FC$ = $EG:GD$ = $BE:CD$. Show that GF is parallel to the angle bisector of $\angle BAC$.

Solution. Set the origin at A. Then $E = pB$ and $D = qC$ for some $p, q \in (0, 1)$. Let $t = \frac{BF}{FC}$, then $F = \frac{tC+B}{t+1}$ and G = $\frac{tD+E}{t+1}$ = $\frac{tqC+pB}{t+1}$.

Since $BE = tCD$, so $(1 – p)|B| = t(1 – q)|C|$. Thus,

$$F-G=\frac{t(1-q)}{t+1}C+\frac{1-p}{t+1}B$$
$$=\frac{(1-p)|B|}{t+1}\left(\frac{C}{|C|}+\frac{B}{|B|}\right).$$

This is parallel to $\dfrac{C}{|C|}+\dfrac{B}{|B|}$, which is in the direction of the angle bisector of $\angle BAC$.

* *

The *dot product* of two vectors X and Y is the number $X \cdot Y = |X||Y| \cos\theta$, where θ is the angle between the vectors. Dot product has the following properties:

(1) $X \cdot Y = Y \cdot X, (X+Y)\cdot Z = X \cdot Z + Y \cdot Z$ and $(cX)\cdot Y = c(X \cdot Y)$.

(2) $|X|^2 = X \cdot X$, $|X \cdot Y| \le |X||Y|$ and $OX \perp OY$ if and only if $X \cdot Y = 0$.

Example 3. (*1975 USAMO*) Let A, B, C, D denote four points in space and AB the distance between A and B, and so on. Show that

$$AC^2 + BD^2 + AD^2 + BC^2 \ge AB^2 + CD^2.$$

<u>Solution</u>. Set the origin at A. The inequality to be proved is

$$C\cdot C + (B-D)\cdot(B-D) + D \cdot D + (B-C)\cdot(B-C) \ge B\cdot B + (C-D)\cdot(C-D).$$

After expansion and regrouping, this is the same as $(B-C-D)\cdot(B-C-D) \ge 0$, with equality if and only if $B - C = D = D - A$, i.e. is $BCAD$ is a parallelogram.

* *

For a triangle ABC, the position vectors of its centroid is

$$G = \frac{A+B+C}{3}.$$

If we take the circumcenter O as the origin, then the position of the orthocenter is $H = A + B + C$ as $\overrightarrow{OH} = 3\overrightarrow{OG}$. Now for the incenter I, let a, b, c be the side lengths and AI intersect BC at D. Since $BD{:}CD = c{:}b$ and $DI{:}AI = \dfrac{ca}{b+c}{:}c = a{:}b + c$, so $D = \dfrac{bB+cC}{b+c}$ and $I = \dfrac{aA+bB+cC}{a+b+c}$.

Example 4. (*2[nd] Balkan Math Olympiad*) Let O be the center of the circle through the points A, B, C, and let D be the midpoint of AB. Let E be the centroid of triangle ACD. Prove that the line CD is perpendicular to line OE if and only if $AB = AC$.

<u>Solution</u>. Set the origin at O. Then

$$D = \frac{A+B}{2},$$

$$E = \frac{A+C+D}{3} = \frac{3A+B+2C}{6},$$

$$D - C = \frac{A+B-2C}{2}.$$

Hence $CD \perp OE$ if and only if $(A + B - 2C)\cdot(3A+B+2C) = 0$. Since $A \cdot A = B \cdot B = C \cdot C$, this is equivalent to $A \cdot (B - C) = A\cdot B - A \cdot C = 0$, which is the same as $OA \perp BC$, i.e. $AB = AC$.

Example 5. (*1990 IMO Usused Problem, Proposed by France*) Given ΔABC with no side equal to another side, let G, I and H be its centroid, incenter and orthocenter, respectively. Prove that $\angle GIH > 90°$.

<u>Solution</u>. Set the origin at the circumcenter. Then

$$H = A + B + C, \quad G = \frac{A+B+C}{3},$$

$$I = \frac{aA+bB+cC}{a+b+c}.$$

We need to show $(G-I)\cdot(H-I) = G\cdot H + I \cdot I - I\cdot(G+H) < 0$. Now $A \cdot A = B\cdot B = C \cdot C = R^2$ and $2B\cdot C = B \cdot B + C\cdot C - (B-C)\cdot(B-C) = 2R^2 - a^2$, … . Hence,

$$G\cdot H = \frac{(A+B+C)\cdot(A+B+C)}{3} = 3R^2 - \frac{a^2+b^2+c^2}{3},$$

$$I \cdot I = \frac{(aA+bB+cC)\cdot(aA+bB+cC)}{(a+b+c)^2} = R^2 - \frac{abc}{a+b+c},$$

$$I\cdot(G+H) = \frac{4(aA+bB+cC)\cdot(A+B+C)}{3(a+b+c)} = 4R^2 - \frac{2[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{3(a+b+c)}.$$

Thus, it is equivalent to proving $(a + b + c)(a^2+b^2+c^2) + 3abc > 2[a^2(b + c) + b^2(c + a) + c^2(a + b)]$, which after expansion and regrouping will become $a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) > 0$. To obtain this inequality, without loss of generality, assume $a \ge b \ge c$. Then $a(a-b)(a-c) \ge b(a-b)(b-c)$ so that the sum of the first two terms is nonnegative. As the third term is also nonnegative, the above inequality is true.

* *

The *cross product* of two vectors X and Y is a vector $X \cdot Y$ having magnitude $|X||Y| \sin\theta$, where θ is the angle between the vectors, and direction perpendicular to the plane of X and Y satisfying the right hand rule. Cross product has the following properties:

(1) $X \cdot Y = -Y \cdot X$, $(X + Y)\cdot Z = X \cdot Z + Y \cdot Z$ and $(cX)\cdot Y = c(X \cdot Y)$.

(2) $\dfrac{|X \cdot Y|}{2}$ is the area of triangle XOY. When $X, Y \ne O$, $X \cdot Y = 0$ if and only if X, O, Y are collinear.

Example 6. (*1984 Annual Greek High School Competition*) Let $A_1\ A_2\ A_3\ A_4\ A_5\ A_6$ be a convex hexagon having its opposite sides parallel. Prove that triangles $A_1\ A_3\ A_5$ and $A_2\ A_4\ A_6$ have equal areas.

<u>Solution</u>. Set the origin at any point. As the opposite sides are parallel, $(A_1 - A_2)\cdot(A_4 - A_5) = 0$, $(A_3 - A_2)\cdot(A_5 - A_6) = 0$ and $(A_3 - A_4)\cdot(A_6 - A_1) = 0$. Expanding these equations and adding them, we get $A_1 \cdot A_3 + A_3 \cdot A_5 + A_5 \cdot A_1 = A_2 \cdot A_4 + A_4 \cdot A_6 + A_6 \cdot A_2$. Now

$$[A_1\ A_3\ A_5] = \frac{|(A_1 - A_3)\cdot(A_1 - A_5)|}{2} = \frac{|A_1 \cdot A_3 + A_3 + A_5 + A_5 \cdot A_1|}{2}.$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***March 23, 2002***.

Problem 141. Ninety-eight points are given on a circle. Maria and José take turns drawing a segment between two of the points which have not yet been joined by a segment. The game ends when each point has been used as the endpoint of a segment at least once. The winner is the player who draws the last segment. If José goes first, who has a winning strategy? (*Source: 1998 Iberoamerican Math Olympiad*)

Problem 142. $ABCD$ is a quadrilateral with $AB \parallel CD$. P and Q are on sides AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Prove that P and Q are equal distance from the intersection point of the diagonals of the quadrilateral. (*Source: 1994 Russian Math Olympiad, Final Round*)

Problem 143. Solve the equation $\cos\cos\cos\cos x = \sin\sin\sin\sin x$. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Problem 144. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all (non-degenerate) triangles ABC with consecutive integer sides a, b, c and such that $C = 2A$.

Problem 145. Determine all natural numbers $k > 1$ such that, for some distinct natural numbers m and n, the numbers $k^m + 1$ and $k^n + 1$ can be obtained from each other by reversing the order of the digits in their decimal representations. (*Source: 1992 CIS Math Olympiad*)

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Solutions

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Problem 136. For a triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are rational, prove that $\cos A$, $\cos B$, $\cos C$ must also be rational. If $\cos A$, $\cos B$, $\cos C$ are rational, must at least one of $\sin A$, $\sin B$, $\sin C$ be rational?

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **LO Chi Fai** (STFA Leung Kau Kui College, Form 6), **WONG Tak Wai Alan** (University of Toronto), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

If $\sin A$, $\sin B$, $\sin C$ are rational, then by cosine law and sine law,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}-\frac{a}{b}\frac{a}{c}\right)$$

$$= \frac{1}{2}\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}-\frac{\sin A}{\sin B}\frac{\sin A}{\sin C}\right)$$

is rational. Similarly, $\cos B$ and $\cos C$ are rational. In the case of an equilateral triangle, $\cos A = \cos B = \cos C = \cos 60^\circ = \frac{1}{2}$ is rational, but $\sin A = \sin B = \sin C = \sin 60^\circ = \frac{\sqrt{3}}{2}$ is irrational.

Other commended solvers: **LEE Tsun Man Clement** (St. Paul's College, Form 3), **LOONG King Pan Campion** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TANG Chun Pong** (La Salle College, Form 4).

Problem 137. Prove that for every positive integer n,

$$(\sqrt{3}+\sqrt{2})^{1/n} + (\sqrt{3}-\sqrt{2})^{1/n}$$

is irrational.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let $x = (\sqrt{3}+\sqrt{2})^{1/n}$. Since $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right) = 1$, $x^{-1} = (\sqrt{3}-\sqrt{2})^{1/n}$. If $x + x^{-1}$ is rational, then $x^2 + x^{-2} = (x + x^{-1})^2 - 2$ is also rational. Since

$$x^{k+1} + x^{-(k+1)} = (x+x^{-1})\left(x^k + x^{-k}\right) - \left(x^{k-1} + x^{-(k-1)}\right),$$

by math induction, $x^n + x^{-n} = 2\sqrt{3}$ would be rational, a contradiction. Therefore, $x + x^{-1}$ is irrational.

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 138. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) If $a + b$ and $a - b$ are relatively prime integers, find the greatest common divisor (or the highest common factor) of $2a + (1+2a)(a^2 - b^2)$ and $2a(a^2 + 2a - b^2)(a^2 - b^2)$.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let (r, s) denote the greatest common divisor (or highest common factor) of r and s. If $(r, s) = 1$, then for any prime p dividing rs, either p divides r or p divides s, but not both. In particular p does not divide $r + s$. So $(r + s, rs) = 1$. Let $x = a + b$ and $y = a - b$. Then

$$\begin{aligned} &2a + (1+2a)(a^2-b^2) \\ &= x + y + (1 + x + y)xy \\ &= (x + y + xy) + (x + y)xy \end{aligned}$$

and

$$\begin{aligned} &2a(a^2 + 2a - b^2)(a^2-b^2) \\ &= (x + y)(xy + x + y)xy. \end{aligned}$$

Now $(x, y) = 1$ implies $(x + y, xy) = 1$. Repeating this twice, we get

$$(x + y + xy, (x + y)xy) = 1$$

and

$$\begin{aligned} &((x + y + xy + (x + y)xy, \\ &(x + y + xy)(x + y)xy) = 1. \end{aligned}$$

So the answer to the problem is 1.

Other commended solvers: **LEE Tsun Man Clement** (St. Paul's College, Form 3), **POON Yiu Keung** (HKUST, Math Major, Year 1), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TANG Chun Pong** (La Salle College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 139. Let a line intersect a pair of concentric circles at points A, B, C, D in that order. Let E be on the outer circle and F be on the inner circle such that chords AE and BF are parallel. Let

G and H be points on chords BF and AE that are the feet of perpendiculars from C to BF and from D to AE, respectively. Prove that $EH = FG$. (*Source: 1958 Shanghai City Math Competition*)

Solution. **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Let M be the midpoint of BC (and AD). Since $\angle DHA = 90^\circ$, $\angle ADH = \angle DHM$. Since $BF \parallel AE$, $\angle BAE = \angle FEA$ by symmetry with respect to the diameter perpendicular to BF and AE. Now $\angle FEA = \angle BAE = 90^\circ - \angle ADH = 90^\circ - \angle DHM = \angle AHG$. So $EF \parallel HG$. Since $EH \parallel FG$ also, $EFGH$ is a parallelogram. Therefore, $EH = FG$.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7).

Problem 140. A convex pentagon has five equal sides. Prove that the interior of the five circles with the five sides as diameters do not cover the interior of the pentagon.

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let the pentagon be $A_1A_2A_3A_4A_5$ and $2r$ be the common length of the sides. Let M_{ij} be the midpoint of A_iA_j and C_i be the circle with diameter A_iA_{i+1} for i = 1, 2, 3, 4, 5 (with $A_6 = A_1$). Since $540 - 3\cdot 60 = 2\cdot 180$ and $\angle A_i <$ 180°, there are at least 3 interior angles (in particular, two adjacent angles) greater than 60°. So we may suppose $\angle A_1, \angle A_2 > 60^\circ$. Since $A_3A_4 = A_5A_4$, we get $A_4M_{35} \perp A_3A_5$. Then M_{35} is on C_3, C_4 and the points on the ray from A_4 to M_{35} lying beyond M_{35} is outside C_3, C_4.

Next, since $\angle A_1 > 60^\circ$ and $A_1A_2 = A_1A_5$, A_2A_5 is the longest side of $\Delta A_1A_2A_5$. By the midpoint theorem, $M_{23}M_{35} = \frac{A_2A_5}{2} > \frac{A_1A_2}{2} = r$ so that M_{35} is outside C_2. Similarly, M_{35} is outside C_5. If M_{35} is not outside C_1, then $A_2M_{35} < A_1A_2 = A_2A_3$ and $\angle A_1M_{35}A_2 \ge 90^\circ$. Since $A_3M_{35} < A_3A_4 = A_2A_3$ also, A_2A_3 must be the longest side of $\Delta A_2A_3M_{35}$. Then $\angle A_2M_{35}A_3 > 60^\circ$. Similarly, $\angle A_1M_{35}A_5 > 60^\circ$. Then, we have $\angle A_1M_{35}A_2 < 60^\circ$, a contradiction. So M_{35} is outside C_1, too.

For i = 1, 2, 5 let $d_i = M_{35}M_{i,i+1} - r > 0$. Let d be the distance from M_{35} to the intersection point of the pentagon with the ray from A_4 to M_{35} lying beyond M_{35}. Choose a point X beyond M_{35} on the ray from A_4 to M_{35} with $XM_{35} < d, d_1, d_2$ and d_5. Then X is inside the pentagon and is outside C_3, C_4. Also, for i = 1, 2, 5,

$$XM_{i,i+1} > M_{35}M_{i,i+1} - XM_{35}$$
$$= r + d_i - XM_{35} > r$$

so that X is outside C_1, C_2, C_5.

Comments: The point M_{35} is enough for the solution as it is not in the interior of the 5 circles. The point X is better as it is not even on any of the circles.

Olympiad Corner

(continued from page 1)

Problem 4. Let O be the center of excircle of ΔABC touching the side BC internally. Let M be the midpoint of AC, P the intersection point of MO and BC. Prove that $AB = BP$, if $\angle BAC = 2\angle ACB$.

Problem 5. Given that 21 regular pentagons P_1, P_2, …, P_{21} are such that for any $k \in$ {1, 2, 3, …, 20}, all the vertices of P_{k+1} are the midpoints of the sides of P_k. Let S be the set of the vertices of $P_1, P_2, ..., P_{21}$. Determine the largest positive integer n for which there always exist four points A, B, C, D from S such that they are the vertices of an isosceles trapezoid and with the same color if we use n kinds of different colors to paint the element of S.

Vector Geometry

(continued from page 2)

Similarly,

$$[A_2A_4A_6] = \frac{|A_2 \;\; A_4 + A_4 \cdot \;\; A_6 + A_6 \cdot A_2|}{2}.$$

So $[A_1 \; A_3 \; A_5] = [A_2 \; A_4 \; A_6]$.

Example 7. (*1996 Balkan Math Olympiad*) Let $ABCDE$ be a convex pentagon and let M, N, P, Q, R be the midpoints of sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, show that this point also lies on EN.

<u>Solution.</u> Set the origin at the commom point. Since, A, P and the origin are collinear,

$$0 = A \;\; P = A \cdot \left(\frac{C+D}{2}\right) = \frac{A \;\; C + A \cdot D}{2}.$$

So $A \;\; C = D \cdot A$. Similarly, $B \cdot D = E \cdot B$, $C \;\; E = A \cdot C$, $D \;\; A = B \cdot D$. Then $E \;\; B = C \cdot E$. So $E \;\; N = E \cdot \left(\frac{B+C}{2}\right) = 0$, which implies E, N and the origin are collinear.

Example 8. (*16th Austrian Math Olympiad*) A line interesects the sides (or sides produced) BC, CA, AB of triangle ABC in the points A_1, B_1, C_1, respectively. The points A_2, B_2, C_2 are symmetric to A_1, B_1, C_1 with respect to the midpoints of BC, CA, AB, respectively. Prove that A_2, B_2, C_2 are collinear.

<u>Solution.</u> Set the origin at a vertex, say C. Then $A_1 = c_1B, B_1 = c_2A, C_1 = A + c_3(B - A)$ for some constants c_1, c_2, c_3. Since A_1, B_1, C_1, are collinear,

$$0 = (B_1 - A_1) \cdot (C_1 - A_1)$$
$$= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)A \cdot B.$$

Since

$$A_2 = B - A_1 = (1 - c_1)B,$$
$$B_2 = A - B_1 = (1 - c_2)A$$

and

$$C_2 = (A + B) - C_1 = c_3A + (1 - c_3)B,$$

so A_2, B_2, C_2, are collinear if and only if

$$0 = (B_2 - A_2) \cdot (C_2 - A_2)$$
$$= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)A \cdot B,$$

which is true.

Mathematical Excalibur

Volume 7, Number 1 *March 2002 – April 2002*

Olympiad Corner

The 32nd Austrian Mathematical Olympiad 2001.

Problem 1. Prove that

$$\frac{1}{25}\sum_{k=0}^{2001}\left[\frac{2^k}{25}\right]$$

is an integer. ($[x]$ denotes the largest integer less than or equal to x.)

Problem 2. Determine all triples of positive real numbers x, y and z such that both $x + y + z = 6$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2 - \frac{4}{xyz}$ hold.

Problem 3. We are given a triangle ABC and its circumcircle with mid-point U and radius r. The tangent c' of the circle with mid-point U and radius $2r$ is determined such that C lies between $c = AB$ and c', and a' and b' are defined analogously, yielding the triangle $A'B'C'$. Prove that the lines joining the mid-points of corresponding sides of ΔABC and $\Delta A'B'C'$ pass through a common point.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 15, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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對數表的構造

李永隆

在現今計算工具發達的年代，要找出如 $ln2$ 這個對數值只需一指之勞。但是大家有沒有想過，在以前計算機尚未出現的時候，那些厚厚成書的對數表是如何精確地構造出來的？ 當然，在歷史上曾出現很多不同的構造方法，各有其所長，但亦各有其所限。下面我們將會討論一個比較有系統的方法，它只需要用上一些基本的微積分技巧，就能夠有效地構造對數表到任意的精確度。

首先注意，$ln(xy) = ln\, x + ln\, y$, 所以我們只需求得所有質數 p 的對數值便可以由此算得其他正整數的對數值。 由 $ln(1 + t)$ 的微分運算和幾何級數公式直接可得

$$\frac{d}{dt}\ln(1+t) = \frac{1}{1+t} = 1 - t + t^2 - t^3$$
$$+\cdots+(-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t}$$

運用微積分基本定理（亦即微分和積分是兩種互逆的運算），即得下式：

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$
$$+\cdots+(-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}dt$$

能夠對於所有正整數 n 皆成立。 現在我們去估計上式中的積分餘項的大小。 設 $|x| < 1$，則有：

$$\left|\int_0^x \frac{(-1)^n t^n}{1+t}dt\right| \le \left|\int_0^x \left|\frac{t^n}{1+t}\right|dt\right|$$
$$\le \left|\int_0^x \frac{t^n}{1-|x|}dt\right| = \frac{|x|^{n+1}}{(n+1)(1-|x|)}$$

由此可見，這個餘項的絕對值會隨著 n 的增大而趨向 0。換句話說，只要 n 選得足夠大， $\ln(1+x)$ 和 $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}$ 之間的誤差就可以小到任意小，所以我們不妨改用下式表達這個情況：

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ (|x|<1)$$

總之 n 的選取總是可以讓我們忽略兩者的誤差。 把上式中的 x 代以 $-x$ 然後將兩式相減，便可以得到下面的公式：

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) \quad (*)$$

可惜的是若直接代入 $x = \frac{p-1}{p+1}$ 使得 $\frac{1+x}{1-x} = p$ 時，$(*)$-式並不能有效地計算 $\ln p$。例如取 $p = 29$，則 $x = \frac{29-1}{29+1} = \frac{14}{15}$，在此時即使計算了 100 項至 $\frac{2x^{199}}{199} \approx 1.1\times10^{-8}$ ，$\ln p$ 的數值還未必能準確至第 8 個小數位（嚴格來說，應該用 $(*)$-式的積分餘項來做誤差估計，不過在這裏我們只是想大約知道其大小）；又例如取 $p = 113$，則 $x = \frac{56}{57}$ 而 $\frac{2x^{199}}{199} \approx 3\times10^{-4}$ ，$\ln p$ 的準確度則更差。 但是我們可以取 $x = \frac{1}{2p^2-1}$ ，則有

$$\ln\left(\frac{1+x}{1-x}\right) = \ln\frac{2p^2-1+1}{2p^2-1-1}$$
$$= \ln\frac{p^2}{(p+1)(p-1)}$$
$$= 2\ln p - \ln(p+1)(p-1)$$

而當質數 $p > 2$ 時，$(p + 1)$ 和 $(p - 1)$ 的質因數都必定小於 p，所以如果我們已算得小於 p 的質數的對數值，就可以用上式來計算 $\ln p$ 的值：

$$2\ln p = \ln\left(\frac{1+x}{1-x}\right) + \ln(p+1) + \ln(p-1)$$

而未知的 $\ln\left(\frac{1+x}{1-x}\right)$ 是能夠有效計算的，因為現在所選的 x 的絕對值很小。例如當 $p = 29$ 時，$x = \frac{1}{2\cdot 29^2 - 1} = \frac{1}{1681}$，所以只需計算到 $\frac{2x^5}{5} \approx 3 \times 10^{-17}$，便能夠準確至十多個小數位了。

經過上面的討論，假設現在我們想構造一個 8 位對數表，則可以依次序地求 2, 3, 5, 7, 11, 13, … 的對數值，而後面質數的對數值都可以用前面的質數的對數值來求得。由此可見，在開始時的 ln 2 是需要算得準確一些：

$$\ln 2 = \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$$

$$\approx 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \dots + \frac{\left(\frac{1}{3}\right)^{21}}{21}\right)$$

$$= 0.6931471805589\ldots$$

這個和確實數值

$$\ln 2 = 0.693147180559945\ldots$$

相比其精確度已到達第 11 位小數。

接著便是要計算 ln 3。取 $x = \frac{1}{2\cdot 3^2 - 1} = \frac{1}{17}$，則有

$$\ln\left(\frac{1+\frac{1}{17}}{1-\frac{1}{17}}\right) \approx 2\left(\frac{1}{17} + \frac{\left(\frac{1}{17}\right)^3}{3} + \frac{\left(\frac{1}{17}\right)^5}{5} + \frac{\left(\frac{1}{17}\right)^7}{7}\right)$$

$$= 0.117783035654504\ldots$$

注意 $\frac{2\left(\frac{1}{17}\right)^9}{9} \approx 1.9\times 10^{-12}$，在 8 位的精確度之下大可以不用考慮。 所以

$$\ln 3 \approx \frac{1}{2}(0.11778303565\ldots + \ln 4 + \ln 2)$$

$$= 1.098612288635\ldots$$

（續於第四頁）

Pell's Equation (II)

Kin Y. Li

For a fixed nonzero integer *N*, as the case $N = -1$ shows, the generalized equation $x^2 - dy^2 = N$ may not have a solution. If it has a least positive solution (x_1, y_1), then $x^2 - dy^2 = N$ has infinitely many positive solutons given by (x_n, y_n), where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})(a + b\sqrt{d})^{n-1}$$

and *(a, b)* is the least positive solution of $x^2 - dy^2 = 1$. However, in general these do not give all positive solutions of $x^2 - dy^2 = N$ as the following example will show.

Example 9. Consider the equation $x^2 - 23y^2 = -7$. It has $(x_1, y_1) = (4, 1)$ as the least positive solution. The next two solutions are (19, 4) and (211, 44). Now the least positive solution of $x^2 - 23y^2 = 1$ is *(a, b)* = (24, 5). Since $(4 + \sqrt{23})(24 + 5\sqrt{23}) = 211 + 44\sqrt{23}$, the solution (19, 4) is skipped by the formula above.

In case $x^2 - dy^2 = N$ has positive solutions, how do we get them all? A solution *(x, y)* of $x^2 - dy^2 = N$ is called *primitive* if *x* and *y* (and *N*) are relatively prime. For $0 \le s < |N|$, we say the solution belong to class C_s if $x \equiv sy \pmod{|N|}$. As *x, y* are relatively prime to *N*, so is *s*. Hence, there are at most $\phi(|N|)$ classes of primitive solutions, where $\phi(k)$ is *Euler's* ϕ *-function* denoting the number of positive integers $m \le k$ that are relatively prime to *k*. Also, for such *s*, $(s^2 - d)y^2 \equiv x^2 - dy^2 \equiv 0 \pmod{|N|}$ and *y, N* relatively prime imply $s^2 \equiv d \pmod{|N|}$.

Theorem. *Let* (a_1, b_1) *be a* C_s *primitive solutions of* $x^2 - dy^2 = N$. *A pair* (a_2, b_2) *is also a* C_s *primitive solution of* $x^2 - dy^2 = N$ *if and only if* $a_2 + b_2\sqrt{d} = (a_2 - b_2\sqrt{d})/(a_1 - b_1\sqrt{d})$. *Multiplying these two equations, we get* $u^2 - dv^2 = N/N = 1$.

To see *u, v* are integers, note $a_1a_2 - db_1b_2 \equiv (s^2 - d)b_1b_2 \equiv 0 \pmod{|N|}$, which implies *u* is an integer. Since $a_1b_2 - b_1a_2 \equiv sb_1b_2 - b_1sb_2 = 0 \pmod{|N|}$, v is also an integer.

For the converse, multiplying the equation with its conjugate shows (a_2, b_2) solves $x^2 - dy^2 = N$. From $a_2 = ua_1 + dvb_1$ and $b_2 = ub_1 + va_1$, we get $a_2 = ua_2 - dvb_2$ and $b_1 = ub_2 - va_2$. Hence, common divisors of a_2, b_2 are also common divisors a_1, b_1. So a_2, b_2 are relatively prime. Finally, $a_2 - sb_2 \equiv (usb_1 + dvb_1) - s(ub_1 + vsb_1) = (d - s^2)vb_1 \equiv 0 \pmod{|N|}$ concludes the proof.

Thus, all primitive solutions of $x^2 - dy^2 = N$ can be obtained by finding a solution (if any) in each class, then multiply them by solutions of $x^2 - dy^2 = 1$. For the nonprimitive solutions, we can factor the common divisors of *a* and *b* to reduce *N*.

Example 10. (1995 IMO proposal by USA leader T. Andreescu) Find the smallest positive integer *n* such that 19*n* + 1 and 95*n* + 1 are both integer squares.

Solution. Let $95n + 1 = x^2$ and $19n + 1 = y^2$, then $x^2 - 5y^2 = -4$. Now $\phi(4) = 2$ and (1, 1), (11, 5) are C_1, C_3 primitive solutions, respectively. As (9, 4) is the least positive solution of $x^2 - 5y^2 = 1$ and $9 + 4\sqrt{5} = (2 + \sqrt{5})^2$, so the primitive positive solutions are pairs *(x, y)*, where $x + y\sqrt{5} = (1 + \sqrt{5})(2 + \sqrt{5})^{2n-2}$ or $(11 + 5\sqrt{5})(2 + \sqrt{5})^{2n-2}$.

Since the common divisors of *x, y* divide 4, the nonprimitive positive solutions are the cases *x* and *y* are even. This reduces to considering $u^2 - 5v^2 = -1$, where we take *u* = *x*/2 and *v* = *y*/2. The least positive solution for $u^2 - 5v^2 = -1$ is (2, 1). So $x + y\sqrt{5} = 2(u + v\sqrt{5}) = 2(2 + \sqrt{5})^{2n-1}$.

In attempt to combine these solutions, we look at the powers of $1 + \sqrt{5}$ coming from the least positive solutions (1, 1).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***May 15, 2002***.

Problem 146. Is it possible to partition a square into a number of congruent right triangles each containing an 30° angle? (*Source: 1994 Russian Math Olympiad, 3rd Round*)

Problem 147. Factor x^8+4x^2+4 into two nonconstant polynomials with integer coefficients.

Problem 148. Find all distinct prime numbers *p, q, r, s* such that their sum is also prime and both p^2+qs, p^2+qr are perfect square numbers. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Problem 149. In a 2000×2000 table, every square is filled with *a* 1 or –1. It is known that the sum of these numbers is nonnegative. Prove that there are 1000 columns and 1000 rows such that the sum of the numbers in these intersection squares is at least 1000. (*Source: 1994 Russian Math Olympiad, 5th Round*)

Problem 150. Prove that in a convex *n*-sided polygon, no more than *n* diagonals can pairwise intersect. For what *n*, can there be *n* pairwise intersecting diagonals? (Here intersection points may be vertices.) (*Source: 1962 Hungarian Math Olympiad*)

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Solutions

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Problem 141. Ninety-eight points are given on a circle. Maria and José take turns drawing a segment between two of the points which have not yet been joined by a segment. The game ends when each point has been used as the endpoint of a segment at least once. The winner is the player who draws the last segment. If José goes first, who has a winning strategy? (*Source: 1998 Iberoamerican Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), 何思銳 (大角嘴天主教小學, Primary 5), **LAM Sze Yui** (Carmel Divine Grace Foundation Secondary School, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), LEUNG Chi Man (Cheung Sha Wan Catholic Secondary School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Yiu Keung** (HKUST, Math Major, Year 1), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **Ricky TANG** (La Salle College, Form 4), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

José has the following winning strategy. He will let Maria be the first person to use the ninety-sixth unused point. Since there are $C_2^{95}=4465$ segments joining pairs of the first ninety-five points, if Maria does not use the ninety-sixth point, José does not have to use it either. Once Maria starts using the ninety-sixth point, José can win by joining the ninety-seventh and ninety-eighth points.

Problem 142. *ABCD* is a quadrilateral with $AB \parallel CD$. *P* and *Q* are on sides *AD* and *BC* respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Prove that *P* and *Q* are equal distance from the intersection point of the diagonals of the quadrilateral. (*Source: 1994 Russian Math Olympiad, Final Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Let *O* be the intersection point of the diagonals. Since $\Delta AOB, \Delta COD$ are similar, *AO:CO = AB:CD = BO:DO.* By sine law,

$$\frac{AB}{BP}=\frac{\sin\angle APB}{\sin\angle BAP}=\frac{\sin\angle CPD}{\sin\angle CDP}=\frac{CD}{CP}.$$

So *AB:CD = BP:CP*. Let *S* be on *BC* so that $SP\perp AD$ and *R* be on *AD* so that $RQ\perp BC$. Then *SP* bisects $\angle BPC$, *BS:CS = BP:CP = AB:CD = AO:CO.* This implies $OS\parallel AB$. Then *AB:OS = CA:CO.* Similarly, *AB:RO = DB:DO*. However,

$$\frac{CA}{CO}=1+\frac{AO}{CO}=1+\frac{BO}{DO}=\frac{DB}{DO}.$$

So *OS = RO*. Since *O* is the midpoint of *RS* and ΔSPR, ΔRQS are right triangles, *PO = OS = QO*.

Other commended solvers: **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Problem 143. Solve the equation cos cos cos cos *x* = sin sin sin sin *x*. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7**)**.

Let *f*(*x*) = sin sin *x* and *g*(*x*) = cos cos *x*. Now

$$\begin{aligned} g(x)-f(x) &= \sin\left(\frac{\pi}{2}-\cos x\right)-\sin\sin x \\ &= 2\cos\left(\frac{\pi}{4}-\frac{\cos x}{2}+\frac{\sin x}{2}\right) \\ &\quad\times\sin\left(\frac{\pi}{4}-\frac{\cos x}{2}-\frac{\sin x}{2}\right)\end{aligned}$$

and

$$\left|\frac{\cos x\pm\sin x}{2}\right|=\frac{\sqrt{2}\left|\sin(x\pm{}^{\pi}\!/_{4})\right|}{2}<\frac{\pi}{4}.$$

So $g(x)-f(x)>0$ (hence *g*(*x*) > *f*(*x*)) for all *x*. Since sin *x*, *f*(*x*), *g*(*x*) ∈ [−1, 1] $\subset[-\pi/2,\ \pi/2]$ and sin *x* is strictly increasing in $[-\pi/2, \pi/2]$, so *f(x)* is strictly increasing in $[-\pi/2,\ \pi/2]$ and

$$f\big(f(x)\big)<f\big(g(x)\big)<g\big(g(x)\big)$$

for all *x*. Therefore, the equation has no solution.

*Other commended solvers***:** **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **OR Kin** (HKUST, Year 1) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Problem 144. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all (non-degenerate) triangles *ABC* with consecutive integer sides *a, b, c* and such that $C=2A$.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 4), **LAM Wai Pui Billy** (STFA Leung

Kau Kui College, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **YEUNG Wing Fung** (STFA Leung Kau Kui College).

Let $a=BC$, $b=CA$, $c=AB$. By sine and cosine laws,

$$\frac{c}{a}=\frac{\sin C}{\sin A}=2\cos A=\frac{b^2+c^2-a^2}{bc}.$$

This gives $bc^2=ab^2+ac^2-a^3$. Factoring, we get $(a-b)(c^2-a^2-ab)=0$. Since the sides are consecutive integers and $C>A$ implies $c>a$, we have $(a, b, c)=(n, n-1, n+1)$, $(n-1, n+1, n)$ or $(n-1, n, n+1)$ for some positive integer $n>1$. Putting these into $c^2-a^2-ab=0$, the first case leads to $-n^2+3n+1=0$, which has no integer solution. The second case leads to $2n-n^2=0$, which yields a degenerate triangle with sides 1, 2, 3. The last case leads to $5n-n^2=0$, which gives $(a, b, c)=(4, 5, 6)$.

Other commended solvers: **CHENG Ka Wai** (STFA Leung Kau Kui College, Form 4), **Clark CHONG Fan Fei** (Queen's College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 145. Determine all natural numbers $k>1$ such that, for some distinct natural numbers m and n, the numbers k^m+1 and k^n+1 can be obtained from each other by reversing the order of the digits in their decimal representations. (*Source: 1992 CIS Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **Ricky TANG** (La Salle College, Form 4) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Without loss of generality, suppose such numbers exist and $n>m$. By the required property, both numbers are not power of 10. So k^n and k^m have the same number of digits. Then $10>\dfrac{k^n}{k^m}=k^{n-m}\ge k$. Since every number and the sum of its digits are congruent (mod 9), we get $k^n+1\equiv k^m+1$ (mod 9). Then $k^n-k^m=k^m(k^{n-m}-1)$ is divisible by 9. Since the two factors are relatively prime, $10>k$ and $9>k^{n-m}-1$, we can only have $k=3$, 6 or 9.

Now $3^3+1=28$ and $3^4+1=82$ show $k=3$ is an answer. The case $k=6$ cannot work as numbers of the form 6^i+1 end in 7 so that both k^m+1 and k^n+1 would begin and end with 7, which makes $k^n/k^m\ge k$ impossible. Finally, the case $k=9$ also cannot work as numbers of the form 9^i+1 end in 0 or 2 so that both numbers would begin and end with 2, which again makes $k^n/k^m\ge k$ impossible.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 4. Determine all real valued functions $f(x)$ in one real variable for which

$$f(f(x)^2+f(y))=xf(x)+y$$

holds for all real numbers x and y.

Problem 5. Determine all integers m for which all solutions of the equation

$$3x^3-3x^2+m=0$$

are rational.

Problem 6. We are given a semicircle with diameter AB. Points C and D are marked on the semicircle, such that $AC=CD$ holds. The tangent of the semicircle in C and the line joining B and D interect in a point E, and the line joining A and E intersects the semicircle in a point F. Show that $CF<FD$ must hold.

對數表的構造

（續第二頁）

這個和確實數值

$$\ln 3=1.09861228866811\ldots$$

相比其精確度也到達第 10 位小數。 讀者不妨自行試算 ln 5, ln 7 等等的數值，然後再和計算機所得的作一比較。

回看上述極為巧妙的計算方法，真的令人佩服當年的數學家們對於數字關係和公式運算的那種創意與觸覺！

【參考文獻】:

項武義教授分析學講座筆記第三章

http://ihome.ust.hk/~malung/391.html

Pell's Equation (II)

(continued from page 2)

The powers are $1+\sqrt{5}$, $6+2\sqrt{5}$, $16+8\sqrt{5}=8(2+\sqrt{5})$, $56+24\sqrt{5}$, $176+80\sqrt{5}=16(11+5\sqrt{5})$, … . Thus, the primitive positive solutions are (x, y) with $x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-5}$ or $2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-1}$. The nonprimitive positive solutions are (x, y) with $x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-3}$. So the general positive solutions are (x, y) with

$$x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{k} \text{ for odd } k.$$

Then

$$y=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right)=F_k,$$

where F_k is the k-th term of the famous *Fibonacci sequence*. Finally, $y^2\equiv 1$ (mod 19) and k should be odd. The smallest such $y=F_{17}=1597$, which leads to $n=\left(F_{17}^2-1\right)/19=134232$.

Comments: For the readers not familiar with the Fibonacci sequence, it is defined by $F_1=1$, $F_2=1$ and $F_{n+1}=F_n+F_{n-1}$ for $n>1$. By math induction, we can check that they satisfy *Binet's formula* $F_n=\left(r_1^n-r_2^n\right)/\sqrt{5}$, where $r_1=(1+\sqrt{5})/2$ and $r_2=(1-\sqrt{5})/2$ are the roots of the *characteristic equation* $x^2=x+1$. (Check cases $n=1$, 2 and in the induction step, just use $r_i^{n+1}=r_i^n+r_i^{n-1}$.)

Mathematical Excalibur

Volume 7, Number 2 *May 2002 – June 2002*

Olympiad Corner

The 31st United States of America Mathematical Olympiad 2002

Problem 1. Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

(a) the union of any two white subsets is white;

(b) the union of any two black subsets is black;

(c) there are exactly N white subsets.

Problem 2. Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***Sep 20, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Problem Solving I

Kin-Yin LI

George Polya's famous book *How to Solve It* is a book we highly recommend every student who is interested in problem solving to read. In solving a difficult problem, Polya teaches us to ask the following questions. What is the condition to be satisfied? Have you seen a similar problem? Can you restate the problem in another way or in a related way? Where is the difficulty? If you cannot solve it, can you solve a part of the problem if the condition is relaxed. Can you solve special cases? Is there any pattern you can see from the special cases? Can you guess the answer? What clues can you get from the answer or the special cases? Below we will provide some examples to guide the student in analyzing problems.

Example 1. (*Polya, How to Solve It, pp. 23-25*) Given ΔABC with AB the longest side. Construct a square having two vertices on side AB and one vertex on each of sides BC and CA using a compass and a straightedge (i.e. a ruler without markings).

Analysis. (*Where is the difficulty?*) The difficulty lies in requiring all four vertices on the sides of the triangle. If we relax *four* to *three*, the problem becomes much easier. On CA, take a point P close to A. Draw the perpendicular from P to AB and let the foot be Q. With Q as center and PQ as radius, draw a circle and let it intersect AB at R. Draw the perpendicular line to AB through R and let S be the point on the line which is PQ units from R and on the same side of AB as P. Then $PQRS$ is a square with P on CA and Q, R on AB.

(*What happens if you move the point P on side CA?*) You get a square similar to $PQRS$. (*What happens in the special case P = A?*) You get a point. (*What happens to S if you move P from A toward C?*) As P moves along AC, the triangles APQ will be similar to each other. Then the triangles APS will also be similar to each other and S will trace a line segment from A. This line AS intersects BC at a point S', which is the fourth vertex we need. From S', we can find the three other vertices dropping perpendicular lines and rotating points.

Example 2. (*1995 Russian Math Olympiad*) There are $n > 1$ seats at a merry-go-around. A boy takes n rides. Between each ride, he moves clockwise a certain number (less than n) of places to a new horse. Each time he moves a different number of places. Find all n for which the boy ends up riding each horse.

Analysis. (*Can you solve special cases?*) The cases n = 2, 4, 6 work, but the cases n = 3, 5 do not work. (*Can you guess the answer?*) The answer should be n is even. (*What clues can you get from the special cases?*) From experimenting with cases, we see that if $n > 1$ is odd, then the last ride seems to always repeat the first horse. (*Why?*) From the first to the last ride, the boy moved $1 + 2 + \cdots + (n-1) = n(n-1)/2$ places. If $n > 1$ is odd, this is a multiple of n and so we repeat the first horse.

(*Is there any pattern you can see from the special cases when n is even?*) Name the horses 1, 2, …, n in the clockwise direction. For n = 2, we can ride horses 1, 2 in that order and the move sequence is 1. For n = 4, we can ride horses 1, 2, 4, 3 in that order and the move sequence is 1, 2, 3. For n = 6, we can ride horses 1, 2, 6, 3, 5, 4 and the move sequence is 1, 4, 3, 2, 5. Then for the general even cases n, we can ride horses 1, 2, n, 3, $n-1$, …, $(n/2)+1$ in that order with move sequence 1, $n-2$, 3, $n-4$, …, 2, $n-1$. The numbers in the move sequence are all distinct as it is the result of merging odd numbers 1, 3, …, $n-1$ with even numbers $n-2$, $n-4$, …, 2.

Example 3. (*1982 Putnam Exam*) Let $K(x, y, z)$ be the area of a triangle with sides x, y, z. For any two triangles with sides a, b, c and a', b', c' respectively, show that

$$\sqrt{K(a,b,c)}+\sqrt{K(a',b',c')} \le \sqrt{K(a+a',b+b',c+c')}$$

and determine the case of equality.

Analysis. (*Can you restate the problem in another way?*) As the problem is about the area and sides of a triangle, we bring out Heron's formula, which asserts the area of a triangle with sides x, y, z is given by

$$K(x,y,z)=\sqrt{s(s-x)(s-y)(s-z)},$$

where s is half the perimeter, i.e. $s=\frac{1}{2}(x+y+z)$. Using this formula, the problem becomes showing

$$\sqrt[4]{stuv}+\sqrt[4]{s't'u'v'} \le \sqrt[4]{(s+s')(t+t')(u+u')(v+v')},$$

where $s=\frac{1}{2}(a+b+c)$, $t=s-a$, $u=s-b$, $v=s-c$ and similarly for s', t', u', v'.

(*Have you seen a similar problem or can you relax the condition?*) For those who saw the forward-backward induction proof of the AM-GM inequality before, this is similar to the proof of the case $n=4$ from the case $n=2$. For the others, having groups of four variables are difficult to work with. We may consider the more manageable case $n=2$. If we replace 4 by 2, we get a simpler inequality

$$\sqrt{xy}+\sqrt{x'y'} \le \sqrt{(x+x')(y+y')}.$$

This is easier. Squaring both sides, canceling common terms, then factoring, this turns out to be just $\left(\sqrt{xy'}-\sqrt{x'y}\right)^2 \ge 0$. Equality holds if and only if $x:x'=y:y'$. Applying this simpler inequality twice, we easily get the required inequality

$$\begin{aligned}&\sqrt[4]{stuv}+\sqrt[4]{s't'u'v'}\\ &\le \sqrt{(\sqrt{st}+\sqrt{s't'})(\sqrt{uv}+\sqrt{u'v'})}\\ &\le \sqrt{\sqrt{(s+s')(t+t')}\sqrt{(u+u')(v+v')}}.\end{aligned}$$

Tracing the equality case back to the simpler inequality, we see equality holds if and only if $a:b:c=a':b':c'$, i.e. the triangles are similar.

Example 4. Is there a way to pack 250 $1\times 1\times 4$ bricks into a $10\times 10\times 10$ box?

Analysis. (*Where is the difficulty?*) 10 is large for a 3 dimensional cube. We can relax the problem a bit by considering a two dimensional analogous problem with smaller numbers, say 1×2 cards pack into a 8×8 board. This is clearly possible. (*What if we relax the board to be a square, say by taking out two squares from the board?*) This may become impossible. For example, if the 8×8 board is a checkerboard and we take out two black squares, then since every 1×2 card covers exactly one white and one black square, any possible covering must require the board to have equal number of white and black squares.

(*What clue can you get from the special cases?*) Coloring a board can help to solve the problem. (*Can we restate the problem in a related way?*) Is it possible to color the cubes of the $10\times 10\times 10$ box with four colors in such a way that in every four consecutive cubes each color occurs exactly once, where consecutive cubes are cubes sharing a common face? Yes, we can put color 1 in a corner cube, then extend the coloring to the whole box by putting colors 1, 2, 3, 4 periodically in each of the three perpendicular directions parallel to the edges of the box. However, a counting shows that for the $10\times 10\times 10$ box, there are 251 color 1 cubes, 251 color 2 cubes, 249 color 3 cubes and 249 color 4 cubes. So the required packing is impossible.

Example 5. (*1985 Moscow Math Olympiad*) For every integer $n\ge 3$, show that $2^n=7x^2+y^2$ for some odd positive integers x and y.

Analysis. (*cf. Arthur Engel, Problem-Solving Strategies, pp. 126-127*) (*Can you solve special cases?*) For $n = 3, 4, \ldots, 10$, we have the table:

n	3	4	5	6	7	8	9	10
$x=x_n$	1	1	1	3	1	5	7	3
$y=y_n$	1	3	5	1	11	9	13	31

(*Is there any pattern you can see from the special cases?*) In cases $n = 3, 5, 8$, it seems that x_{n+1} is the average of x_n and y_n. For cases $n = 4, 6, 7, 9, 10$, the average of x_n and y_n is even and it seems that $|x_n-y_n|=2x_{n+1}$. (*Can you guess the answer?*) The answer should be

$$x_{n+1}=\begin{cases}\frac{1}{2}(x_n+y_n) & \text{if } \frac{1}{2}(x_n+y_n) \text{ is odd}\\ \frac{1}{2}|x_n-y_n| & \text{if } \frac{1}{2}(x_n+y_n) \text{ is even}\end{cases}$$

and

$$\begin{aligned}&y_{n+1}\\ &=\sqrt{2^{n+1}-7x_{n+1}^2}\\ &=\sqrt{2(7x_n^2+y_n^2)-7x_{n+1}^2}\\ &=\begin{cases}\frac{1}{2}|7x_n-y_n| & \text{if } \frac{1}{2}(x_n+y_n) \text{ is odd}\\ \frac{1}{2}(7x_n+y_n) & \text{if } \frac{1}{2}(x_n+y_n) \text{ is even}\end{cases}.\end{aligned}$$

(*Is this correct?*) The case $n = 3$ is correct. If $2^n=7x_n^2+y_n^2$, then the choice of y_{n+1} will give $2^{n+1}=7x_{n+1}^2+y_{n+1}^2$. (*Must x_{n+1} and y_{n+1} be odd positive integers?*) Yes, this can be checked by writing x_n and y_n in the form $4k\pm 1$.

IMO 2002

IMO 2002 will be held in Glasgow, United Kingdom from July 19 to July 30 this summer. Based on the selection test performances, the following students have been chosen to represent Hong Kong:

CHAO Khek Lun (*St. Paul's College*)
CHAU Suk Ling (*Queen Elizabeth School*)
CHENG Kei Tsi (*La Salle College*)
IP Chi Ho (*St. Joseph College*)
LEUNG Wai Ying (*Queen Elizabeth School*)
YU Hok Pun (*SKH Bishop Baker Secondary Sch*)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***September 20, 2002***.

Problem 151. Every integer greater than 2 can be written as a sum of distinct positive integers. Let $A(n)$ be the maximum number of terms in such a sum for n. Find $A(n)$. (*Source: 1993 German Math Olympiad*)

Problem 152. Let $ABCD$ be a cyclic quadrilateral with E as the intersection of lines AD and BC. Let M be the intersection of line BD with the line through E parallel to AC. From M, draw a tangent line to the circumcircle of $ABCD$ touching the circle at T. Prove that $MT = ME$. (*Source: 1957 Nanjing Math Competition*)

Problem 153. Let R denote the real numbers. Find all functions $f: R \rightarrow R$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all pairs of real numbers x, y. (*source: 1997 Czech-Slovak Match*)

Problem 154. For nonnegative numbers a, d and positive numbers b, c satisfying $b + c \geq a + d$, what is the minimum value of $\frac{b}{c+d}+\frac{c}{a+b}$?

(*Source: 1988 All Soviet Math Olympiad*)

Problem 155. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise relatively prime numbers among them. (*Source: 1997 Hungarian Math Olympiad*)

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Solutions

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Problem 146. Is it possible to partition a square into a number of congruent right triangles each containing a 30° angle? (*Source: 1994 Russian Math Olympiad, 3rd Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHEUNG Chung Yeung** (STFA Leung Kau Kui College, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 4).

Without loss of generality, let the sides of the triangles be 2, 1, $\sqrt{3}$. Assume n such triangles can partition a square. Since the sides of the square are formed by sides of these triangles, so the sides of the square are of the form $a+b\sqrt{3}$, where a, b are nonnegative integers. Considering the area of the square, we get $(a+b\sqrt{3})^2 = \frac{n\sqrt{3}}{2}$, which is the same as $2(a^2+3b^2) = (n-4ab)\sqrt{3}$. Since a, b are integers and $\sqrt{3}$ is irrational, we must have $a^2+3b^2 = 0$ and $n - 4ab = 0$. The first equation implies $a = b = 0$, which forces the sides of the square to be 0, a contradiction.

Other commended solver: **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7).

Problem 147. Factor x^8+4x^2+4 into two nonconstant polynomials with integer coefficients.

Solution. **CHENG Ka Wai** (STFA Leung Kau Kui College, Form 4), **CHEUNG CHUNG YEUNG** (STFA Leung Kau Kui College, Form 4), **FUNG Yi** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TANG Sze Ming** (STFA Leung Kau Kui College, Form 4).

$$\begin{aligned} & x^8+4x^2+4 \\ &= (x^8+4x^6+8x^4+8x^2+4) \\ &\qquad -(4x^6+8x^4+4x^2) \\ &= (x^4+2x^2+2)^2-(2x^3+2x)^2 \\ &= (x^4+2x^3+2x^2+2x+2) \\ &\qquad \cdot (x^4-2x^3+2x^2+2). \end{aligned}$$

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **HUI Chun Yin John** (Hong Kong Chinese Women's Club College, Form 6), **LAW Siu Lun Jack** (CCC Ming Kei College, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **Tak Wai Alan WONG** (University of Toronto, Canada), **WONG Wing Hong** (La Salle College, Form 4) & **YEUNG Kai Tsz Max** (Ju Ching Chu Secondary School, Form 5).

Problem 148. Find all distinct prime numbers p, q, r, s such that their sum is also prime and both $p^2 + qs$, $p^2 + qr$ are perfect square numbers. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHEUNG CHUNG YEUNG** (STFA Leung Kau Kui College, Form 4), **LAW Siu Lun Jack** (CCC Ming Kei College, Form 7), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TANG Chun Pong Ricky** (La Salle College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **WONG Wing Hong** (La Salle College, Form 4), **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 4) and **YUEN Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 6).

Since the sum of the primes p, q, r, s is a prime greater than 2, one of p, q, r, s is 2. Suppose $p \neq 2$. Then one of q, r, s is 2 so that one of $p^2 + qs$, $p^2 + qr$ is of the form $(2m+1)^2 + 2(2n + 1) = 4(m^2 + m + n) + 3$, which cannot be a perfect square as perfect squares are of the form $(2k)^2 = 4k^2$ or $(2k+1)^2 = 4(k^2+k) + 1$. So $p = 2$.

Suppose $2^2 + qs = a^2$, then q, s odd implies a odd and $qs = (a + 2)(a - 2)$. Since q, s are prime, the smaller factor $a - 2 = 1$, q or s. In the first case, $a = 3$ and $qs = 5$, which is impossible. In the remaining two cases, either $q = a-2$, $s = a+2 = q+4$ or $s = a-2$, $q = a+2 = s+4$. Next $2^2+qr = b^2$ will similarly implies q, r differe by 4. As q, r, s are distinct primes, one of r, s is $q-4$ and the other is $q = 4$. Note that $q - 4$, q, $q + 4$ have different remainders when they are divided by 3. One of them is 3 and it must be $q-4$. Thus there are two solutions $(p, q, r, s) = (2, 7, 3, 11)$ or $(2, 7, 11, 3)$. It is easy to check both solutions satisfy all conditions.

Other commended solvers: **WONG Wai Yi** (True Light Girl's College, Form 4)

Problem 149. In a $2000 \cdot 2000$ table, every square is filled with $a + 1$ or $a - 1$. It is known that the sum of these numbers is nonnegative. Prove that there are 1000 columns and 1000 rows such that the sum of the numbers in these intersection squares is at least 1000. (*Source: 1994 Russian Math Olympiad, 5th Round*)

***Solution 1*.** **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Since the numbers have a nonnegative sum, there is a column with a nonnegative sum. Hence there are at least one thousand squares in that column filled with +1. Thus, without loss of generality we may assume the squares in rows 1 to 1000 of column 1 are filled with +1. Evaluate the sums of the numbers in the squares of rows 1 to 1000 for each of the remaining columns. Pick the 999 columns with the largest sums in these evaluations. If these 999 columns have a nonnegative total sum S, then we are done (simply take rows 1 to 1000 and the first column with these 999 columns). Otherwise, $S < 0$ and at least one of the 999 columns has a negative sum. Since the sum of the first 100 squares in each column must be even, the sum of the first 100 squares in that column is at most –2. Then the total sum of all squares in rows 1 to 1000 is at most $1000 + S + (-2)1000 < -1000$.

Since the sum of the whole table is nonnegative, the sum of all squares in rows 1001 to 2000 would then be greater than 1000. Then choose the squares in these rows and the 1000 columns with the greatest sums. If these squares have a sum at least 1000, then we are done. Otherwise, assume the sum is less than 1000, then at least one of these 1000 columns will have a nonpositive sum. Thus, the remaining 1000 columns will each have a nonpositive sum. This will lead to the sum of all squares in rows 1001 to 2000 be less than $1000 + (0)1000 = 1000$, a contradiction.

Solution 2. **CHAO Khek Lun Harold** (St. Paul's College, Form 7).

We first prove that for a $n \cdot n$ square filled with +1 and -1 and the sum is at least m, where m, n are of the same parity and $m < n$, there exists a $(n-1) \cdot (n-1)$ square the numbers there have a sum at least $m + 1$. If the sum of the numbers in the $n \cdot n$ square is greater than m, we may convert some of the +1 squares to -1 to make the sum equal m. Let the sum of the numbers in rows 1 to n be $r_1, \ldots, r_n$. Since $r_1 + \cdots + r_n = m < n$, there is a $r_j \le 0$. For each square in row j, add up the numbers in the row and column on which the square lies. Let them be $a_1, \ldots, a_n$. Now $a_1 + \cdots + a_n = m + (n-1)r_j \le m < n$. Since a_i is the sum of the numbers in $2n - 1$ squares, each a_i is odd. So there exists some $a_k \le -1$. Removing row j and column k, the sum of the numbers in the remaining $(n-1) \cdot (n-1)$ square is $m - a_k \ge m + 1$. Finally convert back the –1 squares to +1 above and the result follows.

For the problem, start with $n = 2000$ and $m = 0$, then apply the result above 1000 times to get the desired statement.

Problem 150. Prove that in a convex n-sided polygon, no more than n diagonals can pairwise intersect. For what n, can there be n pairwise intersecting diagonals? (Here intersection points may be vertices.) (*Source: 1962 Hungarian Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **TANG Sze Ming** (STFA Leung Kau Kui College, Form 4.

For $n = 3$, there is no diagonal and for $n = 4$, there are exactly two intersecting diagonals. So let $n \ge 5$. Note two diagonals intersect if and only if the pairs of vertices of the diagonals share a common vertex or separate each other on the boundary. Thus, without loss of generality, we may assume the polygon is regular. For each diagonal, consider its perpendicular bisector. If n is odd, the perpendicular bisectors are exactly the n lines joining a vertex to the midpoint of its opposite side. If n is even, the perpendicular bisectors are either lines joining opposite vertices or lines joining the midpoints of opposite edges and again there are exactly n such lines. Two diagonals intersect if and only if their perpendicular bisectors do not coincide. So there can be no more than n pairwise intersecting diagonals. For $n \ge 5$, since there are exactly n different perpendicular bisectors, so there are n pairwise intersecting diagonals.

Other commended solvers: **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Problem 4. Let R be the set of real numbers. Determine all functions $f : R \to R$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real numbers x and y.

Problem 5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each $i (1 \le i < k)$.

Problem 6. I have an $n \cdot n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \le b(n) \le \frac{1}{5}n^2 + dn$$

for all $n > 0$.

Mathematical Excalibur

Volume 7, Number 3 *July 2002 – October 2002*

Olympiad Corner

The 43rd International Mathematical Olympiad 2002.

Problem 1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Problem 2. Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

Problem 3. Find all pairs of integers such that there exist infinitely many positive integers a for which

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 2, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Mathematical Games (I)

Kin Y. Li

An *invariant* is a quantity that does not change. A *monovariant* is a quantity that keeps on increasing or keeps on decreasing. In some mathematical games, winning often comes from understanding the invariants or the monovariants that are controlling the games.

Example 1. (*1974 Kiev Math Olympiad*) Numbers 1, 2, 3, …, 1974 are written on a board. You are allowed to replace any two of these numbers by one number, which is either the sum or the difference of these numbers. Show that after 1973 times performing this operation, the only number left on the board cannot be 0.

Solution. There are 987 odd numbers on the board in the beginning. Every time the operation is performed, the number of odd numbers left either stay the same (when the numbers taken out are not both odd) or decreases by two (when the numbers taken out are both odd). So the number of odd numbers left on the board after each operation is always odd. Therefore, when one number is left, it must be odd and so it cannot be 0.

Example 2. In an 8×8 board, there are 32 white pieces and 32 black pieces, one piece in each square. If a player can change all the white pieces to black and all the black pieces to white in any row or column in a single move, then is it possible that after finitely many moves, there will be exactly one black piece left on the board?

Solution. No. If there are exactly k black pieces in a row or column before a move is made to that row or column, then after the moves, the number of black pieces in the row or in the column will become $8 - k$, a change of $(8 - k) - k = 8 - 2k$ black pieces on the board. Since $8 - 2k$ is even, the parity of the number of black pieces stay the same before and after the move. Since at the start, there are 32 black pieces, there cannot be 1 black piece left at any time.

Example 3. Four x's and five o's are written around the circle in an arbitrary order. If two consecutive symbols are the same, then insert a new x between them. Otherwise insert a new o between them. Remove the old x's and o's. Keep on repeating this operation. Is it possible to get nine o's?

Solution. If we let $x = 1$ and $o = -1$, then note that consecutive symbols are replaced by their product. If we consider the product P of the nine values before and after each operation, we will see that the new P is the square of the old P. Hence, P will always equal 1 after an operation. So nine o's yielding $P = -1$ can never happen.

Example 4. There are three piles of stones numbering 19, 8 and 9, respectively. You are allowed to choose two piles and transfer one stone from each of these two piles to the third piles. After several of these operations, is it possible that each of the three piles has 12 stones?

Solution. No. Let the number of stones in the three piles be a, b and c, respectively. Consider (mod 3) of these numbers. In the beginning, they are 1, 2, 0. After one operation, they become 0, 1, 2 no matter which two piles have stones transfer to the third pile. So the remainders are always 0, 1, 2 in some order. Therefore, all piles having 12 stones are impossible.

Example 5. Two boys play the following game with two piles of candies. In the first pile, there are 12 candies and in the second pile, there are 13 candies. Each boy takes turn to make a move consisting of eating two candies from one of the piles or transferring a candy from the first pile to the second. The boy who cannot make a move loses. Show that the boy who played second cannot lose. Can he win?

Solution. Consider S to be the number of candies in the second pile minus the first. Initially, $S = 13 - 12 = 1$. After each move, S increases or decreases by 2. So $S \pmod 4$ has the pattern 1, 3, 1, 3, …. Every time after the boy who played first made a move, $S \pmod 4$ would always be 3. Now a boy loses if and only if there are no candies left in the second pile, then $S = 1 - 0 = 1$. So the boy who played second can always make a move, hence he cannot lose.

Since either the total number of candies decreases or the number of candies in the first pile decreases, so eventually the game must stop, so the boy who played second must win.

Example 6. Each member of a club has at most three enemies in the club. (Here enemies are mutual.) Show that the members can be divided into two groups so that each member in each group has at most one enemy in the group.

Solution. In the beginning, randomly divide the members into two groups. Consider the number S of the pairs of enemies in the same group. If a member has at least two enemies in the same group, then the member has at most one enemy in the other group. Transferring the member to the other group, we will decrease S by at least one. Since S is a nonnegative integer, it cannot be decreased forever. So after finitely many transfers, each member can have at most one enemy in the same group.

(Continued on page 4)

IMO 2002

Kin Y. Li

The International Mathematical Olympiad 2002 was held in Glasgow, United Kingdom from July 19 to 30. There were a total of 479 students from 84 countries and regions participated in the Olympiad.

The Hong Kong team members were

Chao Khek Lun (St. Paul's College)
Chau Suk Ling (Queen Elizabeth School)
Cheng Kei Tsi (La Salle College)
Ip Chi Ho (St. Joseph's College)
Leung Wai Ying (Queen Elizabeth School)
Yu Hok Pun (SKH Bishop Baker Secondary School).

The team leader was *K. Y. Li* and the deputy leaders were *Chiang Kin Nam* and *Luk Mee Lin*.

The scores this year ranged from 0 to 42. The cutoffs for medals were 29 points for gold, 24 points for silver and 14 points for bronze. The Hong Kong team received 1 gold medal (*Yu Hok Pun*), 2 silver medals (*Leung Wai Ying* and *Cheng Kei Tsi*) and 2 bronze medals (*Chao Khek Lun* and *Ip Chi Ho*). There were 3 perfect scores, two from China and one from Russia. After the 3 perfect scores, the scores dropped to 36 with 9 students! This was due to the tough marking schemes, which intended to polarize the students' performance to specially distinguish those who had close to complete solutions from those who should only deserve partial points.

The top five teams are China (212), Russia (204), USA (171), Bulgaria (167) and Vietnam (166). Hong Kong came in 24th (120), ahead of Australia, United Kingdom, Singapore, New Zealand, but behind Canada, France and Thailand this year.

One piece of interesting coincidence deserved to be pointed out. Both Hong Kong and New Zealand joined the IMO in 1988. Both won a gold medal for the first time this year and both gold medallists scored 29 points.

The IMO will be hosted by Japan next year at Keio University in Tokyo and the participants will stay in the Olympic village. Then Greece, Mexico, Slovenia will host in the following years.

Addendum. After the IMO, the German leader Professor Gronau sent an email to inform all leaders about his updated webpage
http://www.Mathematik-Olympiaden.de/
which contains IMO news and facts. Clicking *Internationale Olympiaden* on the left, then on that page, scrolling down and clicking *Top-Mathematikern*, *Die erfolgreichsten IMO-Teilnehmer* in blue on the right, we could find the following past IMO participants who have also won the Fields medals, the Nevanlinna prizes and the Wolf prizes:

Richard Borcherds (1977 IMO silver, 1978 IMO gold, 1998 Fields medal)

Vladmir Drinfeld (1969 IMO gold, 1990 Fields medal)

Tim Gowers (1981 IMO gold, 1998 Fields medal)

Laurent Lafforgue (1984 IMO silver, 1985 IMO silver, 2002 Fields medal)

Gregori Margulis (1959 IMO member, 1962 IMO silver, 1978 Fields medal)

Jean-Christoph Yoccoz (1974 IMO gold, 1994 Fields medal)

Alexander Razborov (1979 IMO gold, 1990 Nevanlinna prize)

Peter Shor (1977 IMO silver, 1998 Nevanlinna prize)

László Lovász (1963 IMO silver, 1964 IMO gold, 1965 IMO gold, 1966 IMO gold, 1999 Wolf prize)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***November 2, 2002***.

Problem 156. If $a, b, c > 0$ and $a^2+b^2+c^2=3$, then prove that

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\ge\frac{3}{2}.$$

Problem 157. In base 10, the sum of the digits of a positive integer n is 100 and of $44n$ is 800. What is the sum of the digits of $3n$?

Problem 158. Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on BC such that $BD = 2DC$ and let P be a point on AD such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\angle DPC.$$

Problem 159. Find all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

Problem 160. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective pressures.) What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

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Solutions

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Problem 151. Every integer greater than 2 can be written as a sum of distinct positive integers. Let $A(n)$ be the maximum number of terms in such a sum for n. Find $A(n)$. (*Source: 1993 German Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **Poon Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Tsui Ka Ho** (CUHK, Year 1), **Tak Wai Alan WONG** (University of Toronto) and **WONG Wing Hong** (La Salle College, Form 5).

Let $a_m = m(m+1)/2$. This is the sum of $1, 2, \ldots, m$ and hence the sequence a_m is strictly increasing to infinity. So for every integer n greater than 2, there is a positive integer m such that $a_m \le n < a_{m+1}$. Then n is the sum of the m positive integers

$$1, 2, \ldots, m-1,\ n-m(m-1)/2.$$

Assume $A(n) > m$. Then

$$a_{m+1} = 1+2+\cdots+(m+1) \le n,$$

a contradiction. Therefore, $A(n) = m$. Solving the quadratic inequality

$$a_m = m(m+1)/2 \le n,$$

we find m is the greatest integer less than or equal to $\left(-1+\sqrt{8n+1}\right)/2$.

Other commended solvers: **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7).

Problem 152. Let $ABCD$ be a cyclic quadrilateral with E as the intersection of lines AD and BC. Let M be the intersection of line BD with the line through E parallel to AC. From M, draw a tangent line to the circumcircle of $ABCD$ touching the circle at T. Prove that $MT = ME$. (*Source: 1957 Nanjing Math Competition*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **Poon Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), TANG Sze Ming (STFA Leung Kau Kui College), **Tsui Ka Ho** (CUHK, Year 1) and **WONG Wing Hong** (La Salle College, Form 5).

Since ME and AC are parallel, we have

$$\angle MEB = \angle ACB = \angle ADB = \angle MDE.$$

Also, $\angle BME = \angle EMD$. So triangles BME and EMD are similar. Then

$$MB / ME = ME / MD.$$

So $ME^2 = MD \cdot MB$. By the intersecting chord theorem, also $MT^2 = MD \cdot MB$. Therefore, $MT = ME$.

Problem 153. Let R denote the real numbers. Find all functions $f: R \to R$ such that the equality

$$f(f(x)+y) = f(x^2-y)+4f(x)y$$

holds for all pairs of real numbers x, y. (*Source: 1997 Czech-Slovak Match*)

Solution. **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12),

Setting $y = x^2$, we have

$$f(f(x)+x^2) = f(0)+4x^2f(x).$$

Setting $y = -f(x)$, we have

$$f(0) = f(f(x)+x^2)+4f(x)^2.$$

Comparing these, we see that for each x, we must have $f(x)=0$ or $f(x)=x^2$. Suppose $f(a) = 0$ for some nonzero a. Putting $x = a$ into the given equation, we get

$$f(y) = f(a^2-y).$$

For $y \ne a^2/2$, we have

$$y^2 \ne (a^2-y)^2,$$

which will imply $f(y) = 0$. Finally, setting $x = 2a$ and $y = a^2/2$, we have

$$f(a^2/2) = f(7a^2/2) = 0.$$

So either $f(x) = 0$ for all x or $f(x) = x^2$ for all x. We can easily check both are solutions.

Comments: Many solvers submitted incomplete solutions. Most of them got $\forall x\ (f(x)=0$ or $x^2)$, which is not the same as the desired conclusion that $(\forall x\ f(x)=0)$ or $(\forall x\ f(x)=x^2)$.

Problem 154. For nonnegative numbers a, d and positive numbers b, c satisfying $b + c \ge a + d$, what is the minimum value of $\dfrac{b}{c+d}+\dfrac{c}{a+b}$?

(*Source: 1988 All Soviet Math Olympiad*)

Solution. Without loss of generality, we may assume that $a \ge d$ and $b \ge c$. From $b + c \ge a + d$, we get

$$b+c \geq (a+b+c+d)/2.$$

Now

$$\frac{b}{c+d}+\frac{c}{a+b}$$

$$=\frac{b+c}{c+d}-c\left(\frac{1}{c+d}-\frac{1}{a+b}\right)$$

$$\geq \frac{a+b+c+d}{2(c+d)}$$

$$-(c+d)\left(\frac{1}{c+d}-\frac{1}{a+b}\right)$$

$$=\frac{a+b}{2(c+d)}+\frac{c+d}{a+b}-\frac{1}{2}$$

$$\geq 2\sqrt{\frac{a+b}{2(c+d)}\cdot\frac{c+d}{a+b}}-\frac{1}{2}$$

$$=\sqrt{2}-\frac{1}{2},$$

where the AM-GM inequality was used to get the last inequality. Tracing the equality conditions, we need $b+c=a+d$, $c=c+d$ and $a+b=\sqrt{2}\,c$. So the minimum $\sqrt{2}-1/2$ is attained, for example, when $a=\sqrt{2}+1$, $b=\sqrt{2}-1$, $c=2$, $d=0$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Problem 155. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise relatively prime numbers among them. (*Source: 1997 Hungarian Math Olympiad*)

Solution. **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12) and **WONG Wing Hong** (La Salle College, Form 5).

The answer is 9. Suppose there were 10 pairwise relatively prime numbers $a_1, a_2, \ldots, a_{10}$ among them. Being pairwise relatively prime, their least common multiple is their product *M*. Then the least common multiple of $b, a_2, \ldots, a_{10}$ for any other *b* in the set is also *M*. Since a_1 is relatively prime to each of $a_2, \ldots, a_{10}$, so *b* is divisible by a_1. Similarly, *b* is divisible by the other a_i. Hence *b* is divisible by *M*. Since *M* is a multiple of *b*, so $b = M$, a contradiction to having 1997 distinct integers.

To get an example of 9 pairwise relatively prime integers among them, let p_n be the *n*-th prime number, $a_i = p_i$ (for $i = 1, 2, \ldots, 8$), $a_9 = p_9 p_{10} \cdots p_{1988}$ and

$$b_i = p_1 p_2 \cdots p_{1988} / p_i$$

for $i = 1, 2, \ldots, 1988$. It is easy to see that the a_i's are pairwise relatively prime and any 10 of these 1997 numbers have the same least common multiple.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. (*cont.*)

$$\frac{a^m+a-1}{a^n+a^2-1}$$

is an integer.

Problem 4. Let *n* be an integer greater than 1. The positive divisors of *n* are $d_1, d_2, \ldots, d_k$ where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define

$$D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k.$$

(a) Prove that $D < n^2$.
(b) Determine all *n* for which *D* is a divisor of n^2.

Problem 5. Find all functions *f* from the set ℝ of real numbers to itself such that

$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all *x, y, z, t* in ℝ.

Problem 6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centers by $O_1, O_2, \ldots, O_n$ respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1\leq i<j\leq n}\frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

Mathematical Games (I)

(Continued from page 2)

Remarks. This method of proving is known as the *method of infinite descent*. It showed that you cannot always decrease a quantity when it can only have finitely many possible values.

Example 7. (*1961 All-Russian Math Olympiad*) Real numbers are written in an $m \times n$ table. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations, we can make the sum of the numbers along each line (row or column) nonnegative.

Solution. Let *S* be the sum of all the *mn* numbers in the table. Note that after an operation, each number stay the same or turns to its negative. Hence there are at most 2^{mn} tables. So *S* can only have finitely many possible values. To make the sum of the numbers in each line nonnegative, just look for a line whose numbers have a negative sum. If no such line exists, then we are done. Otherwise, reverse the sign of all the numbers in the line. Then *S* increases. Since *S* has finitely many possible values, *S* can increase finitely many times. So eventually the sum of the numbers in every line must be nonnegative.

Example 8. Given 2*n* points in a plane with no three of them collinear. Show that they can be divided into *n* pairs such that the *n* segments joining each pair do not intersect.

Solution. In the beginning randomly pair the points and join the segments. Let *S* be the sum of the lengths of the segments. (Note that since there are finitely many ways of connecting 2*n* points by *n* segments, there are finitely many possible values of *S*.) If two segments *AB* and *CD* intersect at *O*, then replace pairs *AB* and *CD* by *AC* and *BD*. Since

$$AB + CD = AO + OB + CO + OD > AC + BD$$

by the triangle inequality, whenever there is an intersection, doing this replacement will always decrease *S*. Since there are only finitely many possible values of *S*, so eventually there will not be any intersection.

Mathematical Excalibur

Volume 7, Number 4 *October 2002 – November 2002*

Olympiad Corner

The 2002 Canadian Mathematical Olympiad

Problem 1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5. What is the maximum number of elements that S can contain?

Problem 2. Call a positive integer n **practical** if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

For example, the divisors of 6 are **1**, **2**, **3**, and **6**. Since

$$1 = \mathbf{1},\quad 2 = \mathbf{2},\quad 3 = \mathbf{3},\quad 4 = \mathbf{1} + \mathbf{3},$$
$$5 = \mathbf{2} + \mathbf{3},\quad 6 = \mathbf{6}$$

we see that 6 is practical.

Prove that the product of two practical numbers is also practical.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 15, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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簡介費馬數

梁達榮

考慮形狀如 $2^m + 1$ 的正整數，如果它是質數，則 m 一定是 2 的正次幕。否則的話，設 $m = 2^n s$，其中 s 是 3 或以上的奇數，我們有 $2^m + 1 = 2^{2^n s} + 1 = (2^{2^n})^s + 1 = (2^{2^n} + 1)((2^{2^n})^{s-1} - (2^{2^n})^{s-2} + \cdots \pm 1)$，容易看到 $2^m + 1$ 分解成兩個正因子的積。 業餘數學家費馬 (1601-1665) 曾經考慮過以下這些“費馬”整數，設 $F_n = 2^{2^n} + 1$, $n = 0, 1, 2, \ldots$，費馬看到 $F_0 = 2^{2^0} + 1 = 3$，$F_1 = 2^{2^1} + 1 = 5$，$F_2 = 2^{2^2} + 1 = 17$，$F_3 = 2^{2^3} + 1 = 257$，$F_4 = 2^{2^4} + 1 = 65537$，都是質數，(最後一個是質數，需要花些功夫證明)，他據此而猜想，所有形如 $2^{2^n} + 1$ 的正整數都是質數。

不幸的是，大概一百年後，歐拉 $(1707 - 1783)$ 發現，F_5 不是質數，事實上，直到現在，已知的 F_n，$n \geq 5$，都不是質數。對於 F_5 不是質數，有一個簡單的證明。事實上 $641 = 5^4 + 2^4 = 5 \times 2^7 + 1$，因此 641 整除 $(5^4 + 2^4)2^{28} = 5^4 \times 2^{28} + 2^{32}$。另一方面，由於 $641 = 5 \times 2^7 + 1$，因此 641 也整除 $(5 \times 2^7 + 1)(5 \times 2^7 - 1) = 5^2 \times 2^{14} - 1$，由此，得到 641 整除 $(5^2 \times 2^{14} - 1)(5^2 \times 2^{14} + 1) = 5^4 \times 2^{28} - 1$。最後 641 整除 $5^4 \times 2^{28} + 2^{32}$ 和 $5^4 \times 2^{28} - 1$ 之差，即是 $2^{32} + 1 = F_5$。

這個證明很簡潔，但並不自然，首先，如何知道一個可能的因子是 641，其二，641 能夠寫成兩種和式，實有點幸運。或許可以探究一下，歐拉是怎樣發現 F_5 不是質數。我們相信大概的過程是這樣的，歐拉觀察到，如果 p 是 $F_n = 2^{2^n} + 1$ 的質因子，則 p 一定是 $k \cdot 2^{n+1} + 1$ 的形式。用模算術的言語，如果 p 整除 $2^{2^n} + 1$，則 $2^{2^n} \equiv -1 \pmod{p}$，取平方，得出 $2^{2^{n+1}} \equiv 1 \pmod{p}$。另外，用小費馬定理，(歐拉時已經存在)，知 $2^{p-1} \equiv 1 \pmod{p}$。如果 d 是最小的正整數，使得 $2^d \equiv 1 \pmod{p}$，可以證明(請自証)，d 整除 $p - 1$，也整除 2^{n+1}，但 d 不整除 2^n，(因為 $2^{2^n} \equiv -1 \pmod{p}$)，所以 $d = 2^{n+1}$，再因 d 整除 $p - 1$，所以 $p - 1 = k \cdot 2^{n+1}$，或者 $p = k \cdot 2^{n+1} + 1$。(如果用到所謂的二次互反律，還可以證明，p 實際上是 $k \cdot 2^{n+2} + 1$ 的形式。)

例如考慮 F_4，它的質因子一定是 $32k + 1$ 的形式，取 $k = 1, 2, \ldots$, 等，得可能的因子是 97, 193，(小於 $\sqrt{65537}$，以 $32k + 1$ 形式出現的質數)。但 97 和 193 都不整除 65537，所以 65537 是質數。另外，F_5 的質因子一定是 $64k + 1$ 的形式，取 $k = 1, 2, \ldots$, 等，得可能的因子是 193, 257, 449, 577, 641, …，經幾次嘗試，得出 $2^{2^5} + 1 = 4294967297 = 641 \times 6700417$，這樣快就找出 F_5 的一個質因子，也算幸運，事實上第二個因子也是質數，不過要證明就比較麻煩。

但是如果試圖用這樣的方法找尋其他費馬數的因子，很快就遇上問題。舉例說，$F_6 = 2^{2^6} + 1$ 是一個二十位數，它的平方根是一個十位數 $(\approx 4.29 \times 10^9)$，其中形狀如 $k \cdot 2^7 + 1 = 128k + 1$ 的數有三百多萬個，要從中找尋 F_6 的因子，可不是易事。讀者可以想像一下，F_5 的完全分解歐拉在 1732 年已找到，而在一百年後 Landry 和 Le Lasseur (1880) 才找到 F_6 的完全分解，再過約一百年， Morrison 和 Brillhart (1970) 發現 F_7 的完全分解，因此找尋費馬數的因子分解肯定不是易事。另一方面由於找尋費馬數不是

易事，Pepin在1877年找到費馬數是否質數的一個判斷：$N>3$是一個形如$2^{2^n}+1$的費馬數，則N是質數的一個充分必須條件是 $3^{\frac{N-1}{2}} \equiv -1\,(\mathrm{mod}\, N)$。 考慮到 $\frac{N-1}{2}=2^{2^n-1}$ ，因此是對3不斷取平方，然後求對N的摸。近代對求費馬數是否一個質數上，許多都以此為起點。也因此，曾經有一段長時間，已經知道F_7不是質數，但它的任一因子都不知道。

再簡述一下近代的結果，現在已知由F_5至F_{11}，都是合數，並且已完全分解。 F_{12},F_{13},F_{15}至F_{19}是合數，並且知道部分因子。但F_{14}, F_{20}, F_{22}等，知道是合數，但一個因子也不知道。最大的費馬合數，並且找到一個因子的是F_{382447}，讀者可想像一下，如果以十進制形式寫下這個數，它是多少個位數。另外如F_{33},F_{34},F_{35}等，究竟是合數或質數，一點也不知道。有興趣的話，可參考網頁
http://www.fermatsearch.org/status.htm。

由於費馬數和相關的數有特定的形式，而且具備很多有趣的性質,因此也常在競賽中出現。舉例如下：

例一： 給定費馬數 $F_0,F_1,\ldots,F_n$，有以下的關係 $F_0F_1\cdots F_{n-1}+2=F_n$。

證明： 事實上 $F_n=2^{2^n}+1=2^{2^n}-1+2$ $=2^{2^{n-1}2}-1+2=(2^{2^{n-1}}+1)(2^{2^{n-1}}-1)+2$ $=(2^{2^{n-1}}-1)F_{n-1}+2$。

對於 $2^{2^{n-1}}-1$ ，可以再分解下去，就可以得到要求的結果。當然嚴格證明可以用歸納法。

例二： 給定費馬數 F_m,F_n, $m>n$，則 F_m,F_n 是互質的。

證明： 因為 $F_m=F_{m-1}\cdots F_n\cdots F_0+2$。設 d 整除 F_m 和 F_n，則 d 也 整除2，所以$d=1$或2。但 $d\neq 2$ ，因為 F_m,F_n 都是奇數，因此 $d=1$，即 F_m,F_n 互質。

(因此知道， $F_0,F_1,F_2,\ldots$，是互質的，即他們包括無限多數個質因子，引申是有無限多個質數。)

例三： 有無限多個 n，使得 F_n+2 不是質數。

證明： 只要嘗試幾次就可以觀察到 $F_1+2=7$, $F_3+2=259$, 都是7的倍數。事實上，對於$n=0,1,2,\ldots$, $2^{2^n}\equiv 2,4,2,4,\ldots$ (mod 7)。因此對於奇數n, $F_n+2\equiv 2^{2^n}+1+2\equiv 4+1+2\equiv 0$ (mod 7)，所以不是質數。

另一個容易看到的事實是：

例四： 對於 $n>1$, F_n 最尾的數字是 7。

證明: 對於 $n>1$, 2^n 是 4 的倍數，設 $2^n=4k$, 得 $F_n=2^{2^n}+1=2^{4k}+1=(2^4)^k+1$ $\equiv 1^k+1\equiv 2\,(\mathrm{mod}\,5)$。因此 F_n 最尾的數字是 2 或 7，它不可以是 2，因為 F_n 不是偶數。

例五： 證明存在一個正整數 k，使得對任何正整數 n， $k\cdot 2^n+1$ 都不是質數。

（如果n固定，但容許k在正整數中變動，由一個重要的定理 (Dirichlet) 知道在序列中存在無限多個質數。但若果k固定,而n變動,在序列中究竟有多少個質數，是否無限多個，一般都不大清楚。事實上，反可以找到一個k，對於任何正整數n， $k\cdot 2^n+1$ 都不是質數。這原是波蘭數學家Sierpinski (1882-1969)的一個結果，後來演變成美國數學奧林匹克(1982)的一個題目，直到現在，基本是只有一種證明方法，並且與費馬數有關。）

(續於第四頁)

The 2002 Hong Kong IMO team at the Hong Kong Chek Lap Kok Airport taken on August 1, 2002. From left to right, *Chau Suk Ling*, *Chao Khek Lun*, *Cheng Kei Tsi*, *Chiang Kin Nam* (Deputy Leader), *Yu Hok Pun*, *Ip Chi Ho*, *Leung Wai Ying*, *Li Kin Yin* (Leader).

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***December 15, 2002***.

Problem 161. Around a circle are written all of the positive integers from 1 to N, $N \geq 2$, in such a way that any two adjacent integers have at least one common digit in their base 10 representations. Find the smallest N for which this is possible.

Problem 162. A set of positive integers is chosen so that among any 1999 consecutive positive integers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other.

Problem 163. Let a and n be integers. Let p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be a product of two nonconstant polynomials with integer coefficients.

Problem 164. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$.

Problem 165. For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $n_1, n_2, \ldots, n_{50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

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Solutions

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Problem 156. If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$, then prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

(*Source: 1999 Belarussian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 5).

By the AM-GM and AM-HM inequalities, we have

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca}$$

$$\geq \frac{1}{1+\frac{a^2+b^2}{2}} + \frac{1}{1+\frac{b^2+c^2}{2}} + \frac{1}{1+\frac{c^2+a^2}{2}}$$

$$\geq \frac{9}{3+a^2+b^2+c^2} = \frac{3}{2}.$$

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 5), **LAM Ho Yin** (South Tuen Mun Government Secondary School, Form 6), **LAM Wai Pui** (STFA Leung Kau Kui College, Form 6), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6),**Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LO Chi Fai** (STFA Leung Kau Kui College, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5), **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School, teacher), **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6), **TANG Sze Ming** (STFA Leung Kau Kui College, Form 5), **YAU Chun Biu** and **YIP Wai Kiu (**Jockey Club Ti-I College, Form 5) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Problem 157. In base 10, the sum of the digits of a positive integer n is 100 and of $44n$ is 800. What is the sum of the digits of $3n$? (*Source: 1999 Russian Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LO Chi Fai** (STFA Leung Kau Kui College, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6), and **WONG Wing Hong** (La Salle College, Form 5).

Let $S(x)$ be the sum of the digits of x in base 10. For digits a and b, if $a + b > 9$, then $S(a + b) = S(a) + S(b) - 9$. Hence, if we have to carry in adding x and y, then $S(x + y) < S(x) + S(y)$. So in general, $S(x + y) \leq S(x) + S(y)$. By induction, we have $S(kx) \leq kS(x)$ for every positive integer k. Now

$$\begin{aligned} 800 &= S(44n) = S(40n + n) \\ &\leq S(40n) + S(4n) = 2S(4n) \\ &\leq 8S(n) = 800 . \end{aligned}$$

Hence equality must hold throughout and there can be no carry in computing $4n = n + n + n + n$. So there is no carry in $3n = n + n + n$ and $S(3n) = 300$.

Other commended solvers: **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7).

Problem 158. Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on BC such that $BD = 2DC$ and let P be a point on AD such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\angle DPC.$$

(*Source: 1999 Turkish Math Olympiad*)

Solution. **LAM Wai Pui** (STFA Leung Kau Kui College, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) , **WONG Wing Hong** (La Salle College, Form 5) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Let E be a point on AD extended so that $PE=PB$. Since $\angle CAB=\angle EPB$ and $CA/AB=1=EP/PB$, triangles CAB and EPB are similar. Then $\angle ACB=\angle PEB$, which implies A, C, E, B are concyclic. So $\angle AEC=\angle ABC=\angle AEB$. Therefore, AE bisects $\angle CEB$.

Let M be the midpoint of BE. By the angle bisector theorem, $CE/EB = CD/DB = 1/2$. So $CE = ½EB = ME$. Also, $PE = PE$ and PE bisects $\angle CEM$. It follows triangles CEP and MEP are congruent. Then $\angle BAC = \angle BPE = 2\angle MPE = 2\angle CPE = 2\angle DPC$.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 159. Find all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

(*Source: 1999 Italian Math Olympiad*)

Solution. (*Official Solution*)

For $n = 1$, the solutions are $(x, k, n) = (3^k - 1, k, 1)$, where k is for any positive integer.

For $n > 1$, if n is even, then $x^n + 1 \equiv 1$ or 2 (mod 3) and hence cannot be $3^k \equiv 0 \pmod 3$. So n must be odd. Now $x^n + 1$ can be factored as

$$(x+1)(x^{n-1} - x^{n-2} + \cdots + 1).$$

If $3^k = x^n + 1$, then both of these factors are powers of 3, say they are 3^s, 3^t, respectively. Since

$$x + 1 \le x^{n-1} - x^{n-2} + \cdots + 1,$$

so $s \le t$. Then

$$0 \equiv 3^t \equiv (-1)^{n-1} - (-1)^{n-2} + \cdots + 1$$
$$= n \pmod{x + 1}$$

implying n is divisible by $x + 1$ (and hence also by 3). Let $y = x^{n/3}$. Then

$$3^k = y^3 + 1 = (y + 1)(y^2 - y + 1).$$

So $y + 1$ is also a power of 3, say it is 3^r. If $r = 1$, then $y = 2$ and $(x, k, n) = (2, 2, 3)$ is a solution. Otherwise, $r > 1$ and

$$3^k = y^3 + 1 = 3^{3r} - 3^{2r+1} + 3^{r+1}$$

is strictly between 3^{3r-1} and 3^{3r}, a contradiction.

Other commended solvers: **LEE Pui Chung** (Wah Yan College, Kowloon, Form 7), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Problem 160. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective pressures.) What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

(*Source:1999 Russian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

For $k = 5$, it is always possible. We equalize balloons 1 to 5, then 6 to 10, and so on (five at time). Now take one balloon from each of these 8 groups. We have eight balloons, say a, b, c, d, e, f, g, h. We can equalize a, b, c, d, then e, f, g, h, followed by a, b, e, f and finally c, d, g, h. This will equalize all 8 balloons. Repeat getting one balloon from each of the 8 groups for 4 more times, then equalize them similarly. This will make all 40 balloons having the same pressure.

For $k < 5$, it is not always possible. If the i-th balloon has initial pressure $p_i = \pi^i$, then after equalizing operations, their pressures will always have the form $c_1 p_1 + \cdots + c_{40} p_{40}$ for some rational numbers $c_1, \ldots, c_{40}$. The least common multiple of the denominators of these rational numbers will always be of the form $2^r 3^s$ as $k = 1, 2, 3$ or 4 implies we can only change the denominators by a factor of 2, 3 or 4 after an operation. So $c_1, \ldots, c_{40}$ can never all be equal to 1/40.

Olympiad Corner

(continued from page 1)

Problem 3. Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \ge a + b + c$$

and determine when equality occurs.

Problem 4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with center B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q. Prove that $PQ = r$.

Problem 5. Let $N = \{0, 1, 2, \ldots\}$. Determine all functions $f : N \to N$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in N.

簡介費馬數

(續第二頁)

證明：（證明的起點是中國餘式定理，設 $m_1, m_2, \ldots, m_r$ 是互質的正整數，$a_1, a_2, \ldots, a_r$ 是任意整數，則方程組 $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}$, …, $x \equiv a_r \pmod{m_r}$,有解。並且其解對於模 $m = m_1 m_2 \cdots m_r$ 唯一。現在考慮到任意正整數 n，都可以寫成 $2^h q$ 的形式，其中 q 是奇數。如果能夠選擇 k，使得 $k > 1, k \equiv 1 \pmod{2^{2^h} + 1}$, 則 $k \cdot 2^n + 1 = k \cdot 2^{2^h q} + 1 \equiv (1)(2^{2^h})^q + 1 \equiv (1)(-1)^q + 1 \equiv (-1) + 1 \equiv 0 \pmod{2^{2^h} + 1}$，所以 $k2^n + 1$ 不是質數。留意到這裡用到 q 是奇數的性質。不過，如果這樣做的話，h 會因 n 而變，而 k 隨 h 而變，這是不容許的，k 要在起先之前決定，而不受 n 影響。）

解決的方法是這樣的，我們可以先選擇 k，使得 $k > 1,\ k \equiv 1 \pmod{2^{2^h} + 1}$ ，其中 $h = 0, 1, 2, 3, 4$。這是可能的，因為我們知道 F_0, F_1, F_2, F_3 和 F_4 是不同的質數。這樣的話，可以證明對於所有 $n = 2^h q$ ，其中 $h < 5$，q 是奇數，$k \cdot 2^n + 1$ 都不是質數。對於 $n = 2^h q$，$h \ge 5$ ，又可以怎樣處理呢。留意到所有這樣的數，都可以寫成 $n = 2^h q = 2^5 m$ 的形式，其中 m 可以是奇數，也可以是偶數。另一方面，我們知道 $F_5 = 2^{2^5} + 1 = (641) \times (6700417)$，其中 $P = 641$， $Q = 6700417$ 是不同的質數。如果我們選擇 k，使得 $k > 1$，和 $k \equiv -1 \pmod P$ ，$k \equiv 1 \pmod Q$，則 $k2^n + 1 = k2^{2^5 m} + 1 \equiv (-1)(2^{2^5})^m + 1 \equiv (-1)(-1)^m + 1 \equiv (-1)^{m+1} + 1 \pmod P$，另一方面 $k2^n + 1 = k2^{2^5 m} + 1 \equiv (1)(2^{2^5})^m + 1 \equiv (-1)^m + 1 \pmod Q$。如果 m 是偶數，則 $k2^n + 1$ 是 P 的倍數，如果 m 是奇數，則 $k2^n + 1$ 是 Q 的倍數，因此都不是質數。歸納言之，選擇 k，使得 $k \equiv 1 \pmod x$，$x = 3, 5, 17, 257, 65537, 6700417$, $k \equiv -1 \pmod{641}$ ，則對於所有形如 $k2^n + 1$ 的數，都不是質數。（最後要留意的是，這方程組的最小正整數解不可能是 1，因此所有的 $k2^n + 1$ ，都不是質數。）

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Olympiad Corner

The 19th Balkan Mathematical Olympiad was held in Antalya, Turkey on April 27, 2002. The problems are as follow.

Problem 1. Let $A_1, A_2, \ldots, A_n$ ($n \ge 4$) be points on the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, \ldots, A_n$ are connected by line segments in such a way that each point is connected to three others. Prove that there exists $k > 1$ and distinct points $X_1, X_2, \ldots, X_{2k}$ in $\{A_1, A_2, \ldots, A_n\}$ so that for each $1 \le i \le 2k-1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

Problem 2. The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1 = 20$, $a_2 = 30$, $a_{n+2} = 3a_{n+1} - a_n$, $n > 1$. Find all positive integers n for which $1+5a_na_{n+1}$ is a perfect square.

Problem 3. Two circles with different radii intersect at two pints A and B. The common tangents of these circles are MN and ST where the points M, S are on one of the circles and N, T are on the other. Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the vertices of a rectangle.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 26, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Mathematical Games (II)

Kin Y. Li

There are many mathematical game problems involving strategies to win or to defend. These games may involve number theoretical properties or combinatorial reasoning or geometrical decomposition. Some games may go on forever, while some games come to a stop eventually. A winning strategy is a scheme that allows a player to make moves to win the game no matter how the opponent plays. A defensive strategy cuts off the opponent's routes to winning. The following examples illustrate some standard techniques.

Examples 1. There is a table with a square top. Two players take turn putting a dollar coin on the table. The player who cannot do so loses the game. Show that the first player can always win.

Solution. The first player puts a coin at the center. If the second player can make a move, the first player can put a coin at the position symmetrically opposite the position the second player placed his coin with respect to the center of the table. Since the area of the available space is decreasing, the game must end eventually. The first player will win.

Example 2. (*Bachet's Game*) Initially, there are n checkers on the table, where $n > 0$. Two persons take turn to remove at least 1 and at most k checkers each time from the table. The last person who can remove any checker wins the game. For what values of n will the first person have a winning strategy? For what values of n will the second person have a winning strategy?

Solution. By testing small cases of n, we can easily see that if n is *not* a multiple of $k + 1$ in the beginning, then the first person has a winning strategy, otherwise the second person has a winning strategy. To prove this, let n be the number of checkers on the table. If $n = (k+1)q + r$ with $0 < r < k + 1$, then the first person can win by removing r checkers each time. (Note $r > 0$ every time at the first person's turn since in the beginning it is so and the second person starts with a multiple of $k + 1$ checkers each time and can only remove 1 to k checkers.)

However, if n is a multiple of $k + 1$, then no matter how many checkers the first person takes, the second person can now win by removing r checkers every time.

Example 3. (*Game of Nim*) There are 3 piles of checkers on the table. The first, second and third piles have x, y and z checkers respectively in the beginning, where $x, y, z > 0$. Two persons take turn choosing one of the three piles and removing at least one to all checkers in that pile each time from the table. The last person who can remove any checker wins the game. Who has a winning strategy?

Solution. In base 2 representations, let

$$x = (a_1a_2\ldots a_n)_2, \quad y = (b_1b_2\ldots b_n)_2,$$
$$z = (c_1c_2\ldots c_n)_2, \quad N = (d_1d_2\ldots d_n)_2,$$

where $d_i \equiv a_i + b_i + c_i \pmod 2$. The first person has a winning strategy if and only if N is not 0, i.e. not all d_i's are 0.

To see this, suppose N is not 0. The winning strategy is to remove checkers so N becomes 0. When the d_i's are not all zeros, look at the smallest i such that $d_i = 1$, then one of a_i, b_i, c_i equals 1, say $a_i = 1$. Then remove checkers from the first pile so that $x = (e_ie_{i+1}\ldots e_n)_2$ checkers are left, where $e_j = a_j$ if $d_j = 0$, otherwise $e_j = 1 - a_j$.

(For example, if $x = (1000)_2$ and $N = (1001)_2$, then change x to $(0001)_2$.) After the move, N becomes 0. So the first person can always make a move. The second person will always have $N = 0$ at his turn and making any move will result in at least one d_i not 0, i.e. $N \neq 0$. As the number of checkers is decreasing, eventually the second person cannot make a move and will lose the game.

Example 4. Twenty girls are sitting around a table and are playing a game with n cards. Initially, one girl holds all the cards. In each turn, if at least one girl holds at least two cards, one of these girls must pass a card to each of her two neighbors. The game ends if and only if each girl is holding at most one card.

(*a*) Prove that if $n \geq 20$, then the game cannot end.
(*b*) Prove that if $n < 20$, the game must end eventually.

Solution. (*a*) If $n > 20$, then by the pigeonhole principle, at every moment there exists a girl holding at least two cards. So the game cannot end.

If $n = 20$, then label the girls $G_1, G_2, \ldots, G_{20}$ in the clockwise direction and let G_1 be the girl holding all the cards initially. Define the current value of a card to be i if it is being held by G_i. Let S be the total value of the cards. Initially, $S = 20$.

Consider before and after G_i passes a card to each of her neighbors. If $i = 1$, then S increases by $-1-1+2+20=20$. If $1 < i < 20$, then S does not change because $-i-i+(i-1)+(i+1)=0$. If $i = 20$, then S decreases by 20 because $-20-20+1+19=-20$. So before and after moves, S is always a multiple of 20. Assume the game can end. Then each girl holds a card and $S = 1+2+\cdots+20 = 210$, which is not a multiple of 20, a contradiction. So the game cannot end.

(*b*) To see the game must end if $n < 20$, let's have the two girls sign the card when it is the first time one of them passes card to the other. Whenever one girl passes a card to her neighbor, let's require the girl to use the signed card between the pair if available. So a signed card will be stuck between the pair who signed it. If $n < 20$, there will be a pair of neighbors who never signed any card, hence never exchange any card.

If the game can go on forever, record the number of times each girl passed cards. Since the game can go on forever, not every girl passed card finitely many time. So starting with a pair of girls who have no exchange and moving clockwise one girl at a time, eventually there is a pair G_i and G_{i+1} such that G_i passed cards finitely many times and G_{i+1} passed cards infinitely many times. This is clearly impossible since G_i will eventually stopped passing cards and would keep on receiving cards from G_{i+1}.

Example 5. (*1996 Irish Math Olympiad*) On a 5×9 rectangular chessboard, the following game is played. Initially, a number of discs are randomly placed on some of the squares, no square containing more than one disc. A turn consists of moving all of the discs subject to the following rules:

(*i*) each disc may be moved one square up, down, left or right;
(*ii*) if a disc moves up or down on one turn, it must move left or right on the next turn, and vice versa;
(*iii*) at the end of each turn, no square can contain two or more discs.

The game stops if it becomes impossible to complete another turn. Prove that if initially 33 discs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 discs on the board so that the game can continue forever.

Solution. If 32 discs are placed in the lower right 4×8 rectangle, they can all move up, left, down, right, repeatedly and the game can continue forever.

To show that a game with 33 discs must stop eventually, label the board as shown below:

1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1

Note that there are only eight squares labeled with 3's. A disc on 1 goes to a 3 after two moves, a disc on 2 goes to a 1 or 3 immediately, and a disc on 3 goes to a 2 immediately. Thus if k discs start on 1 and $k > 8$, the game stops because there are not enough 3's to accommodate these discs after two moves. Thus we assume $k \leq 8$, in which case there are at most sixteen discs on squares with 1's or 3's at the start, and at least seventeen discs on squares with 2's. Of these seventeen discs, at most eight can move onto squares with 3's after one move, so at least nine end up on squares with 1's. These discs will not all be able to move onto squares with 3's two moves later. So the game must eventually stop.

Example 6. (*1995 Israeli Math Olympiad*) Two players play a game on an infinite board that consists of 1×1 squares. Players I chooses a square and marks it with an O. Then, player II chooses another square and marks with an X. They play until one of the players marks a row or a column of five consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning.

Solution: Label the squares as shown below.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	1	2	3	3	1	2	3	3	⋯
⋯	1	2	4	4	1	2	4	4	⋯
⋯	3	3	1	2	3	3	1	2	⋯
⋯	4	4	1	2	4	4	1	2	⋯
⋯	1	2	3	3	1	2	3	3	⋯
⋯	1	2	4	4	1	2	4	4	⋯
⋯	3	3	1	2	3	3	1	2	⋯
⋯	4	4	1	2	4	4	1	2	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Note that each number occurs in a pair. The 1's and 2's are in vertical pairs and the 3's and 4's are in horizontal pairs. Whenever player I marks a square, player II will mark the other square in the pair. Since any five consecutive vertical or horizontal squares must contain a pair of these numbers, so player I cannot win.

Example 7. (*1999 USAMO*) The Y2K Game is played on a 1×2000 grid as follow. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell *SOS* wins. If all boxes are filled without producing any *SOS,* then the game is a draw. Prove that the second player has a winning strategy.

Solution. Call an empty square *bad* if playing an S or an O in that square will let the next player gets *SOS* in the next move.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***January 26, 2003***.

Problem 166. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\,[a/b] - [c/a]\,[c/b] \ \le\ c\,\min\{1/a, 1/b\}.$$

Problem 167. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the sum of their digits in base 10 representation.

Problem 168. Let AB and CD be nonintersecting chords of a circle and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the part of segment CD lying inside triangle ABP.

Problem 169. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

Problem 170. (*Proposed by Abderrahim Ouardini, Nice, France*) For any (nondegenerate) triangle with sides a, b, c, let $\sum' h\,(a, b, c)$ denote the sum $h\,(a, b, c) + h\,(b, c, a) + h\,(c, a, b)$. Let $f\,(a, b, c) = \sum' (a\,/\,(b + c - a))^2$ and $g\,(a, b, c) = \sum' j(a, b, c)$, where $j(a,b,c) = (b + c - a)\,/\sqrt{(c+a-b)(a+b-c)}$. Show that $f\,(a, b, c) \ge \max\{3, g(a, b, c)\}$ and determine when equality occurs. (Here $\max\{x,y\}$ denotes the maximum of x and y.)

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Solutions

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Problem 161. Around a circle are written all of the positive integers from 1 to N, $N \ge 2$, in such a way that any two adjacent integers have at least one common digit in their base 10 representations. Find the smallest N for which this is possible. (*Source: 1999 Russian Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yan Sau** (True Light Girls' College, Form 6), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **LAM Wai Pui** (STFA Leung Kau Kui College, Form 5), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Note one of the numbers adjacent to 1 is at least 11. So $N \ge 11$. Then one of the numbers adjacent to 9 is at least 29. So $N \ge 29$. Finally $N = 29$ is possible by writing 1, 11, 12, 2, 22, 23, 3, 13, 14, 4, 24, 25, 5, 15, 16, 6, 26, 27, 7, 17, 18, 8, 28, 29, 9, 19, 21, 20, 10 around a circle. Therefore, the smallest N is 29.

Problem 162. A set of positive integers is chosen so that among any 1999 consecutive positive integers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other. (*Source: 1999 Russian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Define $A(1, i) = i$ for i=1,2,…, 1999. For $k \ge 2$, let $B(k)$ be the product of $A(k–1, 1)$, $A(k–1, 2)$, …, $A(k–1, 1999)$ and define $A(k, i) = B(k) + A(k–1, i)$ for i = 1,2,…, 1999. Since $B(k)$ is a multiple of $A(k–1, i)$, so $A(k, i)$ is also a multiple of $A(k–1, i)$. Then $m < n$ implies $A(n, i)$ is a multiple of $A(m, i)$.

Also, by simple induction on k, we can check that $A(k, 1), A(k, 2), \ldots, A(k, 1999)$ are consecutive integers. So for k = 1,2, …, 2000, among $A(k, 1), A(k, 2), \ldots, A(k, 1999)$, there is a chosen number $A(k, i_k)$. Since $1 \le i_k \le 1999$, by the pigeonhole principle, two of the i_k's are equal. Therefore, among the chosen numbers, there are two numbers with one dividing the other.

Comments: The condition "among any 1999 consecutive positive integers, there is a chosen number" is meant to be interpreted as "among any 1999 consecutive positive integers, there exists at least one chosen number." The solution above covered this interpretation.

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 163. Let a and n be integers. Let p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be the product of two nonconstant polynomials with integer coefficients. (*Source: 1999 Romanian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School).

Assume we have $f(x) = g(x)\,h(x)$, where $g(x)$ and $h(x)$ are two nonconstant polynomials with integer coefficients. Since $p = f(0) = g(0)\,h(0)$, we have either

$$g(0) = \pm p,\quad h(0) = \pm 1$$
$$\text{or}\quad g(0) = \pm 1,\quad h(0) = \pm p.$$

Without loss of generality, assume $g(0) = \pm 1$. Then $g(x) = \pm x^m + \cdots \pm 1$. Let $z_1, z_2, \ldots, z_m$ be the (possibly complex) roots of $g(x)$. Since $1 = |g(0)| = |z_1|\,|z_2| \cdots |z_m|$, so $|z_i| \le 1$ for some i. Now $0 = f(z_i) = z_i^n + az_i + p$ implies

$$p = -z_i^n - az_i \le |z_i|^n + |a|\,|z_i| \le 1 + |a|,$$

a contradiction.

Other commended solvers: **FOK Kai Tung** (Yan Chai Hospital No. 2 Secondary School, Form 6).

Problem 164. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$. (*Source: 1999 Belarussian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Let AO cut BC at D and AP extended cut OC at E. By Ceva's theorem (ΔAOC and point P), we have

$$\frac{AM}{MC} \times \frac{CE}{EO} \times \frac{OD}{DA} = 1.$$

Since $AM = MC$, we get $OD/DA = OE/EC$, which implies $DE \parallel AC$. Then $\angle EDC = \angle DCA = \angle DAC = \angle ODE$, which implies DE bisects $\angle ODC$. In ΔACD, since CE and DE are external angle bisectors at $\angle C$ and $\angle D$ respectively, so E is the excenter of ΔACD opposite A. Then AE bisects $\angle OAC$ so that $\angle DAP = \angle CAP$. Finally,

$$\begin{aligned}\angle BAP &= \angle BAD + \angle DAP \\ &= \angle DCA + \angle CAP \\ &= \angle BPA.\end{aligned}$$

Therefore, $AB = BP$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 165. For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $n_1, n_2, \ldots, n_{50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

(*Source: 1999 Polish Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

We will prove the statement that for $m > 1$, there are positive integers $n_1 < n_2 < \cdots < n_m$ such that all $n_i + S(n_i)$ are equal and n_m is of the form $10\cdots08$ by induction.

For the case $m = 2$, take $n_1 = 99$ and $n_2 = 108$, then $n_i + S(n_i) = 117$.

Assume the case $m = k$ is true and $n_k = 10\cdots08$ with h zeros. Consider the case $m = k + 1$. For $i = 1, 2, \ldots, k$, define

$$N_i = n_i + C, \text{ where } C = 99\cdots900\cdots0$$

(C has $n_k - 8$ nines and $h + 2$ zeros) and $N_{k+1} = 10\cdots08$ with $n_k - 7 + h$ zeros. Then for $i = 1, 2, \ldots, k$,

$$N_i + S(N_i) = C + n_i + S(n_i) + 9(n_k - 8)$$

are all equal by the case $m = k$. Finally,

$$\begin{aligned}N_k + S(N_k) &= C + n_k + 9 + 9(n_k - 8) \\ &= 10\cdots017 \text{ } (n_k - 8 + h \text{ zeros}) \\ &= 10\cdots008 + 9 \\ &= N_{k+1} + S(N_{k+1})\end{aligned}$$

completing the induction.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5).

Olympiad Corner

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Problem 4. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$,

$$\begin{aligned}2n + 2001 \le\ & f(f(n)) + f(n) \\ \le\ & 2n + 2003.\end{aligned}$$

($\mathbb{N}$ is the set of all positive integers.)

Mathematical Games II

(continued from page 2)

<u>Key Observation</u>: A square is bad if and only if it is in a block of 4 consecutive squares of the form $S{*}{*}S$, where * denotes an empty square.

(*<u>Proof.</u>* Clearly, the empty squares in $S{*}{*}S$ are bad. Conversely, if a square is bad, then playing an O there will allow an SOS in the next move by the other player. Thus the bad square must have an S on one side and an empty square on the other side. Playing an S there will also lose the game in the next move, which means there must be another S on the other side of the empty square.)

Now the second player's winning strategy is as follow: after the first player made a move, play an S at least 4 squares away from either end of the grid and from the first player's first move. On the second move, the second player will play an S three squares away from the second player's first move so that the squares in between are empty. (If the second move of the first player is next to or one square away from the first move of the second player, then the second player will place the second S on the other side.) After the second move of the second player, there are 2 bad squares on the board. So eventually somebody will fill these squares and the game will not be a draw.

On any subsequent move, when the second player plays, there will be an odd number of empty squares and an even number of bad squares, so the second player can always play a square that is not bad.

Example 8. (*1993 IMO*) On an infinite chessboard, a game is played as follow. At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece that has been jumped over is then removed. Find those values of n for which the game can end with only one piece remaining on the board.

Solution. Let $\mathbb{Z}$ denotes the set of integers. Consider the pieces placed at the *lattice points* $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$. For $k = 0, 1, 2$, let $C_k = \{(x, y) \in \mathbb{Z}^2 : x+y \equiv k \pmod 3\}$. Let a_k be the number of pieces placed at lattice points in C_k.

A horizontal move takes a piece at (x, y) to an unoccupied point $(x \pm 2, y)$ jumping over a piece at $(x \pm 1, y)$. After the move, each a_k goes up or down by 1. So each a_k changes parity. If n is divisible by 3, then $a_0 = a_1 = a_2 = n^2/3$ in the beginning. Hence at all time, the a_k's are of the same parity. So the game cannot end with one piece left causing two a_k's 0 and the remaining 1.

If n is not divisible by 3, then the game can end. We show this by induction. For $n = 1$ or 2, this is easily seen. For $n \ge 4$, we introduce a trick to reduce the $n \times n$ square pieces to $(n-3) \times (n-3)$ square pieces.

<u>*Trick:*</u> Consider pieces at (0,0), (0,1), (0,2), (1,0). The moves $(1,0) \to (-1,0)$, $(0,2) \to (0,0)$, $(-1,0) \to (1,0)$ remove three consecutive pieces in a column and leave the fourth piece at its original lattice point.

We can apply this trick repeatedly to the $3 \times (n-3)$ pieces on the bottom left part of the $n \times n$ squares from left to right, then the $n \times 3$ pieces on the right side from bottom to top. This will leave $(n-3) \times (n-3)$ pieces. Therefore, the $n \times n$ case follows from the $(n-3) \times (n-3)$ case, completing the induction.

Mathematical Excalibur

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Olympiad Corner

The Fifth Hong Kong (China) Mathematical Olympiad was held on December 21, 2002. The problems are as follow.

Problem 1. Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

(a) Prove that the tangent to the second circle at R is parallel to AK.

(b) Prove that these two tangents are concurrent with KM.

Problem 2. Let $n \geq 3$ be an integer. In a conference there are n mathematicians. Every pair of mathematicians communicate in one of the n official languages of the conference. For any three different official languages, there exist three mathematicians who communicate with each other in these three languages. Determine all n for which this is possible. Justify your claim.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Functional Equations

Kin Y. Li

A *functional equation* is an equation whose variables are ranging over functions. Hence, we are seeking all possible functions satisfying the equation. We will let $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}^+$ or $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0$ denote the nonnegative integers, $\mathbb{Q}$ denote the rational numbers, $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{C}$ denote the complex numbers.

In simple cases, a functional equation can be solved by introducing some substitutions to yield more information or additional equations.

Example 1. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Solution. Replacing x by $1-x$, we have

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4.$$

Since $f(1-x) = 2x - x^4 - x^2 f(x)$ by the given equation, substituting this into the last equation and solving for $f(x)$, we get $f(x) = 1 - x^2$.

Check: For $f(x) = 1 - x^2$,

$$x^2 f(x) + f(1-x) = x^2(1-x^2) + (1-(1-x)^2) = 2x - x^4.$$

For certain types of functional equations, a standard approach to solving the problem is to determine some special values (such as $f(0)$ or $f(1)$), then inductively determine $f(n)$ for $n \in \mathbb{N}_0$, follow by the values $f(1/n)$ and use density to find $f(x)$ for all $x \in \mathbb{R}$. The following are examples of such approach.

Example 2. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that the *Cauchy equation*

$$f(x+y) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{Q}$.

Solution. *Step 1* Taking $x = 0 = y$, we get $f(0) = f(0) + f(0) + f(0)$, which implies $f(0) = 0$.

Step 2 We will prove $f(kx) = k f(x)$ for $k \in \mathbb{N}$, $x \in \mathbb{Q}$ by induction. This is true for $k = 1$. Assume this is true for k. Taking $y = kx$, we get

$$f((k+1)x) = f(x + kx) = f(x) + f(kx) = f(x) + k f(x) = (k+1) f(x).$$

Step 3 Taking $y = -x$, we get

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x),$$

which implies $f(-x) = -f(x)$. So

$$f(-kx) = -f(kx) = -k f(x) \text{ for } k \in \mathbb{N}.$$

Therefore, $f(kx) = k f(x)$ for $k \in \mathbb{Z}$, $x \in \mathbb{Q}$.

Step 4 Taking $x = 1/k$, we get

$$f(1) = f(k(1/k)) = k f(1/k),$$

which implies $f(1/k) = (1/k) f(1)$.

Step 5 For $m \in \mathbb{Z}$, $n \in \mathbb{N}$,

$$f(m/n) = m f(1/n) = (m/n) f(1).$$

Therefore, $f(x) = cx$ with $c = f(1)$.

Check: For $f(x) = cx$ with $c \in \mathbb{Q}$,

$$f(x+y) = c(x+y) = cx + cy = f(x) + f(y).$$

In dealing with functions on $\mathbb{R}$, after finding the function on $\mathbb{Q}$, we can often finish the problem by using the following fact.

Density of Rational Numbers *For every real number x, there are rational numbers $p_1, p_2, p_3, \ldots$ increase to x and there are rational numbers $q_1, q_2, q_3, \ldots$ decrease to x.*

This can be easily seen from the decimal representation of real numbers. For example, the number $\pi = 3.1415\ldots$ is the limits of 3, 31/10, 314/100, 3141/1000, 31415/10000, … and also 4, 32/10, 315/100, 3142/1000, 31416/10000, ….

(In passing, we remark that there is a similar fact with rational numbers replaced by irrational numbers.)

Example 3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in \mathbb{R}$ and $f(x) \ge 0$ for $x \ge 0$.

Solution. Step 1 By example 2, we have $f(x) = x\,f(1)$ for $x\in\mathbb{Q}$.

Step 2 If $x \ge y$, then $x - y \ge 0$. So

$$f(x)=f((x-y)+y)=f(x-y)+f(y)\ge f(y).$$

Hence, f is increasing.

Step 3 If $x \in\mathbb{R}$, then by the density of rational numbers, there are rational p_n, q_n such that $p_n \le x \le q_n$, the p_n's increase to x and the q_n's decrease to x. So by step 2, $p_n\, f(1) = f(p_n) \le f(x) \le f(q_n) = q_n\, f(1)$. Taking limits, the sandwich theorem gives $f(x) = x\,f(1)$ for all x. Therefore, $f(x) = cx$ with $c \ge 0$. The checking is as in example 2.

Remarks. (1) In example 3, if we replace the condition that "$f(x) \ge 0$ for $x \ge 0$" by "f is monotone", then the answer is essentially the same, namely $f(x) = cx$ with $c = f(1)$. Also if the condition that "$f(x) \ge 0$ for $x \ge 0$" is replaced by "f is continuous at 0", then steps 2 and 3 in example 3 are not necessary. We can take rational p_n's increase to x and take limit of $p_n f(1) = f(p_n) = f(p_n-x) + f(x)$ to get $x\,f(1) = f(x)$ since p_n-x increases to 0.

(2) The Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ has noncontinuous solutions (in particular, solutions not of the form $f(x) = cx$). This requires the concept of a *Hamel basis* of the vector space $\mathbb{R}$ over $\mathbb{Q}$ from linear algebra.

The following are some useful facts related to the Cauchy equation.

Fact 1. *Let $A = \mathbb{R}$, $[0,\infty)$ or $(0,\infty)$. If $f:A\to\mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in A$, then either $f(x) = 0$ for all $x \in A$ or $f(x) = x$ for all $x \in A$.*

Proof. By example 2, we have $f(x) = f(1)\,x$ for all $x\in\mathbb{Q}$. If $f(1) = 0$, then $f(x) = f(x\cdot 1) = f(x)\,f(1)=0$ for all $x\in A$.

Otherwise, we have $f(1) \ne 0$. Since $f(1) = f(1)\,f(1)$, we get $f(1) = 1$. Then $f(x) = x$ for all $x \in A \cap \mathbb{Q}$.

If $y \ge 0$, then $f(y) = f(y^{1/2})^2 \ge 0$ and

$$f(x+y) = f(x) + f(y) \ge f(x),$$

which implies f is increasing. Now for any $x\in A\backslash\mathbb{Q}$, by the density of rational numbers, there are $p_n, q_n\in\mathbb{Q}$ such that $p_n < x < q_n$, the p_n's increase to x and the q_n's decrease to x. As f is increasing, we have $p_n = f(p_n) \le f(x) \le f(q_n) = q_n$. Taking limits, the sandwich theorem gives $f(x) = x$ for all $x\in A$.

Fact 2. *If a function $f:(0,\infty)\to\mathbb{R}$ satisfies $f(xy) = f(x)f(y)$ for all $x, y > 0$ and f is monotone, then either $f(x)=0$ for all $x > 0$ or there exists c such that $f(x) = x^c$ for all $x > 0$.*

Proof. For $x > 0$, $f(x) = f(x^{1/2})^2 \ge 0$. Also $f(1) = f(1)\,f(1)$ implies $f(1) = 0$ or 1. If $f(1) = 0$, then $f(x) = f(x)\,f(1) = 0$ for all $x > 0$. If $f(1) = 1$, then $f(x) > 0$ for all $x > 0$ (since $f(x) = 0$ implies $f(1) = f(x(1/x)) = f(x)\,f(1/x) = 0$, which would lead to a contradiction).

Define g: $\mathbb{R}\to\mathbb{R}$ by $g(w) = \ln f(e^w)$. Then

$$\begin{aligned} g(x+y) &= \ln f(e^{x+y}) = \ln f(e^x e^y) \\ &= \ln f(e^x)\,f(e^y) \\ &= \ln f(e^x) + \ln f(e^y) \\ &= g(x) + g(y). \end{aligned}$$

Since f is monotone, it follows that g is also monotone. Then $g(w) = cw$ for all w. Therefore, $f(x) = x^c$ for all $x > 0$.

As an application of these facts, we look at the following example.

Example 4. (*2002 IMO*) Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all x, y, z, t in $\mathbb{R}$.

Solution. (*Due to Yu Hok Pun, 2002 Hong Kong IMO team member, gold medalist*) Suppose $f(x) = c$ for all x. Then the equation implies $4c^2 = 2c$. So c can only be 0 or 1/2. Reversing steps, we can also check $f(x) = 0$ for all x or $f(x) = 1/2$ for all x are solutions.

Suppose the equation is satisfied by a nonconstant function f. Setting $x = 0$ and $z = 0$, we get $2\,f(0)\,(f(y) + f(t)) = 2\,f(0)$, which implies $f(0) = 0$ or $f(y) + f(t) = 1$ for all y, t. In the latter case, setting $y = t$, we get the constant function $f(y) = 1/2$ for all y. Hence we may assume $f(0) = 0$.

Setting $y = 1$, $z = 0$, $t = 0$, we get $f(x)\,f(1) = f(x)$. Since $f(x)$ is not the zero function, $f(1) = 1$. Setting $z = 0$, $t = 0$, we get $f(x)\,f(y) = f(xy)$ for all x,y. In particular, $f(w) = f(w^{1/2})^2 \ge 0$ for $w > 0$.

Setting $x = 0$, $y = 1$ and $t = 1$, we have $2\,f(1)\,f(z) = f(-z) + f(z)$, which implies $f(z) = f(-z)$ for all z. So f is even.

Define the function g: $(0,\infty)\to\mathbb{R}$ by $g(w)=f(w^{1/2}) \ge 0$. Then for all $x,y>0$,

$$\begin{aligned} g(xy) &= f((xy)^{1/2}) = f(x^{1/2}\,y^{1/2}) \\ &= f(x^{1/2})\,f(y^{1/2}) = g(x)\,g(y). \end{aligned}$$

Next f is even implies $g(x^2) = f(x)$ for all x. Setting $z = y$, $t = x$ in the given equation, we get

$$\begin{aligned} (g(x^2) + g(y^2))^2 &= g((x^2 + y^2)^2) \\ &= g(x^2 + y^2)^2 \end{aligned}$$

for all x,y. Taking square roots and letting $a = x^2$, $b = y^2$, we get $g(a)+g(b) = g(a+b)$ for all $a, b > 0$.

By fact 1, we have $g(w) = w$ for all $w > 0$. Since $f(0) = 0$ and f is even, it follows $f(x) = g(x^2) = x^2$ for all x.

Check: If $f(x) = x^2$, then the equation reduces to

$$(x^2 + z^2)(y^2 + t^2) = (xy-zt)^2 + (xt+yz)^2,$$

which is a well known identity and can easily be checked by expansion or seen from $|p|^2|q|^2 = |pq|^2$, where $p = x + iz$, $q = y + it \in\mathbb{C}$.

The concept of fixed point of a function is another useful idea in solving some functional equations. Its definition is very simple. We say w is a fixed point of a function f if and only if w is in the domain of f and $f(w) = w$. Having information on the fixed points of functions often help to solve certain types of functional equations as the following examples will show.

Example 5. (*1983 IMO*) Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x\,f(y)) = y\,f(x)$ for all $x, y \in \mathbb{R}^+$ and as $x\to+\infty$, $f(x)\to 0$.

Solution. Step 1 Taking $x = 1 = y$, we get $f(f(1)) = f(1)$. Taking $x = 1$ and $y = f(1)$, we get $f(f(f(1))) = f(1)^2$. Then $f(1)^2 = f(f(f(1))) = f(f(1)) = f(1)$, which implies $f(1) = 1$. So 1 is a fixed point of f.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***February 28, 2003***.

Problem 171. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\left[\frac{c}{ab}\right]-\left[\frac{c}{a}\right]\left[\frac{c}{b}\right]\le c\min\left\{\frac{1}{a},\frac{1}{b}\right\}.$$

Problem 172. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the square of the sum of their digits in base 10 representation.

Problem 173. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group.

Problem 174. Let M be a point inside acute triangle ABC. Let A', B', C' be the mirror images of M with respect to BC, CA, AB, respectively. Determine (with proof) all points M such that A, B, C, A', B', C' are concyclic.

Problem 175. A regular polygon with n sides is divided into n isosceles triangles by segments joining its center to the vertices. Initially, $n + 1$ frogs are placed inside the triangles. At every second, there are two frogs in some common triangle jumping into the interior of the two neighboring triangles (one frog into each neighbor). Prove that after some time, at every second, there are at least $[\,(n + 1) / 2\,]$ triangles, each containing at least one frog.

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Solutions

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In the last issue, problems 166, 167 and 169 were stated incorrectly. They are revised as problems 171, 172, 173, respectively. As the problems became easy due to the mistakes, we received many solutions. Regretfully we do not have the space to print the names and affiliations of all solvers. We would like to apologize for this.

Problem 166. Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\,[a/b]-[c/a]\,[c/b]\le c\min\{1/a,1/b\}.$$

Solution. Over 30 solvers disproved the inequality by providing different counter-examples, such as $(a, b, c) = (3, 2, 1)$.

Problem 167. Find all positive integers such that they are equal to the sum of their digits in base 10 representation.

Solution. Over 30 solvers sent in solutions similar to the following. For a positive integer N with digits $a_n, \ldots, a_0$ (from left to right), we have

$$N = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_0 \ge a_n + a_{n-1} + \cdots + a_0$$

because $10^k > 1$ for $k > 0$. So equality holds if and only if $a_n = a_{n-1} = \cdots = a_1 = 0$. Hence, $N = 1, 2, \ldots, 9$ are the only solutions.

Problem 168. Let AB and CD be nonintersecting chords of a circle and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the part of segment CD lying inside triangle ABP. (*Source: 1997 Hungarian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7)

Draw the midpoint M of AB. If $AB \parallel CD$, then draw ray MK to intersect the circle at P. Let AP, BP intersect CD at Q,R, respectively. Since $AB \parallel QR$, $\Delta ABP \sim \Delta QRP$. Then M being the midpoint of AB will imply K is the midpoint of QR.

If AB intersects CD at E, then draw the circumcircle of EMK meeting the original circle at S and S'. Draw the circumcircle of BES meeting CD at R. Draw the circumcircle of AES meeting CD at Q. Let AQ, BR meet at P. Since $\angle PBS = \angle RBS = \angle RES = \angle QES = \angle QAS = \angle PAS$, P is on the original circle.

Next, $\angle SMB = \angle SME = \angle SKE = \angle SKR$ and $\angle SBM = 180^\circ - \angle SBE = 180^\circ - \angle SRE = \angle SRK$ imply $\Delta SMB \sim \Delta SKR$ and $MB/KR = BS/RS$. Replacing M by A and K by Q, similarly $\Delta SAB \sim \Delta SQR$ and $AB/QR = BS/RS$. Since $AB = 2MB$, we get $QR = 2KR$. So K is the midpoint of QR.

Problem 169. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

Solution. Almost all solvers used the following argument. Let m and M be the weights of the lightest and heaviest apple(s). Then $3m \ge M$. If the problem is false, then there are two groups A and B with weights w_A and w_B such that $(11/2)\, w_B < w_A$. Since $4m \le w_B$ and $w_A \le 4M$, we get $(11/2)4m < 4M$ implying $3m \le (11/2)m < M$, a contradiction.

Problem 170. (*Proposed by Abderrahim Ouardini, Nice, France*) For any (nondegenerate) triangle with sides a, b, c, let $\sum' h(a, b, c)$ denote the sum $h(a, b, c) + h(b, c, a) + h(c, a, b)$. Let $f(a, b, c) = \sum' (a / (b + c - a))^2$ and $g(a, b, c) = \sum' j(a, b, c)$, where $j(a,b,c) = (b + c - a) / \sqrt{(c+a-b)(a+b-c)}$. Show that $f(a, b, c) \ge \max\{3, g(a, b, c)\}$ and determine when equality occurs. (Here $\max\{x,y\}$ denotes the maximum of x and y.)

Solution. **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 6), **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School) and **WONG Wing Hong** (La Salle College, Form 5).

Let $x = b + c - a$, $y = c + a - b$ and $z = a + b - c$. Then $a = (y + z)/2$, $b = (z + x)/2$ and $c = (x + y)/2$.

Substituting these and using the *AM-GM* inequality, the rearrangement inequality and the *AM-GM* inequality again, we find

$$f(a,b,c)$$

$$= \left(\frac{y+z}{2x}\right)^2 + \left(\frac{z+x}{2y}\right)^2 + \left(\frac{x+y}{2z}\right)^2$$

$$\ge \left(\frac{\sqrt{yz}}{x}\right)^2 + \left(\frac{\sqrt{zx}}{y}\right)^2 + \left(\frac{\sqrt{xy}}{z}\right)^2$$

$$\ge \frac{\sqrt{yz}\,\sqrt{zx}}{xy}+\frac{\sqrt{zx}\,\sqrt{xy}}{yz}+\frac{\sqrt{xy}\,\sqrt{yz}}{zx}$$

$$=\frac{x}{\sqrt{yz}}+\frac{y}{\sqrt{zx}}+\frac{z}{\sqrt{xy}}=g(a,b,c)$$

$$\ge 3\sqrt[3]{\frac{xyz}{\sqrt{yz}\,\sqrt{zx}\,\sqrt{xy}}}=3.$$

So $f(a,b,c) \ge g(a,b,c) = \max\{3, g(a,b,c)\}$ with equality if and only if $x = y = z$, which is the same as $a = b = c$.

Olympiad Corner

(continued from page 1)

Problem 3. If $a \ge b \ge c \ge 0$ and $a + b + c = 3$, then prove that $ab^2 + bc^2 + ca^2 \le 27/8$ and determine the equality case(s).

Problem 4. Let p be an odd prime such that $p \equiv 1 \pmod 4$. Evaluate (with reason)

$$\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^2}{p} \right\},$$

where $\{x\} = x - [x]$, $[x]$ being the greatest integer not exceeding x.

Functional Equations

(continued from page 2)

Step 2 Taking $y = x$, we get $f(xf(x)) = xf(x)$. So $w = xf(x)$ is a fixed point of f for every $x \in \mathbb{R}^+$.

Step 3 Suppose f has a fixed point $x > 1$. By step 2, $xf(x) = x^2$ is also a fixed point, $x^2 f(x^2) = x^4$ is also a fixed point and so on. So the x^m's are fixed points for every m that is a power of 2. Since $x > 1$, for m ranging over the powers of 2, we have $x^m \to \infty$, but $f(x^m) = x^m \to \infty$, not to 0. This contradicts the given property. Hence, f cannot have any fixed point $x > 1$.

Step 4 Suppose f has a fixed point x in the interval (0,1). Then

$$1 = f((1/x)x) = f((1/x)f(x)) = xf(1/x),$$

which implies $f(1/x) = 1/x$. This will lead to f having a fixed point $1/x > 1$, contradicting step 3. Hence, f cannot have any fixed point x in (0,1).

Step 5 Steps 1, 3, 4 showed the only fixed point of f is 1. By step 2, we get $xf(x) = 1$ for all $x \in \mathbb{R}^+$. Therefore, $f(x) = 1/x$ for all $x \in \mathbb{R}^+$.

Check: For $f(x) = 1/x$, $f(xf(y)) = f(x/y) = y/x = yf(x)$. As $x \to \infty$, $f(x) = 1/x \to 0$.

Example 6. (1996 IMO) Find all functions f: $\mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}_0$.

Solution. _Step 1_ Taking $m = 0 = n$, we get $f(f(0)) = f(f(0)) + f(0)$, which implies $f(0) = 0$. Taking $m = 0$, we get $f(f(n)) = f(n)$, i.e. $f(n)$ is a fixed point of f for every $n \in \mathbb{N}_0$. Also the equation becomes

$$f(m + f(n)) = f(m) + f(n).$$

Step 2 If w is a fixed point of f, then we will show kw is a fixed point of f for all $k \in \mathbb{N}_0$. The cases $k = 0, 1$ are known. If kw is a fixed point, then $f((k+1)w) = f(kw + w) = f(kw) + f(w) = kw + w = (k+1)w$ and so $(k+1)w$ is also a fixed point.

Step 3 If 0 is the only fixed point of f, then $f(n) = 0$ for all $n \in \mathbb{N}_0$ by step 1. Obviously, the zero function is a solution.

Otherwise, f has a least fixed point $w > 0$. We will show the only fixed points are kw, $k \in \mathbb{N}_0$. Suppose x is a fixed point. By the division algorithm, $x = kw + r$, where $0 \le r < w$. We have

$$\begin{aligned} x = f(x) &= f(r + kw) = f(r + f(kw)) \\ &= f(r) + f(kw) = f(r) + kw. \end{aligned}$$

So $f(r) = x - kw = r$. Since w is the least positive fixed point, $r = 0$ and $x = kw$.

Since $f(n)$ is a fixed point for all $n \in \mathbb{N}_0$ by step 1, $f(n) = c_n w$ for some $c_n \in \mathbb{N}_0$. We have $c_0 = 0$.

Step 4 For $n \in \mathbb{N}_0$, by the division algorithm, $n = kw + r$, $0 \le r < w$. We have

$$\begin{aligned} f(n) &= f(r + kw) = f(r + f(kw)) \\ &= f(r) + f(kw) = c_r w + kw \\ &= (c_r + k)w = (c_r + [n/w])w. \end{aligned}$$

Check: For each $w > 0$, let $c_0 = 0$ and let $c_1, \ldots, c_{w-1} \in \mathbb{N}_0$ be arbitrary. The function $f(n) = (c_r + [n/w])w$, where r is the remainder of n divided by w, (and the zero function) are all the solutions. Write $m = kw + r$ and $n = lw + s$ with $0 \le r, s < w$. Then

$$\begin{aligned} f(m + f(n)) &= f(r + kw + (c_s + l)w) \\ &= c_r w + kw + c_s w + lw \\ &= f(f(m)) + f(n). \end{aligned}$$

Other than the fixed point concept, in solving functional equations, the injectivity and surjectivity of the functions also provide crucial informations.

Example 7. (*1987 IMO*) Prove that there is no function f: $\mathbb{N}_0 \to \mathbb{N}_0$ such that $f(f(n)) = n + 1987$.

Solution. Suppose there is such a function f. Then f is injective because $f(a) = f(b)$ implies

$$a = f(f(a)) - 1987 = f(f(b)) - 1987 = b.$$

Suppose $f(n)$ misses exactly k distinct values $c_1, \ldots, c_k$ in $\mathbb{N}_0$, i.e. $f(n) \ne c_1, \ldots, c_k$ for all $n \in \mathbb{N}_0$. Then $f(f(n))$ misses the $2k$ distinct values $c_1, \ldots, c_k$ and $f(c_1), \ldots, f(c_k)$ in $\mathbb{N}_0$. (The $f(c_j)$'s are distinct because f is injective.) Now if $w \ne c_1, \ldots, c_k, f(c_1), \ldots, f(c_k)$, then there is $m \in \mathbb{N}_0$ such that $f(m) = w$. Since $w \ne f(c_j)$, $m \ne c_j$, so there is $n \in \mathbb{N}_0$ such that $f(n) = m$, then $f(f(n)) = w$. This shows $f(f(n))$ misses only the $2k$ values $c_1, \ldots, c_k, f(c_1), \ldots, f(c_k)$ and no others. Since $n + 1987$ misses the 1987 values 0, 1, …, 1986 and $2k \ne 1987$, this is a contradiction.

Example 8. (*1999 IMO*) Determine all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Solution. Let $c = f(0)$. Setting $x = y = 0$, we get $f(-c) = f(c) + c - 1$. So $c \ne 0$. Let A be the range of f, then for $x = f(y) \in A$, we get $c = f(0) = f(x) + x^2 + f(x) - 1$. Solving for $f(x)$, this gives $f(x) = (c + 1 - x^2)/2$.

Next, if we set $y = 0$, we get

$$\begin{aligned} &\{f(x - c) - f(x) : x \in \mathbb{R}\} \\ &= \{cx + f(c) - 1 : x \in \mathbb{R}\} = \mathbb{R} \end{aligned}$$

because $c \ne 0$. Then $A - A = \{y_1 - y_2 : y_1, y_2 \in A\} = \mathbb{R}$.

Now for an arbitrary $x \in \mathbb{R}$, let $y_1, y_2 \in A$ be such that $y_1 - y_2 = x$. Then

$$\begin{aligned} f(x) &= f(y_1 - y_2) = f(y_2) + y_1 y_2 + f(y_1) - 1 \\ &= (c+1-y_2^2)/2 + y_1 y_2 + (c+1-y_1^2)/2 - 1 \\ &= c - (y_1 - y_2)^2/2 = c - x^2/2. \end{aligned}$$

However, for $x \in A$, $f(x) = (c + 1 - x^2)/2$. So $c = 1$. Therefore, $f(x) = 1 - x^2/2$ for all $x \in \mathbb{R}$.

Check: For $f(x) = 1 - x^2/2$, both sides equal $1/2 + y^2/2 - y^4/8 + x - xy^2/2 - x^2/2$.

Mathematical Excalibur

Volume 8, Number 2 *April 2003 – May 2003*

Olympiad Corner

The Final Round of the 51st Czech and Slovak Mathematical Olympiad was held on April 7-10, 2002. Here are the problems.

Problem 1. Solve the system

$$(4x)_5 + 7y = 14,$$
$$(2y)_5 - (3x)_7 = 74,$$

in the domain of the integers, where $(n)_k$ stands for the multiple of the number k closest to the number n.

Problem 2. Consider an arbitrary equilateral triangle KLM, whose vertices K, L and M lie on the sides AB, BC and CD, respectively, of a given square $ABCD$. Find the locus of the midpoints of the sides KL of all such triangles KLM.

Problem 3. Show that a given natural number A is the square of a natural number if and only if for any natural number n, at least one of the differences

$$(A+1)^2 - A,\ (A+2)^2 - A,$$
$$(A+3)^2 - A,\ \ldots,\ (A+n)^2 - A$$

is divisible by n.

(continued on page 4)

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 26, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Countability

Kin Y. Li

Consider the following two questions:

(1) Is there a nonconstant polynomial with integer coefficients which has every prime number as a root?
(2) Is every real number a root of some nonconstant polynomial with integer coefficients?

The first question can be solved easily. Since the set of roots of a nonconstant polynomial is finite and the set of prime numbers is infinite, the roots cannot contain all the primes. So the first question has a negative answer.

However, for the second question, both the set of real numbers and the set of roots of nonconstant polynomials with integer coefficients are infinite. So we cannot answer this question as quickly as the first one.

In number theory, a number is said to be *algebraic* if it is a root of a nonconstant polynomial with integer coefficients, otherwise it is said to be *transcendental*. So the second question asks if every real number is algebraic.

Let's think about the second question. For every rational number a/b, it is clearly the root of the polynomial $P(x) = bx - a$. How about irrational numbers? For numbers of the form $\sqrt[n]{a/b}$, it is a root of the polynomial $P(x) = b\,x^n - a$. To some young readers, at this point they may think, perhaps the second question has a positive answer. We should do more checking before coming to any conclusion. How about π and e? Well, they are hard to check. Are there any other irrational number we can check?

Recall $\cos(3\theta)=4\cos^3\theta-3\cos\theta$. So setting $\theta=20^\circ$, we get $1/2 = 4\cos^3 20^\circ - 3\cos 20^\circ$. It follows that $\cos 20^\circ$ is a root of the polynomial $P(x) = 8x^3 - 6x - 1$. With this, we seem to have one more piece of evidence to think the second question has a positive answer.

So it is somewhat surprising to learn that the second question turns out to have a negative answer. In fact, it is known that π and e are not roots of nonconstant polynomials with integer coefficients, i.e. they are transcendental. Historically, the second question was answered before knowing π and e were transcendental. In 1844, Joseph Liouville proved for the first time that transcendental numbers exist, using continued fractions. In 1873, Charles Hermite showed e was transcendental. In 1882, Ferdinand von Lindemann generalized Hermite's argument to show π was also transcendental. Nowadays we know almost all real numbers are transcendental. This was proved by Georg Cantor in 1874. We would like to present Cantor's countability theory used to answer the question as it can be applied to many similar questions.

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of all rational numbers and $\mathbb{R}$ the set of all real numbers.

Recall a *bijection* is a function f: $A \to B$ such that for every b in B, there is exactly one a in A satisfying $f(a) = b$. Thus, f provides a way to *correspond* the elements of A with those of B in a *one-to-one* manner.

We say a set S is *countable* if and only if S is a finite set or there exists a bijection $f: \mathbb{N} \to S$. For an infinite set, since such a bijection is a one-to-one correspondence between the positive integers and the elements of S, we have

$$1 \leftrightarrow s_1,\ 2 \leftrightarrow s_2,\ 3 \leftrightarrow s_3,\ 4 \leftrightarrow s_4,\ \ldots$$

and so the elements of S can be listed *orderly* as $s_1, s_2, s_3, \ldots$ without repetition or omission. Conversely, any such list of the elements of a set is equivalent to showing the set is countable since assigning $f(1) = s_1$, $f(2) = s_2$, $f(3) = s_3$, … readily provide a bijection.

Certainly, $\mathbb{N}$ is countable as the identity function $f:\mathbb{N}\to\mathbb{N}$ defined by $f(n)=n$ is a bijection. This provides the usual listing of $\mathbb{N}$ as 1, 2, 3, 4, 5, 6, …. Next, for $\mathbb{Z}$, the usual listing would be

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots.$$

However, to be in a one-to-one correspondence with $\mathbb{N}$, there should be a first element, followed by a second element, etc. So we can try listing $\mathbb{Z}$ as

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots.$$

From this we can construct a bijection $g:\mathbb{N}\to\mathbb{Z}$, namely define g as follow:

$$g(n) = (1-n)/2 \text{ if } n \text{ is odd}$$

and

$$g(n) = n/2 \text{ if } n \text{ is even.}$$

For $\mathbb{Q}$, there is no usual listing. So how do we proceed? Well, let's consider listing the set of all positive rational numbers $\mathbb{Q}^+$ first. Here is a table of $\mathbb{Q}^+$.

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

In the m-th row, the numerator is m and in the n-th column, the denominator is n.

Consider the southwest-to-northeast diagonals. The first one has 1/1, the second one has 2/1 and 1/2, the third one has 3/1, 2/2, 1/3, etc. We can list $\mathbb{Q}^+$ by writing down the numbers on these diagonals one after the other. However, this will repeat numbers, for example, 1/1 and 2/2 are the same. So to avoid repetitions, we will write down only numbers whose numerators and denominators are relatively prime. This will not omit any positive rational numbers because we can cancel common factors in the numerator and denominator of a positive rational number to arrive at a number in the table that we will not skip. Here is the list we will get for $\mathbb{Q}^+$:

1/1, 2/1, 1/2, 3/1, 1/3, 4/1, 3/2, 2/3, 1/4, 5/1, 1/5, 6/1, 5/2, 4/3, 3/4, … .

Once we have a listing of $\mathbb{Q}^+$, we can list $\mathbb{Q}$ as we did for $\mathbb{Z}$ from $\mathbb{N}$, i.e.

0, 1/1, –1/1, 2/1, –2/1, 1/2, –1/2, 3/1, –3/1, 1/3, –1/3, 4/1, –4/1, 3/2, ….

This shows $\mathbb{Q}$ is countable, although the bijection behind this listing is difficult to write down.

If a bijection $h:\mathbb{N}\to\mathbb{Q}$ is desired, then we can do the following. Define $h(1) = 0$. For an integer $n > 1$, write down the prime factorization of $g(n)$, where g is the function above. Suppose

$$g(n) = \pm\, 2^a\, 3^b\, 5^c\, 7^d \ldots.$$

Then we define

$$h(n) = \pm\, 2^{g(a+1)}\, 3^{g(b+1)}\, 5^{g(c+1)}\, 7^{g(d+1)} \ldots$$

with $g(n)$, $h(n)$ taking the same sign.

Next, how about $\mathbb{R}$? This is interesting. It turns out $\mathbb{R}$ is _uncountable_ (i.e. not countable). To explain this, consider the function $u:(0,1)\to\mathbb{R}$ defined by $u(x) = \tan \pi(x-1/2)$. It has an inverse function $v(x) = 1/2 + (\operatorname{Arctan} x)/\pi$. So both u and v are bijections. Now assume there is a bijection $f:\mathbb{N}\to\mathbb{R}$. Then $F = v\circ f:\mathbb{N}\to(0,1)$ is also a bijection. Now we write the decimal representations of $F(1)$, $F(2)$, $F(3)$, $F(4)$, $F(5)$, … in a table.

$$\begin{aligned}
F(1) &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\
F(2) &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\
F(3) &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\
F(4) &= 0.a_{41}a_{42}a_{43}a_{44}\ldots \\
F(5) &= 0.a_{51}a_{52}a_{53}a_{54}\ldots \\
F(6) &= 0.a_{61}a_{62}a_{63}a_{64}\ldots
\end{aligned}$$

Consider the number

$$r = 0.\, b_1b_2b_3b_4b_5b_6\ldots,$$

where the digit $b_n = 2$ if $a_{nn} = 1$ and $b_n = 1$ if $a_{nn} \neq 1$. Then $F(n) \neq r$ for all n because $a_{nn} \neq b_n$. This contadicts F is a bijection. Thus, no bijection $f:\mathbb{N}\to\mathbb{R}$ can exist. Therefore, (0,1) and $\mathbb{R}$ are both uncountable.

We remark that the above argument shows no matter how the elements of (0,1) are listed, there will always be numbers omitted. The number r above is one such number.

So some sets are countable and some sets are uncountable.

For more complicated sets, we will use the following theorems to determine if they are countable or not.

Theorem 1. Let A be a subset of B. If B is countable, then A is countable.

Theorem 2. If for every integer n, S_n is a countable set, then their union is countable.

For the next theorem, we introduce some terminologies first. An object of the form $(x_1,\ldots,x_n)$ is called an _ordered n-tuple_. For sets $T_1, T_2, \ldots, T_n$, the _Cartesian product_ $T_1\times\cdots\times T_n$ of these sets is the set of all ordered n-tuples $(x_1,\ldots,x_n)$, where each x_i is an element of T_i for $i = 1,\ldots, n$.

Theorem 3. If $T_1, T_2, \ldots, T_n$ are countable sets, then their Cartesian product is also countable.

We will give some brief explanations for these theorems. For theorem 1, if A is finite, then A is countable. So suppose A is infinite, then B is infinite. Since B is countable, we can list B as $b_1, b_2, b_3, \ldots$ without repetition or omission. Removing the elements b_i that are not in A, we get a list for A without repetition or omission.

For theorem 2, let us list the elements of S_n without repetition or omission in the n-th row of a table. (If S_n is finite, then the row contains finitely many elements.) Now we can list the union of these sets by writing down the diagonal elements as we have done for the positive rational numbers. To avoid repetition, we will not write the element if it has appeared before. Also, if some rows are finite, it is possible that as we go diagonally, we may get to a "hole". Then we simply skip over the hole and go on.

For theorem 3, we use mathematical induction. The case $n = 1$ is trivial. For the case $n = 2$, let $a_1, a_2, a_3, \ldots$ be a list of the elements of T_1 and $b_1, b_2, b_3, \ldots$ be a list of the elements of T_2 without repetition or omission. Draw a table with (a_i, b_j) in the i-th row and j-th column. Listing the diagonal elements as for the positive rational numbers, we get a list for $T_1\times T_2$ without repetition or omission. This takes care the case $n = 2$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***April 26, 2003***.

Problem 176. (*Proposed by Achilleas PavlosPorfyriadis,AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that the fraction

$$\frac{m(n+1)+1}{m(n+1)-n}$$

is irreducible for all positive integers m and n.

Problem 177. A locust, a grasshopper and a cricket are sitting in a long, straight ditch, the locust on the left and the cricket on the right side of the grasshopper. From time to time one of them leaps over one of its neighbors in the ditch. Is it possible that they will be sitting in their original order in the ditch after 1999 jumps?

Problem 178. Prove that if $x < y$, then there exist integers m and n such that

$$x < m + n\sqrt{2} < y.$$

Problem 179. Prove that in any triangle, a line passing through the incenter cuts the perimeter of the triangle in half if and only if it cuts the area of the triangle in half.

Problem 180. There are $n \geq 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least 1/6 of the distances between them are divisible by 3.

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Solutions

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Problem 171. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\left[\frac{c}{ab}\right] - \left[\frac{c}{a}\right]\left[\frac{c}{b}\right] \leq c \min\left\{\frac{1}{a}, \frac{1}{b}\right\}.$$

Solution. **LEE Man Fui** (STFA Leung Kau Kui College, Form 6) and **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6).

Since the inequality is symmetric in a and b, without loss of generality, we may assume $a \geq b$. For every x, $bx \geq b[x]$. Since $b[x]$ is an integer, we get $[bx] \geq b[x]$. Let $x = c/(ab)$. We have

$$\begin{aligned} & c[c/(ab)] - [c/a][c/b] \\ &= c[x] - [bx][c/b] \\ &\leq (c/b)[bx] - [bx][c/b] \\ &= [bx]((c/b) - [c/b]) \\ &< bx \cdot 1 = c/a = c \min\{1/a, 1/b\}. \end{aligned}$$

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Rooney TANG Chong Man** (Hong Kong Chinese Women's Club College, Form 5).

Problem 172. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the square of the sum of their digits in base 10 representation.

Solution. **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and and **WONG Wing Hong** (La Salle College, Form 5).

Suppose there is such an integer n and it has k digits. Then $10^{k-1} \leq n \leq (9k)^2$. However, for $k \geq 5$, we have

$$(9k)^2 = 81k^2 < (5^4/2)2^k \leq (5^{k-1}/2)2^k = 10^{k-1}.$$

So $k \leq 4$. Then $n \leq 36^2$. Since n is a perfect square, we check $1^2, 2^2, \ldots, 36^2$ and find only 1 and $9^2 = 81$ work.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6) and **Rooney TANG Chong Man** (Hong Kong Chinese Women's Club College, Form 5).

Problem 173. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group. (*Source: 1997 Russian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA).

Let $a_1, a_2, \ldots, a_{300}$ be the weights of the apples in increasing order. For $j = 1, 2, \ldots, 75$, let the j-th group consist of the apples with weights $a_j, a_{75+j}, a_{150+j}, a_{225+j}$. Note the weights of the groups are increasing. Then the ratio of the weights of any two groups is at most

$$\begin{aligned} & \frac{a_{75} + a_{150} + a_{225} + a_{300}}{a_1 + a_{76} + a_{151} + a_{226}} \\ &\leq \frac{a_{76} + a_{151} + a_{226} + 3a_1}{a_1 + a_{76} + a_{151} + a_{226}} \\ &= 1 + \frac{2}{1 + (a_{76} + a_{151} + a_{226})/a_1}. \end{aligned}$$

Since $3 \leq (a_{76} + a_{151} + a_{226})/a_1 \leq 9$, so the ratio of groups is at most $1+2/(1+3)=3/2$.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **Terry CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6).

Problem 174. Let M be a point inside acute triangle ABC. Let A', B', C' be the mirror images of M with respect to BC, CA, AB, respectively. Determine (with proof) all points M such that A, B, C, A', B', C' are concyclic.

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

For such M, note the points around the circle are in the order A, B', C, A', B, C'. Now $\angle ACC' = \angle ABC'$ as they are subtended by chord AC'. Also, $AB'=AC'$ because they both equal to AM by symmetry. So $\angle ABC' = \angle ACB'$ as they are subtended by chords AC' and AB' respectively. By symmetry, we also have $\angle ACB' = \angle ACM$. Therefore, $\angle ACC' = \angle ACM$ and so C, M, C' are collinear. Similarly, A, M, A' are collinear. Then $CM \perp AB$ and $AM \perp BC$. So M is the orthocenter of ΔABC.

Conversely, if M is the orthocenter, then $\angle ACB' = \angle ACM = 90° - \angle BAC = \angle ABB'$, which implies A, B, C, B' are concyclic. Similarly, A' and C' are on the circumcircle of ΔABC.

Other commended solvers: **CHEUNG**

Yun Kuen (Hong Kong Chinese Women's Club College, Form 5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 5).

Problem 175. A regular polygon with n sides is divided into n isosceles triangles by segments joining its center to the vertices. Initially, $n + 1$ frogs are placed inside the triangles. At every second, there are two frogs in some common triangle jumping into the interior of the two neighboring triangles (one frog into each neighbor). Prove that after some time, at every second, there are at least $[(n + 1) / 2]$ triangles, each containing at least one frog. (*Source: 1993 Jiangsu Province Math Olympiad*)

Solution. (*Official Solution*)

By the pigeonhole principle, the process will go on forever. Suppose there is a triangle that never contains any frog. Label that triangle number 1. Then label the other triangles in the clockwise direction numbers 2 to n. For each frog in a triangle, label the frog the number of the triangle. Let S be the sum of the squares of all frog numbers. On one hand, $S \le (n + 1)\, n^2$. On the other hand, since triangle 1 never contains any frog, then at every second, some two terms of S will change from $i^2 + i^2$ to $(i+1)^2 + (i-1)^2 = 2i^2 + 2$ with $i < n$. Hence, S will keep on increasing , which contradicts $S \le (n+1)\, n^2$. Thus, after some time T, every triangle will eventually contain some frog at least once.

By the jumping rule, for any pair of triangles sharing a common side, if one of them contains a frog at some second, then at least one of them will contain a frog from then on. If n is even, then after time T, the n triangles can be divided into $n / 2 = [(n + 1) / 2]$ pairs, each pair shares a common side and at least one of the triangles in the pair has a frog. If n is odd, then after time T, we may remove one of the triangles with a frog and divide the rest into $(n - 1)/2$ pairs. Then there will exist $1 + (n-1)/2 = [(n+1)/2]$ triangles, each contains at least one frog.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 4. Find all pairs of real numbers a, b for which the equation in the domain of the real numbers

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has two solutions and the sum of them equals 12.

Problem 5. A triangle KLM is given in the plane together with a point A lying on the half-line opposite to KL. Construct a rectangle $ABCD$ whose vertices B, C and D lie on the lines KM, KL and LM, respectively. (We allow the rectangle to be a square.)

Problem 6. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all $x, y \in \mathbb{R}^+$ the equality

$$f(x f(y)) = f(xy) + x.$$

Countability

(continued from page 2)

Assume the case $n = k$ is true. For k+1 countable sets $T_1, \ldots, T_k, T_{k+1}$, we apply the case $n = k$ to conclude $T_1 \times \cdots \times T_k$ is countable. Then $(T_1 \times \cdots \times T_k) \times T_{k+1}$ is countable by the case $n = 2$.

We should remark that for theorem 1, if C is uncountable and B is countable, then C cannot be a subset of B. As for theorem 2, it is also true for finitely many set $S_1, \ldots, S_n$ because we can set $S_{n+1}, S_{n+2}, \ldots$ all equal to S_1, then the union of $S_1, \ldots, S_n$ is the same as the union of $S_1, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots$. However, for theorem 3, it only works for finitely many sets. Although it is possible to define ordered infinite tuples, the statement is not true for the case of infinitely many sets.

Now we go back to answer question 2 stated in the beginning of this article. We have already seen that $C = \mathbb{R}$ is uncountable. To see question 2 has a negative answer, it is enough to show the set B of all algebraic numbers is countable. By the remark for theorem 1, we can conclude that $C = \mathbb{R}$ cannot be a subset of B. Hence, there exists at least one real number which is not a root of any nonconstant polynomial with integer coefficients.

To show B is countable, we will first show the set D of all nonconstant polynomials with integer coefficients is countable.

Observe that every nonconstant polynomial is of degree n for some positive integer n. Let D_n be the set of all polynomials of degree n with integer coefficients. Let $\mathbb{Z}'$ denote the set of all nonzero integers. Since $\mathbb{Z}'$ is a subset of $\mathbb{Z}$, $\mathbb{Z}'$ is countable by theorem 1 (or simply deleting 0 from a list of $\mathbb{Z}$ without repetition or omission).

Note every polynomial of degree n is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \text{ (with } a_n \ne 0),$$

which is uniquely determined by its coefficients. Hence, if we define the function $w : \mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z} \to D_n$ by

$$w(a_n, a_{n-1}, \ldots, a_0) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

then w is a bijection. By theorem 3, $\mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is countable. So there is a bijection $q : \mathbb{N} \to \mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Then $w \circ q : \mathbb{N} \to D_n$ is also a bijection. Hence, D_n is countable for every positive integer n. Since D is the union of $D_1, D_2, D_3, \ldots$, by theorem 2, D is countable.

Finally, let $P_1, P_2, P_3, \ldots$ be a list of all the elements of D. For every n, let R_n be the set of all roots of P_n, which is finite by the fundamental theorem of algebra. Hence R_n is countable. Since B is the union of $R_1, R_2, R_3, \ldots$, by theorem 2, B is countable and we are done.

Historically, the countability concept was created by Cantor when he proved the rational numbers were countable in 1873. Then he showed algebraic numbers were also countable a little later. Finally in December 1873, he showed real numbers were uncountable and wrote up the results in a paper, which appeared in print in 1874. It was this paper of Cantor that also introduced the one-to-one correspondence concept into mathematics for the first time!

Mathematical Excalibur

Volume 8, Number 3 *June 2003 – July 2003*

Olympiad Corner

The XV Asia Pacific Mathematics Olympiad took place on March 2003. The time allowed was 4 hours. No calculators were to be used. Here are the problems.

Problem 1. Let a, b, c, d, e, f be real numbers such that the polynomial

$$P(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

Problem 2. Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of ΔAEF and ΔCGH be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 10, 2003***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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容斥原則和 Turan 定理

梁 達 榮

設 A 為有限集，以$|A|$表示它含元素的個數。如果有兩個有限集 A 和 B，以 $A\cup B$ 表示 A 和 B 的并集（它包含屬於 A 或 B 的元素），以 $A\cap B$ 表示 A 和 B 的交集（它包含同時屬於 A 和 B 的元素）。眾所周知，如果 A 和 B 之間沒有共同元素，則 $|A\cup B|=|A|+|B|$，但是如果 A 和 B 之間有共同元素 x，當數算 A 元素的數目時，x 被算了一次，但數算 B 元素的數目時，x 又再被算了一次。為了抵消這樣的重覆，在計算$|A\cup B|$時，我們要減去重覆數算的次數，即$|A\cap B|$。因此 $|A\cup B|=|A|+|B|-|A\cap B|$。

對於三個集的并集 $A\cup B\cup C$，我們可以先數算 A，B 和 C 的個數，相加起來，發覺是太大了，必須減去一些交集的個數，現在 A，B 和 C 中任兩個集的交集可以是 $A\cap B$，$A\cap C$ 和 $B\cap C$，當我們減去這些交集的元素個數時，發覺又變得太少了，最後我們還要加上三個集的交集的元素個數，最後得 $|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$。

一般來説，如果有 n 個有限集 $A_1, A_2, \ldots, A_n$，則 $|A_1\cup A_2\cup\cdots\cup A_n| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i\prec j\le n}|A_i\cap A_j| + \sum_{1\le i\prec j\prec k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n-1}|A_1\cap A_2\cap\cdots\cap A_n|$ 等式中右邊第一個和式代表 A_1 至 A_n 各集元素個數的總和，第二個和式代表任何兩個集的交集元素個數的總和，餘此類推，直到考慮 $A_1, A_2, \cdots$ 至 A_n 的交集為止。

上面的等式，一般稱為容斥原則 (Inclusion-Exclusion Principle)，其命意義相當明顯。證明可以採歸納法，但也可以利用二項式定理加以證明。過程大概如下。設 x 屬於 $A_1\cup A_2\cup\cdots\cup A_n$，則 x 屬於其中 k 個 A_i，$(k\ge 1)$，為方便計，設 x 屬於 $A_1, A_2, \ldots, A_k$，但不屬於 $A_{k+1}, \ldots, A_n$。這樣的話，x 在 $A_1\cup A_2\cup\cdots\cup A_n$ 的"貢獻"為 1。在右邊第一個和式中，x 的"貢獻"為 $k = C_1^k$。在第二個和式中，由於 x 在 $A_1, A_2, \ldots, A_k$ 中出現，則 x 在它們任兩個集的交集中出現，但不在其他兩個集的交集中出現，因此，x 在第二個和式中的"貢獻"為 C_2^k。這樣分析下去，我們發覺 x 在右邊的"貢獻"總和是 $C_1^k - C_2^k + C_3^k - \cdots + (-1)^{k+1}C_k^k = 1-(1-1)^k = 1$。

留意我們用到了二項式定理，由於 x 在兩邊的貢獻相等，我們獲得了容斥原則成立的證明。

再者二項式系數有以下的性質。C_m^k 在 $m\le\frac{k}{2}$ 時遞增，在 $m\ge\frac{k}{2}$ 時遞減。(例如 $k=5$，有 $C_0^5 < C_1^5 < C_2^5 = C_3^5 > C_4^5 > C_5^5$，$C_m^5$ 在 $m=2, 3$ 時取最大值，$k=6$，$C_0^6 < C_1^6 < C_2^6 < C_3^6 > C_4^6 > C_5^6 > C_6^6$，$C_m^6$ 在 $m=3$ 時最大值。）利用這個關係，讀者可以証明，如果在容斥原則的右邊，略去一個正項及它以後各項，則式的左邊大於右邊，這是因為 x 對於右邊的貢獻非正，或者被略去的貢獻非負。同理，如果在容斥原則的右邊略去一個負項及它以後各項，則式的左邊變為小於右邊。這是一個有用的估計。

容斥原則作為數算集的大小的用途上時常出現，應用廣泛。

例一： 這是一個經典的題目，將 1, 2, …, n 重新安排次序，得到一個排列，如果沒有一個數字在原先的位置上，則稱之為亂序，(例如，4321 是一個亂序，4213 不是)，現在問，有多少個亂序？

解答: 顯而易見，所有的排列數目是 $n! = n\times(n-1)\times\cdots\times 1$。但如果直接找尋亂序的數目，卻不是很容易。因此我們定義 A_i 為 i 在正確位置的排列，$1\le i\le n$。易見 $|A_i| = (n-1)!$，同理 $|A_i\cap A_j| = (n-2)!$，此處 $i\ne j$，等等。因此

$$\begin{aligned}&|A_1\cup A_2\cup\cdots\cup A_n|\\&=\sum_{i=1}^{n}|A_i|-\sum_{1\le i\prec j\le n}|A_i\cap A_j|\\&+\sum_{1\le i\prec j\prec k\le n}|A_i\cap A_j\cap A_k|-\cdots\\&+(-1)^{n-1}|A_1\cap A_2\cap\cdots\cap A_n|\\&=n(n-1)!-C_2^n(n-2)!\\&+C_3^n(n-3)!-\cdots+(-1)^{n-1}1\\&=n!-\tfrac{n!}{2!}+\tfrac{n!}{3!}-\cdots+(-1)^{n-1}\tfrac{n!}{n!}\end{aligned}$$

最後，亂序的數目是

$$\begin{aligned}&n!-|A_1\cup A_2\cup\cdots\cup A_n|\\&=n!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\cdots+(-1)^n\tfrac{1}{n!}\right)\end{aligned}$$ 。

例二： (IMO 1991) 設 $S = \{1, 2, \ldots, 280\}$。求最小的自然數 n，使得 S 的每一個 n 元子集都含有 5 個兩兩互素的數。

解答: 首先利用容斥原則求得 $n\ge 217$。設 A_1, A_2, A_3, A_4 是 S 中分別為 2, 3, 5, 7 的倍數的集，則

$$\begin{aligned}&|A_1|=140,|A_2|=93,|A_3|=56,|A_4|=40,\\&|A_1\cap A_2|=46,|A_1\cap A_3|=28,\\&|A_1\cap A_4|=20,|A_2\cap A_3|=18,\\&|A_2\cap A_4|=13,|A_3\cap A_4|=8,\\&|A_1\cap A_2\cap A_3|=9,|A_1\cap A_2\cap A_4|=6,\\&|A_1\cap A_3\cap A_4|=4,|A_2\cap A_3\cap A_4|=2,\\&|A_1\cap A_2\cap A_3\cap A_4|=1\end{aligned}$$

因此

$$\begin{aligned}&|A_1\cap A_2\cap A_3\cap A_4|=140+93\\&+56+40-46-28-20-18-13\\&-8+9+6+4+2-1=216\end{aligned}$$

對於這個 216 元的集，任取 5 個數，必有兩個同時屬於 A_1, A_2, A_3 或 A_4，因此不互素。按題意，所以必須有 $n\ge 217$。現在要證明 S 中任一 217 元集必有 5 個互素的數，方法是要構造適當的“抽屜”。其中一個比較簡潔的構造是這樣的。設 A 是 S 的一個子集，並且 $|A|\ge 217$。定義

B_1 ={1 和 S 中的素數}，$|B_1| = 60$，

$B_2 = \{2^2,3^2,5^2,7^2,11^2,13^2\}, |B_2| = 6,$

$B_3 = \{2\times131, 3\times89, 5\times53, 7\times37, 11\times23, 13\times19\}, |B_3| = 6,$

$B_4 = \{2\times127, 3\times87, 5\times47, 7\times31, 11\times19, 13\times17\}, |B_4| = 6,$

$B_5 = \{2\times113, 3\times79, 5\times43, 7\times27, 11\times17\}, |B_5| = 5,$

$B_6 = \{2\times109, 3\times73, 5\times41, 7\times23, 11\times13\}, |B_6| = 5.$

易見 B_1 至 B_6 互不相交，並且 $|B_1\cup B_2\cup B_3\cup B_4\cup B_5\cup B_6| = 88$。去掉這 88 個數，$S$ 中尚有 $280-88 = 192$ 個數。現在 A 最小有 217 個元素，$217-192 = 25$，即是說 A 中最小有 25 個元屬於 B_1 至 B_6。易見，不可能每個 B_i 只含 A 中 4 個或以下的元素，即是說最少有 5 個或以上的元素屬於同一個 B_i，因此互素。注意這裏我們用到另一個原則：抽屜原則。

例三： (1989 IMO) 設 n 是正整數。我們說集{1, 2, 3, …, 2n}的一個排列 $(x_1, x_2,\ldots,x_{2n})$ 具有性質 P，如果在{1, 2, 3, …, 2n − 1}中至少有一個 i，使得 $|x_i - x_{i+1}| = n$ 成立。證明具有性質 P 的排列比不具有性質 P 的排列多。

解答: 留意如果 $|x_i - x_{i+1}| = n$，其中一個 x_i 或 x_{i+1} 必小於 $n + 1$。因此對於 k = 1, 2, …, n, 定義 A_k 為 k 與 $k + n$ 相鄰的排列的組合，易見 $|A_k| = 2\times(2n-1)!$。(這是因為 k 與 $k + n$ 并合在一起，但位置可以互相交換，想像它是一個“數”，而另外有 2n - 2 個數，這(2n – 2) + 1 個數位置隨意。) 同時 $|A_k\cap A_h| = 2^2\times(2n-2)!$, $1\le k<h\le n$, (k 與 $k + n$ 合在一起成為一個“數” h 與 $h + n$ 合在一起成為一個“數”。)因此具性質 P 的排列的數目

$$\begin{aligned}&|A_1\cup A_2\cup\cdots\cup A_n|\ge\sum_{k=1}^{n}|A_k|\\&-\sum_{1\le k\prec h\le n}|A_k\cap A_h|\\&=2\times(2n-1)!\times n-C_2^n\times2^2\times(2n-2)!\\&=2n\times(2n-2)!\times n=(2n)!\times\tfrac{n}{2n-1}>(2n)!\times\tfrac{1}{2}\end{aligned}$$

這個數目超過(2n)!的一半，因此具性質 P 的排列比不具性質 P 的排列多。

(這一個問題，當年被視為一個難題，但如果看到它與容斥原則的關係，就變得很容易了。)

例四：設 n 和 k 為正整數， $n>3$, $\frac{n}{2}<k<n$。平面上有 n 個點，其中任意三點不共線，如果其中每個點至少與其它 k 個點用線連結，則連結的線段中至少有三條圍成一個三角形。

解答: 因為 $n>3, k>\frac{n}{2}$，則 $k\ge 2$，所以 n 個點中必中兩個點 v_1 和 v_2 相連結。考慮餘下的點，設與 v_1 相連結的點集為 A，與 v_2 相連結的點集為 B，則 $|A|\ge k-1, |B|\ge k-1$。另外

$$\begin{aligned}&n-2\ge|A\cup B|=|A|+|B|-|A\cap B|\\&\ge 2k-2-|A\cap B|\end{aligned}$$

即 $|A\cap B|\ge 2k-n>0$。因此，存在點 v_3 與 v_1 和 v_2 相連結，構成一個三角形。

例五：一次會議有 1990 位數學家參加，其中每人最少有 1327 位合作者。証明，可以找到 4 位數學家，他們中每兩人都合作過。

証明: 將數學家考慮為一個點集，曾經合作過的連結起來，得到一個圖。如上例，v_1 互 v_2 曾合作過，所以連結起來，餘下的，設 A 為和 v_1 合作過的點集，B 為和 v_2 合作過的點集，則 $|A|\ge 1326, |B|\ge 1326$，同樣，$|A\cup B|\le 1990-2=1998$，因此

$$\begin{aligned}&|A\cap B|=|A|+|B|-|A\cup B|\\&\ge 2\times1326-1998=664>0\end{aligned}$$

即是說，可以找到數學家 v_3，與 v_1 和 v_2 都合作過。設 C 為除 v_1 和 v_2 以外，與 v_3 合作過的數學家，即 $|C|\ge 1325$。同時

$1998\ge|(A\cap B)\cup C|=|A\cap B|+|C|-|A\cap B\cap C|$ 即

$$\begin{aligned}&|A\cap B\cap C|\ge|A\cap B|+|C|-1988\\&\ge 664+1325-1988=1>0\end{aligned}$$ 。

因此 $A\cap B\cap C$ 非空，取 $v_4\in A\cap B\cap C$，則 v_1, v_2, v_3, v_4 都曾經合作過。

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***August 10, 2003***.

Problem 181. (*Proposed by Achilleas PavlosPorfyriadis, AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that in a convex polygon, there cannot be two sides with no common vertex, each of which is longer than the longest diagonal.

Problem 182. Let a_0, a_1, a_2, … be a sequence of real numbers such that

$$a_{n+1} \geq a_n^2 + 1/5 \quad \text{for all } n \geq 0.$$

Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

Problem 183. Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square?

Problem 184. Let $ABCD$ be a rhombus with $\angle B = 60^\circ$. M is a point inside ΔADC such that $\angle AMC = 120^\circ$. Let lines BA and CM intersect at P and lines BC and AM intersect at Q. Prove that D lies on the line PQ.

Problem 185. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided one also changes the state of every d-th bulb after it (where d is a divisor of n and is less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

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Solutions

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Problem 176. (*Proposed by Achilleas PavlosPorfyriadis, AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that the fraction

$$\frac{m(n+1)+1}{m(n+1)-n}$$

is irreducible for all positive integers m and n.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School), **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade) and **Alan T. W. WONG** (Markham, ON, Canada).

If the fraction is reducible, then $m(n+1)+1$ and $m(n+1)-n$ are both divisible by a common factor $d>1$. So their difference $n+1$ is also divisible by d. This would lead to

$$1 = (m(n+1)+1) - m(n+1)$$

divisible by d, a contradiction.

Other commended solvers: **CHEUNG Tin** (STFA Leung Kau Kui College, Form 4), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Alexandre THIERY** (Pothier High School, Orleans, France), **Michael A. VEVE** (Argon Engineering Associates, Inc., Virginia, USA) and **Maria ZABAR** (Trieste College, Trieste, Italy).

Problem 177. A locust, a grasshopper and a cricket are sitting in a long, straight ditch, the locust on the left and the cricket on the right side of the grasshopper. From time to time one of them leaps over one of its neighbors in the ditch. Is it possible that they will be sitting in their original order in the ditch after 1999 jumps?

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade).

Let L, G, C denote the locust, grasshopper, cricket, respectively. There are 6 orders:

$$LCG, CGL, GLC, CLG, GCL, LGC.$$

Let LCG, CGL, GLC be put in one group and CLG, GCL, LGC be put in another group. Note after one leap, an order in one group will become an order in the other group. Since 1999 is odd, the order LGC originally will change after 1999 leaps.

Problem 178. Prove that if $x < y$, then there exist integers m and n such that

$$x < m + n\sqrt{2} < y.$$

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Note $0 < \sqrt{2} - 1 < 1$. For a positive integer

$$k > \frac{\log(b-a)}{\log(\sqrt{2}-1)},$$

we get $0 < (\sqrt{2}-1)^k < b - a$. By the binomial expansion,

$$x = (\sqrt{2}-1)^k = p + q\sqrt{2}$$

for some integers p and q. Next, there is an integer r such that

$$r - 1 \leq \frac{a - [a]}{x} < r.$$

Then a is in the interval

$$I = [[a] + (r-1)x, [a] + rx).$$

Since the length of I is $x < b - a$, we get

$$a < [a] + rx = ([a] + rp) + rq\sqrt{2} < b.$$

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **Alexandre THIERY** (Pothier High School, Orleans, France) and **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade).

Problem 179. Prove that in any triangle, a line passing through the incenter cuts the perimeter of the triangle in half if and only if it cuts the area of the triangle in half.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School), and **Alexandre THIERY** (Pothier High School, Orleans, France).

Let ABC be the triangle, s be its semiperimeter and r be its inradius. Without loss of generality, we may assume the line passing through the incenter cuts AB and AC at P and Q respectively. (If the line passes through a vertex of ΔABC, we may let $Q = C$.)

Let $[XYZ]$ denote the area of ΔXYZ. The line cuts the perimeter of ΔABC in half if and only if $AP + AQ = s$, which is equivalent to

$$\begin{aligned}[APQ] &= [API] + [AQI] \\ &= (r\cdot AP)/2 + (r\cdot AQ)/2 \\ &= rs/2 = [ABC]/2.\end{aligned}$$

i.e. the line cuts the area of ΔABC in half.

Problem 180. There are $n \ge 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least 1/6 of the distances between them are divisible by 3.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

We will first show that for any 4 of the points, there is a pair with distance divisible by 3. Assume A, B, C, D are 4 of the points such that no distance between any pair of them is divisible by 3. Since $x \equiv 1$ or $2 \pmod 3$ implies $x^2 \equiv 1 \pmod 3$, AB^2, AC^2, AD^2, BC^2, BD^2 and CD^2 are all congruent to 1 (mod 3).

Without loss of generality, we may assume that $\angle ACD = \alpha + \beta$, where $\alpha = \angle ACB$ and $\beta = \angle BCD$. By the cosine law,

$$AD^2 = AC^2 + CD^2 - 2AC\cdot CD\cos\angle ACD.$$

Now

$$\begin{aligned}\cos\angle ACD &= \cos(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

By cosine law, we have

$$\cos\alpha = \frac{AC^2 + BC^2 - AB^2}{2AC\cdot BC} \text{ and}$$

$$\cos\beta = \frac{BC^2 + CD^2 - BD^2}{2BC\cdot CD}.$$

Using $\sin x = \sqrt{1-\cos^2 x}$, we can also find $\sin\alpha$ and $\sin\beta$. Then

$$\begin{aligned}2BC^2\cdot AD^2 &= 2BC^2(AC^2 + CD^2) \\ &\quad - (2AC\cdot BC)(2BC\cdot CD)\cos\angle ACD \\ &= P + Q,\end{aligned}$$

where

$$\begin{aligned}P &= 2BC^2(AC^2 + CD^2) \\ &\quad - (AC^2+BC^2-AB^2)(BC^2+CD^2-BD^2)\end{aligned}$$

and

$$\begin{aligned}Q^2 &= (4AC^2\cdot BC^2 - (AC^2 + BC^2 - AB^2)^2) \\ &\quad \times (4BC^2\cdot CD^2 - (BC^2 + CD^2 - BD^2)^2).\end{aligned}$$

However, $2BC^2\cdot AD^2 \equiv 2 \pmod 3$, $P \equiv 0 \pmod 3$ and $Q \equiv 0 \pmod 3$. This lead to a contradiction.

For $n \ge 4$, there are C_4^n groups of 4 points. By the reasoning above, each of these groups has a pair of points with distance divisible by 3. This pair of points is in a total of C_2^{n-2} groups. Since $C_4^n / C_2^{n-1} = \frac{1}{6}C_2^n$, the result follows.

Olympiad Corner

(continued from page 1)

Problem 3. Let $k \ge 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \ge 3k/4$. Let n be a composite integer. Prove:

(a) if $n = 2p_k$, then n does not divide $(n-k)!$;
(b) if $n > 2p_k$, then n divides $(n-k)!$.

Problem 4. Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \ge 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

Problem 5. Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

容斥原則和 Turan 定理

(continued from page 2)

套用圖論的語言，例四和例五的意義正如，給定一個 n 點的圖，最少有多少條線，才可以保證有一個三角形 (K_3) 或一個 K_4（四點的圖，任兩點都相連），或者換另一種說法，設有一個 n 點的圖沒有三角形，則該圖最多有多少條線段，等等。這一範圍的圖論稱為極端圖論。最先的結果是這樣的：

<u>Mantel 定理 (1907)</u>: 設 n 點的簡單圖不含 K_3，則其邊數最大值為 $\left[\frac{n^2}{4}\right]$。

（此處 $[x]$ 是小於或等於 x 的最大整數。在例四中，邊數和多於 $\binom{n}{2}\times n\times\frac{1}{2} > \left[\frac{n^2}{4}\right]$，因此結果立即成立。）

比較精緻的命題是這樣的。

<u>定理</u>： 如果 n 點的圖有 q 條邊，則圖至少有 $\dfrac{4q(q-\frac{n^2}{4})}{3n}$ 個三角形。

<u>例六</u>： 在圓周上有 21 個點，由其中二點引伸至圓心所成的圓心角度，最多有 110 個大於 120°。

<u>解答</u>： 如果兩點與圓心形成的圓心角度大於 120°，則將兩點連結起來，得到一個圖，這個圖沒有三角形，因此邊數最多有 $\left[\frac{21^2}{4}\right] = \left[\frac{441}{4}\right] = 110$ 條，或者最多有 110 個引伸出來的圓心角度大於 120°。

如上所說，定義 K_p 是一個 p 個點的完全圖，即 p 個點任兩點都相連，對於一個 n 點的圖 G，如果沒有包含 K_p，則 G 最多有多少條邊呢?

<u>Turan 定理(1941)</u>: 如果一個 n 點的圖 G 不含 K_p，則該圖最多有 $\frac{p-2}{2(p-1)}n^2 - \frac{r(p-1-r)}{2(p-1)}$ 條邊，其中 r 是由 $n = k(p-1) + r$, $0 \le r < p - 1$ 所定義的。如 Mantel 定理的情況，這個定理是極端圖論的一個起點。

Paul Turan (1910-1976) 猶太裔匈牙利人，當他在考慮這一類問題時，還是被關在一個集中營內的呢！

Mathematical Excalibur

Volume 8, Number 4 *August 2003 – October 2003*

Olympiad Corner

The 2003 International Mathematical Olympiad took place on July 2003 in Japan. Here are the problems.

Problem 1. Let A be a subset of the set S = {1, 2, …, 1000000} containing exactly 101 elements. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ such that the sets

$$a_j = \{x + t_j \mid x \in A\} \text{ for } j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Problem 2. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Problem 3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal. (A convex hexagon $ABCDEF$ has three pairs of opposite sides: AB and DE, BC and EF, CD and FA.)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

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利用 GW-BASIC 繪畫曼德勃羅集的方法

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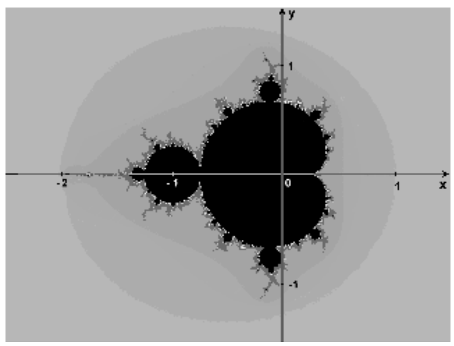
已知一個複數 c_0，並由此定義一個複數數列 $\{c_n\}$，使 $c_{n+1} = c_n^2 + c_0$，其中 n = 0, 1, 2, ……。如果這個數列有界，即可以找到一個正實數 M，使對於一切的 n，$|c_n| < M$，那麼 c_0 便屬於曼德勃羅集（Mandelbrot Set）之內。



可以將以上定義寫成一個 GW-BASIC 程序（對不起！我本人始終都是喜歡最簡單的電腦語言，而且我認為將 GW-BASIC 程序翻譯成其他電腦語言亦不難），方法如下：

```
10   LEFT = 150 : TOP = 380 : W = 360 : M = .833
20   R = 2.64 : S = 2 * R / W
30   RECEN = 0 : IMCEN = 0
40   SCREEN 9 : CLS
50   FOR Y = 0 TO W
60   FOR X = 0 TO W
70   REC = S * (X – W / 2) + RECEN : IMC = S * (Y – W / 2) + IMCEN
80   RE = REC : IM = IMC
90   RE2 = RE * RE : IM2 = IM * IM : J = 0
100  WHILE RE2 + IM2 <= 256 AND J < 15
110  IM = 2 * RE * IM + IMC
120  RE = RE2 – IM2 + REC
130  RE2 = RE * RE : IM2 = IM * IM : J = J + 1
140  WEND
150  IF J < 3 THEN GOTO 220
160  IF J >= 3 AND J < 6 THEN COLOR 14 : REM YELLOW
170  IF J >= 6 AND J < 9 THEN COLOR 1 : REM BLUE
180  IF J >= 9 AND J < 12 THEN COLOR 2 : REM GREEN
190  IF J >= 12 AND J < 15 THEN COLOR 15 : REM WHITE
200  IF J >= 15 THEN COLOR 12 : REM RED
210  PSET (X + LEFT, (TOP – Y)* M)
220  NEXT X
230  NEXT Y
240  COLOR 15 : REM WHITE
250  LINE (LEFT, (TOP – W / 2) * M) – (W + LEFT, (TOP – W / 2) * M)
260  LINE (W / 2 + LEFT, (TOP – W) * M) – (W / 2 + LEFT, TOP * M)
270  END
```

以下是這程序的解釋：

W 紀錄在電腦畫面上將要畫出圖形的大小。現將 W 設定為 360（見第 10 行），表示打算在電腦畫面上一個 360 × 360 的方格內畫出曼德勃羅集（見第 50 及 60 行）。

LEFT 是繪圖時左邊的起點，TOP 是圖的最低的起點（見第 210、250 及 260 行）。注意：在 GW-BASIC 中，畫面坐標是由上至下排列的，並非像一般的理解，將坐標由下至上排，因此要以 “TOP – Y” 的方法將常用的坐標轉換成電腦的坐標。

由於電腦畫面上的一點並非正方形，橫向和縱向的大小並不一樣，故引入 M（= $\frac{5}{6}$）來調節長闊比（見第 10、210、250 及 260 行）。

留意 W 衹是「畫面上」的大小，並非曼德勃羅集內每一個複數點的實際坐標，故需要作出轉換。R 是實際的數值（見第 20 行），即繪畫的範圍實軸由 –R 畫至 +R，同時虛軸亦由 –R 畫至 +R。S 計算 W 與 R 之間的比例，並應用於後面的計算之中（見第 20 及 70 行）。

RECEN 和 IMCEN 用來定出中心點的位置，現在以 (0, 0) 為中心 (見第 30 行)。我們可以通過更改 R、RECEN 和 IMCEN 的值來移動或放大曼德勃羅集。

第 40 行選擇繪圖的模式及清除舊有的畫面。

程序的第 50 及 60 行定出畫面上的坐標 X 和 Y，然後在第 70 行計算出對應複數 c_0 的實值和虛值。

注意：若 $c_0 = a_0 + b_0\,i$，$c_n = a_n + b_n\,i$，則

$$\begin{aligned} c_{n+1} &= c_n^{\,2} + c_0 \\ &= (a_n + b_n\,i)^2 + (a_0 + b_0\,i) \\ &= a_n^{\,2} - b_n^{\,2} + 2a_nb_n\,i + a_0 + b_0\,i \\ &= (a_n^{\,2} - b_n^{\,2} + a_0) \\ &\quad + (2a_nb_n + b_0)i \text{。} \end{aligned}$$

所以 c_{n+1} 的實部等於 $a_n^{\,2} - b_n^{\,2} + a_0$，而虛部則等於 $2a_nb_n + b_0$。

將以上的計算化成程序，得第 110 及 120 行。REC 和 IMC 分別是 c_0 的實值和虛值。RE 和 IM 分別是 c_n 的實值和虛值。RE2 和 IM2 分別是 c_n 的實值和虛值的平方。

J 用來紀錄第 100 至 140 行的循環的次數。第 100 行亦同時計算 c_n 模的平方。若模的平方大於 256 或者循環次數多於 15，循環將會終止。這時候，J 的數值越大，表示該數列較「收斂」，即經過多次計算後，c_n 的模仍不會變得很大。第 150 至 200 行以顏色將收斂情況分類，紅色表示最「收斂」的複數，其次是白色，跟著是綠色、藍色和黃色，而最快擴散的部分以黑色表示。第 210 行以先前選定的顏色畫出該點。

曼德勃羅集繪畫完成後，以白色畫出橫軸及縱軸 (見第 240 至 260 行)，以供參考。程序亦在此結束。

執行本程序所須的時間，要視乎電腦的速度，以現時一般的電腦而言，整個程序應該可以 1 分鐘左右完成。

參考書目

Heinz-Otto Peitgen, Hartmut Jürgens and Dietmar Saupe (1992) *Fractals for the Classroom Part Two: Introduction to Fractals and Chaos*. NCTM, Springer-Verlag.

IMO 2003

T. W. Leung

The 44th International Mathematical Olympiad (IMO) was held in Tokyo, Japan during the period 7 - 19 July 2003. Because Hong Kong was declared cleared from SARS on June 23, our team was able to leave for Japan as scheduled. The Hong Kong Team was composed as follows.

Chung Tat Chi (Queen Elizabeth School)
Kwok Tsz Chiu (Yuen Long Merchants Assn. Sec. School)
Lau Wai Shun (T. W. Public Ho Chuen Yiu Memorial College)
Siu Tsz Hang (STFA Leung Kau Kui College)
Yeung Kai Sing (La Salle College)
Yu Hok Pun (SKH Bishop Baker Secondary School)
Leung Tat Wing (Leader)
Leung Chit Wan (Deputy Leader)

Two former Hong Kong Team members, Poon Wai Hoi and Law Ka Ho, paid us a visit in Japan during this period.

The contestants took two 4.5 Hours contests on the mornings of July 13 and 14. Each contest consisted of three questions, hence contest 1 composed of Problem 1 to 3, contest 2 Problem 4 to 6. In each contest usually the easier problems come first and harder ones come later. After normal coordination procedures and Jury meetings cutoff scores for gold, silver and bronze medals were decided. This year the cutoff scores for gold, silver and bronze medals were 29, 19 and 13 respectively. Our team managed to win two silvers, two bronzes and one honorable mention. (Silver: Kwok Tsz Chiu and Yu Hok Pun, Bronze: Siu Tsz Hang and Yeung Kai Sing, Honorable Mention: Chung Tat Chi, he got a full score of 7 on one question, which accounted for his honorable mention, and his total score is 1 point short of bronze). Among all contestants three managed to obtain a perfect score of 42 on all six questions. One contestant was from China and the other two from Vietnam.

The Organizing Committee did not give official total scores for individual countries, but it is a tradition that scores between countries were compared. This year the top five teams were Bulgaria, China, U.S.A., Vietnam and Russia respectively. The Bulgarian contestants did extremely well on the two hard questions, Problem 3 and 6. Many people found it surprising. On the other hand, despite going through war in 1960s Vietnam has been strong all along. Perhaps they have participated in IMOs for a long time and have a very good Russian tradition.

Among 82 teams, we ranked unofficially 26. We were ahead of Greece, Spain, New Zealand and Singapore, for instance. Both New Zealand and we got our first gold last year. But this year the performance of the New Zealand Team was a bit disappointing. On the other hand, we were behind Canada, Australia, Thailand and U.K.. Australia has been doing well in the last few years, but this year the team was just 1 point ahead of us. Thailand has been able to do quite well in these few years.

IMO 2004 will be held in Greece, IMO 2005 in Mexico, IMO 2006 in Slovenia. IMO 2007 will be held in Vietnam, the site was decided during this IMO in Japan.

For the reader who will try out the IMO problems this year, here are some comments on Problem 3, the hardest problem in the first day of the competitions.

Problem 3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}\,/\,2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal. (A convex hexagon *ABCDEF* has three pairs of opposite sides: *AB* and *DE*, *BC* and *EF*, *CD* and *FA*.)

The problem is hard mainly because one does not know how to connect the given condition with that of the interior angles. Perhaps hexagons are not as rigid as triangles. It also reminded me of No. 5, IMO 1996, another hard problem of polygons.

The main idea is as follows. Given a hexagon *ABCDEF*, connect *AD*, *BE* and *CF* to form the diagonals. From the given condition of the hexagon, it can be proved that the triangles formed by the diagonals and the sides are actually equilateral triangles. Hence the interior angles of the hexagons are 120°. Good luck.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***November 30, 2003***.

Problem 186. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Let α, β, γ be complex numbers such that

$$\alpha+\beta+\gamma=1,$$
$$\alpha^2+\beta^2+\gamma^2=3,$$
$$\alpha^3+\beta^3+\gamma^3=7.$$

Determine the value of $\alpha^{21}+\ \beta^{21}+\ \gamma^{21}$.

Problem 187. Define $f(n)=n!$. Let

$$a=0.f(1)f(2)f(3)\ldots .$$

In other words, to obtain the decimal representation of a write the numbers $f(1)$, $f(2)$, $f(3)$, … in base 10 in a row. Is a rational? Give a proof.

Problem 188. The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles.

Problem 189. $2n+1$ segments are marked on a line. Each of the segments intersects at least n other segments. Prove that one of these segments intersect all other segments.

Problem 190. (*Due to Abderrahim Ouardini*) For nonnegative integer n, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and

$$f(n)=\left\lfloor\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right\rfloor -\left\lfloor\sqrt{9n+1}\right\rfloor.$$

Find the range of f and for each p in the range, find all nonnegative integers n such that $f(n)=p$.

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Solutions

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Problem 181. (*Proposed by Achilleas PavlosPorfyriadis, AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that in a convex polygon, there cannot be two sides with no common vertex, each of which is longer than the longest diagonal.

Proposer's Solution.

Suppose a convex polygon has two sides, say AB and CD, which are longer than the longest diagonal, where A, B, C, D are distinct vertices and A, C are on opposite side of line BD. Since AC, BD are diagonals of the polygon, we have $AB > AC$ and $CD > BD$. Hence,

$$AB + CD > AC + BD.$$

By convexity, the intersection O of diagonals AC and BD is on these diagonals. By triangle inequality, we have

$$AO + BO > AB \text{ and } CO + DO > CD.$$

So $AC + BD > AB + CD$, a contradiction.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **John PANAGEAS** (Kaisari High School, Athens, Greece), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (CUHK, Math Major, Year 1) and **YAU Chi Keung** (CNC Memorial College, Form 6).

Problem 182. Let a_0, a_1, a_2, … be a sequence of real numbers such that

$$a_{n+1} \ge a_n^2 + 1/5 \quad \text{for all } n \ge 0.$$

Prove that $\sqrt{a_{n+5}} \ge a_{n-5}^2$ for all $n \ge 5$. (*Source: 2001 USA Team Selection Test*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5) and **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School).

Adding $a_{n+1} - a_n^2 \ge 1/5$ for nonnegative integers $n = k, k+1, k+2, k+3, k+4$, we get

$$a_{k+5} - \sum_{n=k+1}^{k+4}(a_n^2 - a_n) - a_k^2 \ge 1 .$$

Observe that

$$x^2 - x + 1/4 = (x - 1/2)^2 \ge 0$$

implies $1/4 \ge -(x^2 - x)$. Applying this to the inequality above and simplifying, we easily get $a_{k+5} \ge a_k^2$ for nonnegative integer k. Then $a_{k+10} \ge a_{k+5}^2 \ge a_k^4$ for nonnegative integer k. Taking square root, we get the desired inequality.

Other commended solvers: **POON Ming Fung** (STFA Leung Kau Kui College, Form 6) and **SIU Tsz Hang** (CUHK, Math Major, Year 1).

Problem 183. Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square? (*Source: 1999 Russian Math Olympiad*)

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **SIU Tsz Hang** (CUHK, Math Major, Year 1).

Let $a_1, a_2, \ldots, a_{10}$ be distinct integers and S be their sum. For $i = 1, 2, \ldots, 10$, we would like to have $S - a_i = k_i^2$ for some integer k_i. Let T be the sum of $k_1^2, \ldots, k_{10}^2$. Adding the 10 equations, we get $9S = T$. Then $a_i = S - (S - a_i) = (T/9) - k_i^2$. So all we need to do is to choose integers $k_1, k_2, \ldots, k_{10}$ so that T is divisible by 9. For example, taking $k_i = 3i$ for $i = 1, \ldots, 10$, we get 376, 349, 304, 241, 160, 61, –56, –191, –344, –515 for $a_1, \ldots, a_{10}$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5).

Problem 184. Let $ABCD$ be a rhombus with $\angle B = 60^\circ$. M is a point inside ΔADC such that $\angle AMC = 120^\circ$. Let lines BA and CM intersect at P and lines BC and AM intersect at Q. Prove that D lies on the line PQ. (*Source: 2002 Belarussian Math Olympiad*)

Solution. **John PANAGEAS** (Kaisari High School, Athens, Greece), and **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Since $ABCD$ is a rhombus and $\angle ABC = 60^\circ$, we see $\angle ADC$, $\angle DAC$, $\angle DCA$, $\angle PAD$ and $\angle DCQ$ are all 60°**.**

Now

$$\angle CAM + \angle MCA = 180^\circ - \angle AMC = 60^\circ$$

and

$$\angle DCM + \angle MCA = \angle DCA = 60^\circ$$

imply $\angle CAM = \angle DCM$.

Since $AB \parallel CD$, we get

$$\angle APC = \angle DCM = \angle CAQ.$$

Also, $\angle PAC = 120^\circ = \angle ACQ$. Hence ΔAPC and ΔCAQ are similar. So $PA/AC = AC/CQ$.

Since $AC = AD = DC$, so $PA/AD = DC/CQ$. As $\angle PAD = 60^\circ = \angle DCQ$, so ΔPAD and ΔDCQ are similar. Then

$$\angle PDA + \angle ADC + \angle CDQ$$
$$= \angle PDA + \angle PAD + \angle APD = 180^\circ.$$

Therefore, P, D, Q are collinear.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (CUHK, Math Major, Year 1), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School).

Problem 185. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided one also changes the state of every d-th bulb after it (where d is a divisor of n and is less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

Solution.

Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ and the lights be at $1, \omega, \omega^2, \ldots, \omega^{n-1}$ with the one at 1 on initially. If d is a divisor of n that is less than n and the lights at

$$\omega^a, \omega^{a+d}, \omega^{a+2d}, \cdots, \omega^{a+(n-d)}$$

have the same state, then we can change the state of these n/d lights. Note their sum is a geometric series equal to

$$\omega^a(1-\omega^n)/(1-\omega^d) = 0.$$

So if we add up the numbers corresponding to the lights that are on before and after a move, it will remain the same. Since in the beginning this number is 1, it will never be

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

Therefore, all the lights can never be on at the same time.

Comments: This problem was due to Professor James Propp, University of Wisconsin, Madison (see his website http://www.math.wisc.edu/~propp/) and was selected from page 141 of the highly recommended book by Paul Zeitz titled *The Art and Craft of Problem Solving*, published by Wiley.

Olympiad Corner

(continued from page 1)

Problem 4. Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisector of $\angle ABC$ and $\angle ADC$ meet on AC.

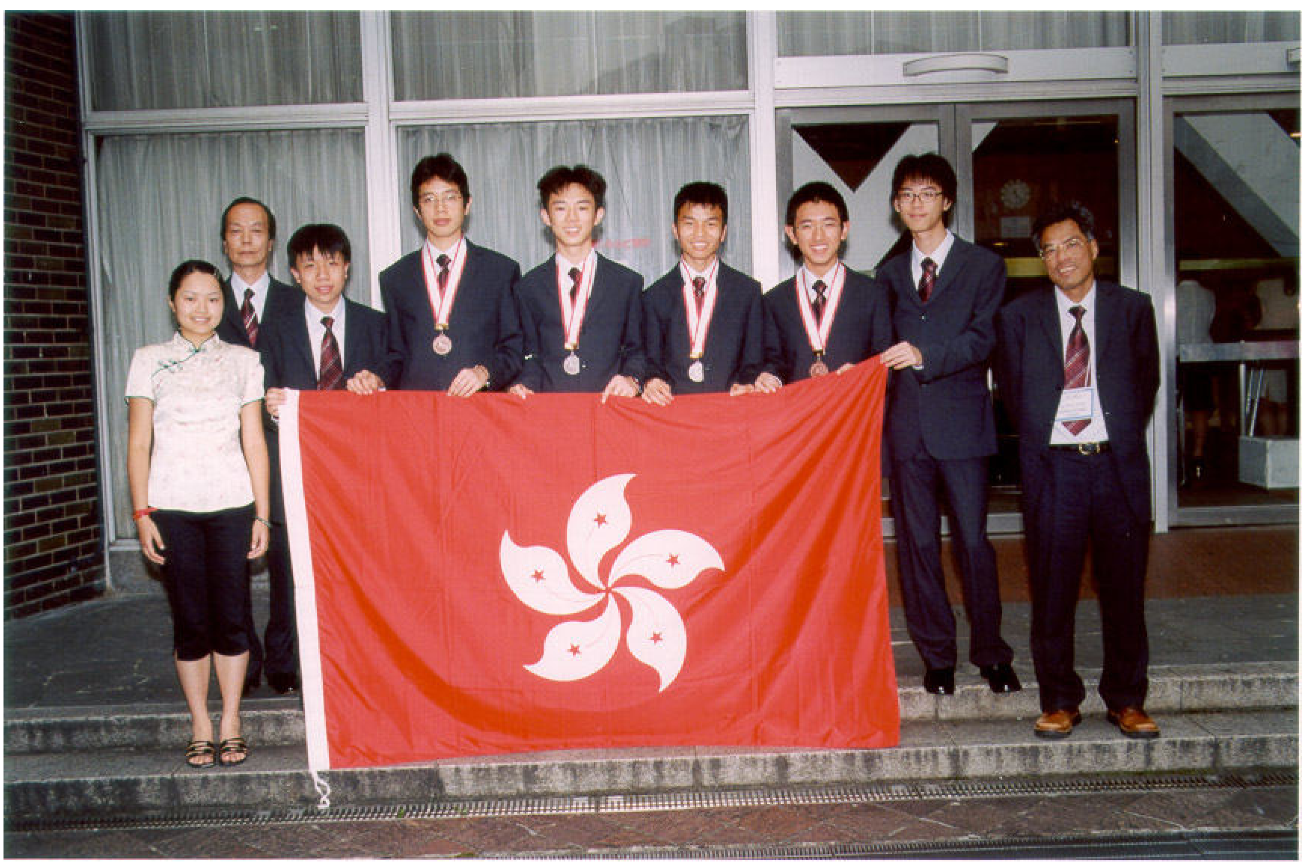
Problem 5. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be real numbers with $x_1 \le x_2 \le \cdots \le x_n$.

(a) Prove that

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i - x_j\right|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

(b) Show that equality holds if and only if $x_1, x_2, \ldots, x_n$ is an arithmetic sequence.

Problem 6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.



The 2003 Hong Kong IMO team from left to right: Wei Fei Fei (Guide), Leung Chit Wan (Deputy Leader), Chung Tat Chi, Siu Tsz Hang, Kwok Tsz Chiu, Yu Hok Pun, Yeung Kai Sing, Lau Wai Shun, Leung Tat Wing (Leader).

Mathematical Excalibur

Volume 8, Number 5 *November 2003 – December 2003*

Olympiad Corner

The 2003 USA Mathematical Olympiad took place on May 1. Here are the problems.

Problem 1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Problem 2. A convex polygon P in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygons P are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Problem 3. Let $n \neq 0$. For every sequence of integers $A = a_0, a_1, a_2, \ldots, a_n$ satisfying $0 \le a_i \le i$, for $i = 0, \ldots, n$, define another sequence $t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$ by setting $t(a_i)$ to be the number of terms in the sequence A that precede the terms a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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集與子集族

梁達榮

眾所周知，如果 S 是一個含 n 個元素的集，則它有 2^n 個子集，(包含空集及 S 本身)。不過如果選取子集的條件有所限制，例如子集只能有最多 k 個元素，或者所選取的兩個子集都必須相交（或不相交）等，則所能選取的子集必相應減少。

反過來說，如果 S 含一固定數目的子集，而這些子集又適合某些條件，則 n 的值不可能太小，又或者可以推到這些子集必須含有一些共同元素等。

這一類問題，泛稱集與子集族的問題，已經有很多有趣的成果。另外這些問題很能考驗學生的分析能力，並且需要的數學知識較少，所以在數學比賽中亦經常出現。先舉一個較簡單的例子。

例一：(蘇聯數學競賽 1965) 有一個委員會共舉行了 40 次會議，每次會議共有 10 人參加。並且每 2 個委員最多共一起參加同一會議 1 次。試證該委會員組成人數必多於 60 人。

證明：每一個會議有 10 人參加，因此共有 $C_2^{10} = 45$ “對”委員。按條件每一對委員不會在其他會議中出現，即 40 個會議共產生 $40 \times 45 = 1{,}800$ 不同的委員對。

如果該委員會有 n 人，則有 $C_2^n = \dfrac{n(n-1)}{2}$ 不同的對。所以必有 $1800 \le \dfrac{n(n-1)}{2}$，解之即得 $n > 60$。

如果所有委員組成一個集，則每一個會議參加的 10 人就是這個集的一個子集。按條件即是說任意這樣的兩個子集的交集最多包含一個元素。現在這樣的子集共 40 個，可以推斷這個集不可能太小、考慮這些問題，一個可能的策略是，尋找一個適當的觀察量，再用兩個不同的角度估計這個量，在這個例子中我們考慮的是共同參加同一會議的委員對。

例一的另一證明：我們也可以從以下的角度考慮此一問題。因為有 40 次會議，每次有 10 人參加，所以共 400 “人次”參加這些會議。假設這個委員會的總人數不多於 60 人，因為 $400/60 \approx 6.67$，則其中 1 人必參加 7 個或以上的會議。但是按照條件，參加這 7 個或以上會議的其他委員都不可能相同，因此共有 $7 \times 9 = 63$ 或以上不同的委員，矛盾！（留意在這裏用到鴿巢原理。）

有時候這一類問題可以另外的形式出現：

例二：(奧地利－波蘭數學競賽 1978) 有 1978 個集，每集含 40 個元素，並且任兩集剛好有 1 個共同元素。試證這 1978 個集必含有 1 個共同元素。

證明：設 A 為其中一個集，考慮其他 1977 個集，每一個集與 A 都有一個共同元素。由於 $1977/40 \approx 49.43$，即是說，A 中必有一個元素 x 在另外 50 個集 $A_1, A_2, \ldots, A_{50}$ 內，且因條件所限，x 是 $A_1, A_2, \ldots, A_{50}$ 的惟一公共元。

考慮另外一個集 B，如果 x 不在 B 內，由於 B 和 $A_1, A_2, ..., A_{50}$ 都相交，且由條件所限，相交的元素都不同，則 B 最少有 51 個元素，這是不可能的。所以 x 在 B 內，且 B 是任意的，所以 x 在任一個集內，證畢。

這個結果可以這樣推廣，且證明完全相似：設有 n 個集，每一個集有 k 個元素，任意兩集剛好有一個共同元素。如果 $n > k^2 - k + 1$，則這 n 個集有一個共同元素。

考慮一個較為困難的例子:

<u>例三</u>：(俄羅斯數學競賽 1996) 由 1600 個議員組成 16000 個委員會，每個委員會由 80 個委員組成。試證明：一定存在兩個委員會，它們之間至少有 4 個相同的議員。

<u>證明</u>：這一次我們不考慮每一個委員會組成委員的對，反過來考慮每一個議員所參加委員會形成的對。設議員 1, 2, … , 1600 分別參加了 $k_1, k_2, …, k_{1600}$ 個委員會，則總共有 $C_2^{k_1} + C_2^{k_2} + ... + C_2^{k_{1600}}$ 個委員會對。如果委員會的數目是 N, 則 $k_1 + k_2 + … + k_{1600} = 80N$，(在題中 $N = 16000$，且每個委員會由 80 人組成。) 現在試圖估計這些委員會對

$$C_2^{k_1} + C_2^{k_2} + \cdots + C_2^{k_{1600}}$$
$$= \frac{\sum_{i=1}^{1600} k_i^2 - \sum_{i=1}^{1600} k_i}{2}$$
$$\geq \frac{(\sum_{i=1}^{1600} k_i)^2}{3200} - \frac{(\sum_{i=1}^{1600} k_i)}{2}$$
$$= \frac{(80N)^2}{3200} - \frac{80N}{2}$$
$$= 2N^2 - 40N = 2N(N - 20)。$$

如果任兩個委員會最多有 3 個共同議員，則最多有

$$3C_2^N = \frac{3N(N-1)}{2}$$

個委員會對。因此

$$2N(N-20) \leq \frac{3}{2}N(N-1)。$$

即 $N \leq 77$，與 $N = 16, 000$ 矛盾。

(留意在估計中用到 Cauchy-Schwarz Inequality。) 無獨有偶，我們有以下的例子:

<u>例四</u>：(IMO1998) 在一次比賽中，有 m 個比賽員和 n 個評判，其中 $n \geq 3$ 是一個奇數。每一個評判對每一個比賽員進行評審為合格或不合格。如果任一對評判最多對 k 個比賽員的評審一致，試證明

$$\frac{k}{m} \geq \frac{n-1}{2n}。$$

<u>證明</u>: 題目已經提醒我們，我們考慮的是評判所成的“對”，這些”對”評判員對某些比賽員的決定一致。對於比賽員 i，$1 \leq i \leq m$，如果有 x_i 個評判認為他合格，y_i 個評判認為他不合格，則評判一致的對是

$$C_2^{x_i} + C_2^{y_i}$$
$$= \frac{(x_i^2 + y_i^2) - (x_i + y_i)}{2}$$
$$\geq \frac{(x_i + y_i)^2}{4} - \frac{(x_i + y_i)}{2}$$
$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[(n-1)^2 - 1\right]。$$

因為 n 是奇數，而 $C_2^{x_i} + C_2^{y_i}$ 是整數，因為 $C_2^{x_i} + C_2^{y_i}$ 最少是 $\frac{1}{4}(n-1)^2$。現在因為有 n 個評判，而任一對評判最多對 k 個比賽員意見一致，因為一致的評判最多是 kC_2^n。所以

$$kC_2^n \geq \sum_{i=1}^{m}\left[C_2^{x_i} + C_2^{y_i}\right] \geq \frac{m(n-1)^2}{4}$$

，化簡結果即為所求。

現在考慮一個形式略為不同的題目。我們的對象是一些長為 n 的數列，這些數列只包括 0 或 1，兩個這樣的數列的“距離”定義為對應位置數字不同的個數。例如 1101011 和 1011000 為兩個長為 7 的數列，它們在位置 2,3,6,7 的數字不同，因此它們的距離是 4。用集的言語來描述是，有 7 個元素 1,2,3,4,5,6 和 7 的一個集，數列一在位置 1,2,4,6,7 非零，因此可想像是包括 1,2,4,6,7 的一個子集，數列二是包括 1,3 和 4 的子集，屬於數列一或數列二，但不同時屬於兩個數列的子集包括 2,3,6,7，稱為兩個子集的對稱差，而“距離”正好是對稱差所含元素的數目。現在可以考慮的是，給定 n 和距離的限制，，這樣的數列最多是多少。

<u>例五</u>：有 m 個包括 0 或 1，長為 n 的數列，如果任兩個數列間的距離最少為 d，試證明

$$m \leq \frac{2d}{2d-n}。$$

<u>證明</u>：現在要考慮的是任兩個數列中“相異對”的數目，因為有 C_2^m 對數列，而任一對數列的“相異對”或“距離最少是 d，因此總距離最少是 dC_2^m。將這些數列排起來成為 m 個橫行，每一直行 j, $1 \leq j \leq n$ 就對應著那些數列的 j 位置。如果 j 直行有 x_j 個 "0"，則有 $m - x_j$ 個 "1"，因此相異對有 $x_j(m - x_j)$ 個。觀察到

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***February 28, 2004***.

Problem 191. Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Problem 192. Inside a triangle ABC, there is a point P satisfies $\angle PAB = \angle PBC = \angle PCA = \varphi$. If the angles of the triangle are denoted by α, β and γ, prove that

$$\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}.$$

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form $3k + 1$ as of the form $3k + 2$? Give a proof of your answer.

Problem 194. (*Due to Achilleas Pavlos PORFYRIADIS, American College of Thessaloniki "Anatolia", Thessaloniki, Greece*) A circle with center O is internally tangent to two circles inside it, with centers O_1 and O_2, at points S and T respectively. Suppose the two circles inside intersect at points M, N with N closer to ST. Show that S, N, T are collinear if and only if $SO_1/OO_1 = OO_2/TO_2$.

Problem 195. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Given n ($n > 3$) points on a plane, no three of them are collinear, x pairs of these points are connected by line segments. Prove that if

$$x \ge \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then there is at least one triangle having these line segments as edges.

Find all possible values of integers $n > 3$ such that $\dfrac{n(n-1)(n-2)+3}{3(n-2)}$ is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

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Solutions

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Problem 186. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Let α, β, γ be complex numbers such that

$$\alpha+\beta+\gamma = 1,$$
$$\alpha^2+\beta^2+\gamma^2 = 3,$$
$$\alpha^3+\beta^3+\gamma^3 = 7.$$

Determine the value of $\alpha^{21}+\beta^{21}+\gamma^{21}$.

Solution. **Helder Oliveira de CASTRO** (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 7), **FOK Kai Tung** (Yan Chai Hospital No. 2 Secondary School, Form 7), **FUNG Chui Ying** (True Light Girls' College, Form 6), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada), **LOK Kin Leung** (Tuen Mun Catholic Secondary School, Form 6), **SIU Ho Chung** (Queen's College, Form 5), **YAU Chi Keung** (CNC Memorial College, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Using the given equations and the identities

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha),$$
$$(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$$
$$= \alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma,$$

we get $\alpha\beta+\beta\gamma+\gamma\alpha = -1$ and $\alpha\beta\gamma = 1$. These imply α, β, γ are the roots of $f(x) = x^3 - x^2 - x - 1 = 0$. Let $S_n = \alpha^n + \beta^n + \gamma^n$, then $S_1 = 1$, $S_2 = 3$, $S_3 = 7$ and for $n > 0$,

$$S_{n+3} - S_{n+2} - S_{n+1} - S_n$$
$$= \alpha^n f(\alpha) - \beta^n f(\beta) - \gamma^n f(\gamma) = 0.$$

Using this recurrence relation, we find $S_4 = 11$, $S_5 = 21$, … , $S_{21} = 361109$.

Problem 187. Define $f(n) = n!$. Let

$$a = 0.f(1)f(2)f(3)\ldots.$$

In other words, to obtain the decimal representation of a write the numbers $f(1)$, $f(2)$, $f(3)$, … in base 10 in a row. Is a rational? Give a proof. (*Source: Israeli Math Olympiad*)

Solution. **Helder Oliveira de CASTRO** (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada) and **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Assume a is rational. Then its decimal representation will eventually be periodic. Suppose the period has k digits. Then for every $n > 10^k$, $f(n)$ is nonzero and ends in at least k zeros, which imply the period cannot have k digits. We got a contradiction.

Problem 188. The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles. (*Source: 2000 Saint Petersburg City Math Olympiad*)

Solution. **SIU Ho Chung** (Queen's College, Form 5).

Let O, G and H be the circumcenter, centroid and orthocenter of triangle ABC respectively. Let T and R be the projections of G and L onto line S. From the Euler line theorem (cf. *Math Excalibur, vol.* 3, *no.* 1, *p.*1), we know that O, G, H are collinear, G is between O and H and $2\ OG = GH$. Then T is between B and K and $2\ BT = TK$.

Also, G is on the median BL and $2\ LG = BG$. So T is between B and R and $2\ RT = BT$. Then $2\ BR = 2\ (BT + RT) = TK + TB = BK$. So $BR = RK$. Since LR is perpendicular to line S, by Pythagorean theorem, $BL=LK$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6) and **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Problem 189. $2n + 1$ segments are marked on a line. Each of the segments intersects at least n other segments. Prove that one of these segments intersect all other segments. (*Source 2000 Russian Math Olympiad*)

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

We imagine the segments on the line as intervals on the real axis. Going from left to right, let I_i be the i-th segment we meet with $i = 1, 2, \ldots, 2n+1$. Let I_i^l and I_i^r be the left and right endpoints of I_i respectively. Now I_1 contains $I_2^l, \ldots, I_{n+1}^l$. Similarly, I_2 which already intersects I_1 must contain $I_3^l, \ldots, I_{n+1}^l$ and so on. Therefore the segments $I_1, I_2, \ldots, I_{n+1}$ intersect each other.

Next let I_k^r be the rightmost endpoint among $I_1^r, I_2^r, \ldots, I_{n+1}^r$ $(1 \le k \le n+1)$. For each of the n remaining intervals $I_{n+2}, I_{n+3}, \ldots, I_{2n+1}$, it must intersect at least one of $I_1, I_2, \ldots, I_{n+1}$ since it has to intersect at least n intervals. This means for every $j \ge n+2$, there is at least one $m \le n+1$ such that $I_j^l \le I_m^r \le I_k^r$, then I_k intersects I_j and hence every interval.

Problem 190. (*Due to Abderrahim Ouardini*) For nonnegative integer n, let $[x]$ be the greatest integer less than or equal to x and

$$f(n) = \left[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}\right] - \left[\sqrt{9n+1}\right].$$

Find the range of f and for each p in the range, find all nonnegative integers n such that $f(n) = p$.

Combined Solution by the Proposer and **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6).

For positive integer n, we claim that

$$\sqrt{9n+8} < g(n) < \sqrt{9n+9},$$

where

$$g(n) = \sqrt{n} + \sqrt{n+1} + \sqrt{n+2}.$$

This follows from

$$g(n)^2 = 3n + 3 + 2(\sqrt{n(n+1)} + \sqrt{(n+1)(n+2)} + \sqrt{(n+2)n})$$

and the following readily verified inequalities for positive integer n,

$$(n+0.4)^2 < n(n+1) < (n+0.5)^2,$$

$$(n+1.4)^2 < (n+1)(n+2) < (n+1.5)^2$$

and $(n+0.7)^2 < (n+2)n < (n+1)^2$. The claim implies the range of f is a subset of nonnegative integers.

Suppose there is a positive integer n such that $f(n) \ge 2$. Then

$$\sqrt{9n+9} > [g(n)] > 1 + \sqrt{9n+1}.$$

Squaring the two extremes and comparing, we see this is false for $n > 1$. Since $f(0) = 1$ and $f(1) = 1$, we have $f(n) = 0$ or 1 for all nonnegative integers n.

Next observe that

$$\sqrt{9n+8} < [g(n)] < \sqrt{9n+9}$$

is impossible by squaring all expressions. So $[g(n)] = [\sqrt{9n+8}]$.

Now $f(n) = 1$ if and only if $p = [g(n)]$ satisfies $[\sqrt{9n+1}] = p-1$, i.e.

$$\sqrt{9n+1} < p \le \sqrt{9n+8}.$$

Considering squares (mod 9), we see that $p^2 = 9n+4$ or $9n+7$.

If $p^2 = 9n+4$, then $p = 9k+2$ or $9k+7$. In the former case, $n = 9k^2+4k$ and $(9k+1)^2 \le 9n+1 = 81k^2+36k+1 < (9k+2)^2$ so that $[\sqrt{9n+1}] = 9k+1 = p-1$. In the latter case, $n = 9k^2+14k+5$ and $(9k+6)^2 \le 9n+1 = 81k^2+126k+46 < (9k+7)^2$ so that $[\sqrt{9n+1}] = 9k+6 = p-1$.

If $p^2 = 9n+7$, then $p = 9k+4$ or $9k+5$. In the former case, $n = 9k^2+8k+1$ and $(9k+3)^2 \le 9n+1 = 81k^2+72k+10 < (9k+4)^2$ so that $[\sqrt{9n+1}] = 9k+3 = p-1$. In the latter case, $n = 9k^2+10k+2$ and $(9k+4)^2 \le 9n+1 = 81k^2+90k+19 < (9k+5)^2$ so that $[\sqrt{9n+1}] = 9k+4 = p-1$.

Therefore, $f(n) = 1$ if and only if n is of the form $9k^2+4k$ or $9k^2+14k+5$ or $9k^2+8k+1$ or $9k^2+10k+2$.

Olympiad Corner

(continued from page 1)

Problem 4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

Problem 6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

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(continued from page 2)

$$x_j(m-x_j) \le \left[\frac{x_j+(m-x_j)}{2}\right]^2 = \frac{m^2}{4}$$

，因此

$$dc_2^m \le \sum_{j=1}^{n} x_j(m-x_j) \le \sum_{j=1}^{n} \frac{m^2}{4} = \frac{nm^2}{4}。$$

化簡即得 $m \le \dfrac{2d}{2d-n}$ 。

例如 $n=7$，$d=4$，得 $\dfrac{2d}{2d-n} = 8$ ，

所以不可能構造 9 個長為 7，而相互間最少距離為 4 的數列。(讀者可試圖構造 8 個這樣的數列。) 這個例子實際上是編碼理論一個結果的特殊情況，這個結果一般稱為 Plotkin 限 (Plotkin Bound)。

集和子集族還有許多有趣的結果，有待研究和討論。

Mathematical Excalibur

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Olympiad Corner

The Sixth Hong Kong (China) Mathematical Olympiad took place on December 20, 2003. Here are the problems. Time allowed: 3 hours

Problem 1. Find the greatest real K such that for every positive u, v and w with $u^2 > 4vw$, the inequality

$$(u^2 - 4vw)^2 > K(2v^2 - uw)(2w^2 - uv)$$

holds. Justify your claim.

Problem 2. Let $ABCDEF$ be a regular hexagon of side length 1, and O be the center of the hexagon. In addition to the sides of the hexagon, line segments are drawn from O to each vertex, making a total of twelve unit line segments. Find the number of paths of length 2003 along these line segments that start at O and terminate at O.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. K, L, M, N are the midpoints of sides AB, BC, CD and DA respectively. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are the vertices of a parallelogram.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance and to Lee Man Fui and Poon Ming Fung for typesetting.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 25, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometry via Complex Numbers

Kin Y. Li

Complex numbers are wonderful. In this article we will look at some applications of complex numbers to solving geometry problems. If a problem involves points and chords on a circle, often we can without loss of generality assume it is the unit circle. In the following discussion, we will use the same letter for a point to denote the same complex number in the complex plane. To begin, we will study the equation of lines through points. Suppose Z is an arbitrary point on the line through W_1 and W_2. Since the vector from W_1 to Z is a multiple of the vector from W_1 to W_2, so in terms of complex numbers, we get $Z - W_1 = t(W_2 - W_1)$ for some real t. Now $t = \bar{t}$ and so

$$\frac{Z - W_1}{W_2 - W_1} = \frac{\overline{Z} - \overline{W_1}}{\overline{W_2} - \overline{W_1}}$$

Reversing the steps, we can see that every Z satisfying the equation corresponds to a point on the line through W_1 and W_2. So this is the equation of a line through two points in the complex variable Z.

Next consider the line passing through a point C and perpendicular to the line through W_1 and W_2. Let Z be on this line. Then the vector from C to Z is perpendicular to the vector from W_1 to W_2. In terms of complex numbers, we get $Z - C = it(W_2 - W_1)$ for some real t. So

$$\frac{Z - C}{i(W_2 - W_1)} = \frac{\overline{Z} - \overline{C}}{\bar{i}(\overline{W_2} - \overline{W_1})}.$$

Again reversing steps, we can conclude this is the equation of the line through C perpendicular to the line through W_1 and W_2.

In case the points W_1 and W_2 are on the unit circle, we have $W_1\overline{W_1} = 1 = W_2\overline{W_2}$. Multiplying the numerators and denominators of the right sides of the two displayed equations above by W_1W_2, we can simplify them to

$$Z + W_1W_2\overline{Z} = W_1 + W_2$$

and $\quad Z - W_1W_2\overline{Z} = C - W_1W_2\overline{C}$

respectively.

By moving W_2 toward W_1 along the unit circle, in the limit, we will get the equation of the tangent line at W_1 to the unit circle. It is $Z + W_1{}^2\overline{Z} = 2W_1$.

Similarly, the equation of the line through C perpendicular to this tangent line is $Z - W_1{}^2\overline{Z} = C - W_1{}^2\overline{C}$.

For a given triangle $A_1A_2A_3$ with the unit circle as its circumcircle, in terms of complex numbers, its circumcenter is the origin O, its centroid is $G = (A_1 + A_2 + A_3)/3$, its orthocenter is $H = A_1 + A_2 + A_3$ (because $OH = 3OG$) and the center of its nine point circle is $N = (A_1 + A_2 + A_3)/2$ (because N is the midpoint of OH).

Let us proceed to some examples.

Example 1. *(2000 St. Petersburg City Math Olympiad, Problem Corner 188)* The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles.

Solution. (*Due to POON Ming Fung, STFA Leung Kau Kui College, Form 6*) Without loss of generality, let the circumcircle of triangle ABC be the unit circle on the plane. Let $A = a + bi$, $B = -i$, $C = c + di$. Then the orthocenter is $H = A + B + C$ and $K = (a + c) - i$, $L = (a + c)/2 + (b + d)i/2$. Since

$$LB = \frac{1}{2}\sqrt{(a+c)^2 + (b+d+2)^2} = KL,$$

triangle BKL is isosceles.

Example 2. Consider triangle ABC and its circumcircle S. Reflect the circle with respect to AB, AC and BC to get three new circles S_{AB}, S_{AC} and S_{BC} (with the same radius as S). Show that these three new circles intersect at a common point. Identify this point.

Solution. Without loss of generality, we may assume S is the unit circle. Let the center of S_{AB} be O', then O' is the mirror image of O with respect to the segment AB. So $O' = A + B$ (because segments OO' and AB bisect each other). Similarly, the centers of S_{AC} and S_{BC} are $A + C$ and $B + C$ respectively. We need to show there is a point Z such that Z is on all three new circles, i.e.

$$|Z-(A+B)| = |Z-(A+C)| = |Z-(B+C)| = 1.$$

We easily see that the orthocenter of triangle ABC, namely $Z = H = A + B + C$, satisfies these equations. Therefore, the three new circles intersect at the orthocenter of triangle ABC.

Example 3. A point A is taken inside a circle. For every chord of the circle passing through A, consider the intersection point of the two tangents at the endpoints of the chord. Find the locus of these intersection points.

Solution. Without loss of generality we may assume the circle is the unit circle and A is on the real axis. Let WX be a chord passing through A with W and X on the circle. The intersection point Z of the tangents at W and X satisfies $Z+W^2\overline{Z}=2W$ and $Z+X^2\overline{Z}=2X$. Solving these equations together for Z, we find $Z=2/(\overline{W}+\overline{X})$.

Since A is on the chord WX, the real number A satisfies the equation for line WX, i.e. $A + WXA = W + X$. Using $W\overline{W} = 1 = X\overline{X}$, we see that

$$\operatorname{Re} Z = \frac{1}{\overline{W}+\overline{X}} + \frac{1}{W+X} = \frac{WX+1}{W+X} = \frac{1}{A}.$$

So the locus lies on the vertical line through $1/A$.

Conversely, for any point Z on this line, draw the two tangents from Z to the unit circle and let them touch the unit circle at the point W and X. Then the above equations are satisfied by reversing the argument. In particular, $A + WXA = W + X$ and so A is on the chord WX. Therefore, the locus is the line perpendicular to OA at a distance $1/OA$ from O.

Example 4. Let A_1, A_2, A_3 be the midpoints of W_2W_3, W_3W_1, W_1W_2 respectively. From A_i drop a perpendicular to the tangent line to the circumcircle of triangle $W_1W_2W_3$ at W_i. Prove that these perpendicular lines are concurrent. Identify this point of concurrency.

Solution. Without loss of generality, let the circumcircle of triangle $W_1W_2W_3$ be the unit circle. The line perpendicular to the tangent at W_1 through $A_1 = (W_2 + W_3)/2$ has equation

$$Z - W_1^2\overline{Z} = \frac{W_2+W_3}{2} - W_1^2\,\frac{\overline{W_2}+\overline{W_3}}{2}.$$

Using $W_1\overline{W_1}=1$, we may see that the right side is the same as

$$\frac{W_1+W_2+W_3}{2} - W_1^2\,\frac{\overline{W_1}+\overline{W_2}+\overline{W_3}}{2}.$$

From this we see that $N = (W_1 + W_2 + W_3)/2$ satisfies the equation of the line and so N is on the line. Since the expression for N is symmetric with respect to W_1, W_2, W_3, we can conclude that N will also lie on the other two lines. Therefore, the lines concur at N, the center of the nine point circle of triangle $W_1W_2W_3$.

Example 5. *(Simson Line Theorem)* Let W be on the circumcircle of triangle $Z_1Z_2Z_3$ and P, Q, R be the feet of the perpendiculars from W to Z_3Z_1, Z_1Z_2, Z_2Z_3 respectively. Prove that P, Q, R are collinear. (This line is called the *Simson line* of triangle $Z_1Z_2Z_3$ from W.)

Solution. Without loss of generality, we may assume the circumcircle of triangle $Z_1Z_2Z_3$ is the unit circle.

Then $|Z_1|=|Z_2|=|Z_3|=|W|=1$. Now P is on the line Z_3Z_1 and the line through W perpendicular to Z_3Z_1. So P satisfies the equations $P+Z_1Z_3\overline{P}=Z_1+Z_3$ and $P - Z_1Z_3\overline{P}=W-Z_1Z_3\overline{W}$. Solving these together for P, we get

$$P = \frac{Z_1 + Z_3 + W - Z_1Z_3\overline{W}}{2}.$$

Similarly,

$$Q = \frac{Z_1 + Z_2 + W - Z_1Z_2\overline{W}}{2}$$

and

$$R = \frac{Z_2 + Z_3 + W - Z_2Z_3\overline{W}}{2}.$$

To show P, Q, R are collinear, it suffices to check that

$$\frac{P-R}{Q-R} = \frac{\overline{P}-\overline{R}}{\overline{Q}-\overline{R}}.$$

Now the right side is

$$\frac{\overline{Z_1} - \overline{Z_2} - \overline{Z_1Z_3W} + \overline{Z_2Z_3W}}{\overline{Z_1} - \overline{Z_3} - \overline{Z_1Z_2W} + \overline{Z_2Z_3W}}.$$

Multiplying the numerator and denominator by $Z_1Z_2Z_3\overline{W}$ and using $Z_i\overline{Z_i} = 1 = W\overline{W}$, we get

$$\frac{Z_2Z_3\overline{W} - Z_1Z_3\overline{W} - Z_2 + Z_1}{Z_2Z_3\overline{W} - Z_1Z_2\overline{W} - Z_3 + Z_1}.$$

This equals the left side $(P-R)/(Q-R)$ and we complete the checking.

Example 6. (*2003 IMO, Problem 4*) Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ meet on AC.

Solution. (*Due to SIU Tsz Hang, 2003 Hong Kong IMO team member*) Without loss of generality, assume A, B, C, D lies on the unit circle and the perpendicular bisector of AC is the real axis. Let $A = \cos\theta + i\sin\theta$, then $C=\overline{A}=\cos\theta - i\sin\theta$ so that $AC = 1$ and $A + C = 2\cos\theta$. Since the bisectors of $\angle ABC$ and $\angle ADC$ pass through the midpoints of the major and minor arc AC, we may assume the bisectors of $\angle ABC$ and $\angle ADC$ pass through 1 and −1 respectively. Let AC intersect the bisector of $\angle ABC$ at Z, then Z satisfies $Z+AC\overline{Z}=A+C$, (which is $Z+\overline{Z}=2\cos\theta$), and $Z+B\overline{Z}=B+1$. Solving for Z, we get

$$Z = \frac{2B\cos\theta - B - 1}{B-1}.$$

Similarly, the intersection point Z' of AC with the bisector of $\angle ADC$ is

$$Z' = \frac{2D\cos\theta + D - 1}{D+1}.$$

Next, R is on the line AB and the line through D perpendicular to AB. So $R+AB\overline{R}=A+B$ and $R-AB\overline{R}=D-AB\overline{D}$. Solving for R, we find

$$R = \frac{A+B+D-AB\overline{D}}{2}.$$

Similarly,

$$P = \frac{B+C+D-BC\overline{D}}{2}$$

and

$$Q = \frac{C+A+D-CA\overline{D}}{2}.$$

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 25, 2004.***

Problem 196. (*Due to John PANAGEAS, High School "Kaisari", Athens, Greece*) Let $x_1, x_2, \ldots, x_n$ be positive real numbers with sum equal to 1. Prove that for every positive integer m,

$$n \le n^m (x_1^m + x_2^m + \ldots + x_n^m).$$

Problem 197. In a rectangular box, the length of the three edges starting at the same vertex are prime numbers. It is also given that the surface area of the box is a power of a prime. Prove that exactly one of the edge lengths is a prime number of the form $2^k - 1$.

Problem 198. In a triangle ABC, $AC = BC$. Given is a point P on side AB such that $\angle ACP = 30^\circ$. In addition, point Q outside the triangle satisfies $\angle CPQ = \angle CPA + \angle APQ = 78^\circ$. Given that all angles of triangles ABC and QPB, measured in degrees, are integers, determine the angles of these two triangles.

Problem 199. Let R^+ denote the positive real numbers. Suppose $f: R^+ \to R^+$ is a strictly decreasing function such that for all $x, y \in R^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$ for every $x > 0$. (*Source: 1997 Iranian Math Olympiad*)

Problem 200. Aladdin walked all over the equator in such a way that each moment he either was moving to the west or was moving to the east or applied some magic trick to get to the opposite point of the Earth. We know that he travelled a total distance less than half of the length of the equator altogether during his westward moves. Prove that there was a moment when the difference between the distances he had covered moving to the east and moving to the west was at least half of the length of the equator.

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Solutions

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Due to an editorial mistake in the last issue, solutions to problems 186, 187, 188 by **POON Ming Fung** (*STFA Leung Kau Kui College, Form 6*) were overlooked and his name was not listed among the solvers. We express our apology to him and point out that his clever solution to problem 188 is printed in example 1 of the article "Geometry via Complex Numbers" in this issue.

Problem 191. Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Solution. **Helder Oliveira de CASTRO** (ITA-Aeronautic Institue of Technology, Sao Paulo, Brazil) and **Yufei ZHAO** (Don Mills Collegeiate Institute, Toronoto, Canada, Grade 10).

If $x < -2$, then the right side of the equation is not defined. If $x > 2$, then

$$x^3 - 3x = \frac{x^3 + 3x(x+2)(x-2)}{4} > \frac{x^3}{4} > \sqrt{x+2}.$$

So the solution(s), if any, must be in [-2, 2]. Write $x = 2\cos a$, where $0 \le a \le \pi$. The equation becomes

$$8\cos^3 a - 6\cos a = \sqrt{2\cos a + 2}.$$

Using the triple angle formula on the left side and the half angle formula on the right side, we get

$$2\cos 3a = 2\cos\frac{a}{2} (\ge 0).$$

Then $3a \pm (a/2) = 2n\pi$ for some integer n. Since $3a \pm (a/2) \in [-\pi/2, 7\pi/2]$, we get $n = 0$ or 1. We easily checked that $a = 0$, $4\pi/5$, $4\pi/7$ yield the only solutions $x = 2$, $2\cos(4\pi/5)$, $2\cos(4\pi/7)$.

Other commended solvers: **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 7), **LEE Man Fui** (CUHK, Year 1), **LING Shu Dung**, **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SINN Ming Chun** (STFA Leung Kau Kui College, Form 4), **SIU Ho Chung** (Queen's College, Form 5), **TONG Yiu Wai** (Queen Elizabeth School), **YAU Chi Keung** (CNC Memorial College, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 192. Inside a triangle ABC, there is a point P satisfies $\angle PAB = \angle PBC = \angle PCA = \varphi$. If the angles of the triangle are denoted by α, β and γ, prove that

$$\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}.$$

Solution. **LEE Tsun Man Clement** (St. Paul's College), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Ho Chung** (Queen's College, Form 5) and **Yufei ZHAO** (Don Mills Collegiate Institute, Tornoto, Canada, Grade 10).

Let AP meet BC at X. Since $\angle XBP = \angle BAX$ and $\angle BXP = \angle AXB$, triangles XPB and XBA are similar. Then $XB/XP = XA/XB$. Using the sine law and the last equation, we have

$$\frac{\sin^2\varphi}{\sin^2\beta} = \frac{\sin^2\angle XAB}{\sin^2\angle XBA} = \frac{XB^2}{XA^2}$$

$$= \frac{XP\cdot XA}{XA^2} = \frac{XP}{XA}$$

Using [ ] to denote area, we have

$$\frac{XP}{XA} = \frac{[XBP]}{[XBA]} = \frac{[XCP]}{[XCA]} = \frac{[BPC]}{[ABC]}$$

Combining the last two equations, we have $\sin^2\varphi/\sin^2\beta = [BPC]/[ABC]$. By similar arguments, we have

$$\frac{\sin^2\varphi}{\sin^2\alpha} + \frac{\sin^2\varphi}{\sin^2\phi} + \frac{\sin^2\varphi}{\sin^2\gamma}$$

$$= \frac{[APB]}{[ABC]} + \frac{[BPC]}{[ABC]} + \frac{[CPA]}{[ABC]}$$

$$= \frac{[ABC]}{[ABC]} = 1$$

The result follows.

Other commended solvers: **CHENG Tsz Chung** (La Salle College, Form 5), **LEE Man Fui** (CUHK, Year 1) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Comments: Professor Murray KLAMKIN (University of Alberta, Edmonton, Canada) informed us that the result $\csc^2\varphi = \csc^2\alpha + \csc^2\beta + \csc^2\gamma$ in the problem is a known relation for the Brocard angle φ of a triangle. Also known is $\cot\varphi = \cot\alpha + \cot\beta + \cot\gamma$. He mentioned these relations and others are given in R.A. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960, pp. 266-267. (For the convenience of interested readers, the Chinese translation of this book can be found in many bookstore.–Ed) LEE Man Fui and Achilleas PORFYRIADIS gave a proof of the cotangent relation and use it to derive the cosecant relation, which is the equation in the problem, by trigonometric manipulations.

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form $3k + 1$ as of the form $3k + 2$? Give a proof of your answer.

Solution 1. **K.C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher), **LEE Tsun Man Clement** (St. Paul's College), **SIU Ho Chung** (Queen's College, Form 5) and **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10).

No. For a perfect square m^2, let $m = 3^a b$ with b not divisible by 3. Then $m^2 = 3^{2a}b^2$. Observe that divisors of the form $3k + 1$ or $3k + 2$ for m^2 and for b^2 consist of the same numbers because they cannot have any factor of 3. Since b^2 has an odd number of divisors and they can only be of the form $3k + 1$ or $3k + 2$, so the number of divisors of the form $3k + 1$ cannot be the same as the number of divisors of the form $3k + 2$. Therefore, the same is true for m^2.

Solution 2. **Helder Oliveira de CASTRO** (ITA-Aeronautic Institute of Technology, Sao Paulo, Brazil), **LEE Man Fui** (CUHK, Year 1), **LING Shu Dung, POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Alan T.W. WONG** (Markham, Ontario, Canada) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

No. For a perfect square, its prime factorization is of the form $2^{2e_1}3^{2e_2}5^{2e_3}\cdots$. Let x, y, z be the number of divisors of the form $3k$, $3k + 1$, $3k + 2$ for this perfect square respectively. Then $x + y + z = (2e_1 + 1)\,(2e_2 + 1)\,(2e_3 + 1)\cdots$ is odd. Now divisor of the form $3k$ has at least one factor 3, so $x = (2e_1 + 1)\,(2e_2)\,(2e_3 + 1)\cdots$ is even. Then $y + z$ is odd. Therefore y cannot equal z.

Other commended solvers: **CHENG Tsz Chung** (La Salle College, Form 5) and **YEUNG Wai Kit** (STFA Leung Kau Kui College).

Problem 194. (*Due to Achilleas Pavlos PORFYRIADIS, American College of Thessaloniki "Anatolia", Thessaloniki, Greece*) A circle with center O is internally tangent to two circles inside it, with centers O_1 and O_2, at points S and T respectively. Suppose the two circles inside intersect at points M, N with N closer to ST. Show that S, N, T are collinear if and only if $SO_1/OO_1 = OO_2/TO_2$.

Solution. **CHENG Tsz Chung** (La Salle College, Form 5), **K. C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher), **Helder Oliveira de CASTRO** (ITA-Aeronautic Institute of Technology, Sao Paulo, Brazil), **LEE Tsun Man** Clement (St. Paul's College), **LING Shu Dung, POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Ho Chung** (Queen's College, Form 5), **YEUNG Wai Kit** (STFA Leung Kau Kui College), **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10) and the proposer.

If S, N, T are collinear, then triangles SO_1N and SOT are isosceles and share the common angle OST, which imply they are similar. Thus $\angle SO_1N = \angle SOT$ and so lines O_1N and OT are parallel. Similarly, lines O_2N and OS are parallel. Hence, OO_1NO_2 is a parallelogram and $OO_2 = O_1N = O_1S$, $OO_1 = O_2N = O_2T$. Therefore, $SO_1/OO_1 = OO_2/TO_2$. Conversely, if $SO_1/OO_1 = OO_2/TO_2$, then using $OO_1 = OS - O_1S$ and $OO_2 = OT - O_2T$, we get

$$\frac{O_1S}{OS - O_1S} = \frac{OT - O_2T}{O_2T},$$

which reduces to $O_1S + O_2T = OS$. Then $OO_1 = OS - O_1S = O_2T = O_2N$ and $OO_2 = OT - O_2T = O_1S = O_1N$. Hence OO_1NO_2 is again a parallelogram. Then

$$\begin{aligned}&\angle O_1NS + \angle O_1NO_2 + \angle O_2NT\\&= \angle O_1SN + \angle O_1NO_2 + \angle O_2TN\\&= \frac{1}{2}\angle OO_1N + \angle O_1NO_2 + \frac{1}{2}\angle OO_2N\\&= 180^\circ.\end{aligned}$$

Therefore, S, N, T are collinear.

Other commended solver: **TONG Yiu Wai** (Queen Elizabeth School).

Problem 195. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Given n ($n > 3$) points on a plane, no three of them are collinear, x pairs of these points are connected by line segments. Prove that if

$$x \ge \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then there is at least one triangle having these line segments as edges. Find all possible values of integers $n > 3$ such that $\frac{n(n-1)(n-2)+3}{3(n-2)}$ is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

Solution. **SIU Ho Chung** (Queen's College, Form 5), **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10) and the proposer.

For every three distinct points A, B, C, form a pigeonhole containing the three segments AB, BC, CA. (Each segment may be in more than one pigeonholes.) There are C_3^n pigeonholes. For each segment joining a pair of endpoints, that segment will be in $n - 2$ pigeonholes. So if $x(n-2) \ge 2C_3^n + 1$, that is

$$x \ge \frac{2C_3^n + 1}{n-2} = \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then by the pigeonhole principle, there is at least one triangle having these line segments as edges.

If $f(n) = (n(n-1)(n-2) + 3) / (3(n-2))$ is an integer, then $3(n-2)f(n) = n(n-1)(n-2) + 3$ implies 3 is divisible by $n - 2$. Since $n > 3$, we must have $n = 5$. Then $f(5) = 7$. For the five vertices A, B, C, D, E of a regular pentagon, if we connected the six segments BC, CD, DE, EA, AC, BE, then there is no triangle. So a minimum of $f(5) = 7$ segments is needed to get a triangle.

Other commended solvers: **K. C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher) and **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 4. Find, with reasons, all integers a, b, and c such that
$\frac{1}{2}(a+b)(b+c)(c+a) + (a+b+c)^3 = 1 - abc.$

Geometry via Complex Numbers

(continued from page 2)

By Simson's theorem, P, Q, R are collinear. So $PQ = QR$ if and only if $Q = (P + R)/2$. In terms of A, B, C, D, this may be simplified to

$$C + A - 2B = (2CA - AB - BC)\overline{D}.$$

In terms of B, D, θ, this is equivalent to $(2\cos\theta - 2B)D = 2 - 2B\cos\theta$. This is easily checked to be the same as

$$\frac{2\cos\theta - B - 1}{B - 1} = \frac{2D\cos\theta + D - 1}{D + 1},$$

i.e. $Z = Z'$.

Comments: The official solution by pure geometry is shorter, but it takes a fair amount of time and cleverness to discover. Using complex numbers as above reduces the problem to straight computations.

Mathematical Excalibur

Volume 9, Number 2 — *May 2004 – July 2004*

Olympiad Corner

The XVI Asian Pacific Mathematical Olympiad took place on March 2004. Here are the problems. Time allowed: 4 hours.

Problem 1. Determine all finite nonempty sets S of positive integers satisfying

$\frac{i+j}{(i,j)}$ is an element of S for all i, j in S,

where (i, j) is the greatest common divisor of i and j.

Problem 2. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, COH is equal to the sum of the areas of the other two.

Problem 3. Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to color the points of S with at most two colors, such that for any points p, q of S, the number

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 9, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inversion

Kin Y. Li

In algebra, the method of logarithm *transforms* tough problems involving multiplications and divisions into simpler problems involving additions and subtractions. For every positive number x, there is a unique real number $\log x$ in base 10. This is a one-to-one correspondence between the positive numbers and the real numbers.

In geometry, there are also transformation methods for solving problems. In this article, we will discuss one such method called *inversion*. To present this, we will introduce the *extended plane*, which is the plane together with a point that we would like to think of as infinity. Also, we would like to think of *all* lines on the plane will go through *this point at infinity*! To understand this, we will introduce the *stereographic projection*, which can be described as follow.

Consider a sphere sitting on a point O of a plane. If we remove the north pole N of the sphere, we get a *punctured* sphere. For every point P on the plane, the line NP will intersect the punctured sphere at a *unique* point S_P. So this gives a one-to-one correspondence between the plane and the punctured sphere. If we consider the points P on a circle in the plane, then the S_P points will form a circle on the punctured sphere. However, if we consider the points P on *any* line in the plane, then the S_P points will form a punctured circle on the sphere with N as the point removed from the circle. *If we move a point P on any line on the plane toward infinity, then S_P will go toward the same point N!* Thus, in this model, all lines can be thought of as going to the same infinity.

Now for the method of inversion, let O be a point on the plane and r be a positive number. The *inversion* with center O and radius r is the function on the extended plane that sends a point $X \neq O$ to the *image* point X' on the ray $\overrightarrow{OX}$ such that

$$OX \cdot OX' = r^2.$$

When $X = O$, X' is taken to be the point at infinity. When X is infinity, X' is taken to be O. The circle with center O and radius r is called the *circle of inversion*.

The method of inversion is based on the following facts.

(1) The function sending X to X' described above is a one-to-one correspondence between the extended plane with itself. (This follows from checking $(X')' = X$.)

(2) If X is on the circle of inversion, then $X' = X$. If X is outside the circle of inversion, then X' is the midpoint of the chord formed by the tangent points T_1, T_2 of the tangent lines from X to the circle of inversion. (This follows from

$$OX \cdot OX' = (r \sec \angle T_1OX)(r \cos \angle T_1OX) = r^2.)$$

(3) A circle not passing through O is sent to a circle not passing through O. In this case, the images of concyclic points are concyclic. The point O, the centers of the circle and the image circle are collinear. *However, the center of the circle is not sent to the center of the image circle!*

(4) A circle passing through O is sent to a line which is not passing through O and is parallel to the tangent line to the circle at O. Conversely, a line not passing through O is sent to a circle passing through O with the tangent line at O parallel to the line.

(5) A line passing through O is sent to itself.

(6) If two curves intersect at a certain angle at a point $P \neq O$, then the image curves will also intersect at the same angle at P'. If the angle is a right angle, the curves are said to be <u>orthogonal</u>. So in particular, orthogonal curves at P are sent to orthogonal curves at P'. A circle orthogonal to the circle of inversion is sent to itself. Tangent curves at P are sent to tangent curves at P'.

(7) If points A, B are different from O and points O, A, B are not collinear, then the equation $OA\cdot OA' = r^2 = OB\cdot OB'$ implies $OA/OB=OB'/OA'$. Along with $\angle AOB = \angle B'OA'$, they imply ΔOAB, $\Delta OB'A'$ are similar. Then

$$\frac{A'B'}{AB} = \frac{OA'}{OB} = \frac{r^2}{OA\cdot OB}$$

so that

$$A'B' = \frac{r^2}{OA\cdot OB} AB .$$

The following are some examples that illustrate the powerful method of inversion. In each example, when we do inversion, it is often that we take the point that plays the <u>most significant role</u> and where <u>many circles and lines intersect</u>.

***Example* 1.** (*Ptolemy's Theorem*) For coplanar points A, B, C, D, if they are concyclic, then

$$AB\cdot CD + AD\cdot BC = AC\cdot BD.$$

Solution. Consider the inversion with center D and any radius r. By fact (4), the circumcircle of ΔABC is sent to the line through A', B', C'. Since $A'B' + B'C' = A'C'$, we have by fact (7) that

$$\frac{r^2}{AD\cdot BD}AB + \frac{r^2}{BD\cdot CD}BC = \frac{r^2}{AD\cdot CD}AC.$$

Multiplying by $(AD\cdot BD\cdot CD)/r^2$, we get the desired equation.

Remarks. The steps can be reversed to get the converse statement that if

$$AB\cdot CD + AD\cdot BC = AC\cdot BD,$$

then A,B,C,D are concyclic.

***Example* 2.** (*1993 USAMO*) Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let O be their intersection point. Prove that the reflections of O across AB, BC, CD, DA are concyclic.

Solution. Let P, Q, R, S be the feet of perpendiculars from O to AB, BC, CD, DA, respectively. The problem is equivalent to showing P, Q, R, S are concyclic (since they are the midpoints of O to its reflections). Note $OSAP$, $OPBQ$, $OQCR$, $ORDS$ are cyclic quadrilaterals. Let their circumcircles be called C_A, C_B, C_C, C_D, respectively.

Consider the inversion with center O and any radius r. By fact (5), lines AC and BD are sent to themselves. By fact (4), circle C_A is sent to a line L_A parallel to BD, circle C_B is sent to a line L_B parallel to AC, circle C_C is sent to a line L_C parallel to BD, circle C_D is sent to a line L_D parallel to AC.

Next C_A intersects C_B at O and P. This implies L_A intersects L_B at P'. Similarly, L_B intersects L_C at Q', L_C intersects L_D at R' and L_D intersects L_A at S'.

Since $AC \perp BD$, $P'Q'R'S'$ is a rectangle, hence cyclic. Therefore, by fact (3), P, Q, R, S are concyclic.

***Example* 3.** (*1996 IMO*) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

Solution. Let lines AP, BD intersect at X, lines AP, CE intersect at Y. We have to show $X = Y$. By the angle bisector theorem, $BA/BP = XA/XP$. Similarly, $CA/CP = YA/YP$. As X, Y are on AP, we get $X = Y$ if and only if $BA/BP = CA/CP$.

Consider the inversion with center A and any radius r. By fact (7), ΔABC, $\Delta AC'B'$ are similar, ΔAPB, $\Delta AB'P'$ are similar and ΔAPC, $\Delta AC'P'$ are similar. Now

$$\begin{aligned}\angle B'C'P' &= \angle AC'P' - \angle AC'B' \\ &= \angle APC - \angle ABC \\ &= \angle APB - \angle ACB \\ &= \angle AB'P - \angle AB'C' \\ &= \angle C'B'P'.\end{aligned}$$

So $\Delta B'C'P'$ is isosceles and $P'B' = P'C'$. From ΔAPB, $\Delta AB'P'$ similar and ΔAPC, $\Delta AC'P'$ similar, we get

$$\frac{BA}{BP} = \frac{P'A}{P'B'} = \frac{P'A}{P'C'} = \frac{CA}{CP}.$$

Therefore, $X = Y$.

***Example* 4.** (*1995 Israeli Math Olympiad*) Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that $AB \perp PQ$ and is tangent to O. Prove that AC bisects $\angle PAB$.

Solution. Consider the inversion with center C and any radius r. By fact (7), ΔCAP, $\Delta CP'A'$ similar and ΔCAB, $\Delta CB'A'$ similar. So AC bisects PAB if and only if $\angle CAP = \angle CAB$ if and only if $\angle CP'A' = \angle CB'A'$.

By fact (5), line PQ is sent to itself. Since circle O passes through C, circle O is sent to a line O' parallel to PQ. By fact (6), since H is tangent to circle O and is orthogonal to line PQ, H is sent to the semicircle H' tangent to line O' and has diameter $P'Q'$. Since segment AB is tangent to circle O and is orthogonal to PQ, segment AB is sent to arc $A'B'$ on the semicircle tangent to line O' and has diameter CB'. Now observe that arc $A'Q'$ and arc $A'C$ are symmetrical with respect to the perpendicular bisector of CQ' so we get $\angle CP'A' = \angle CB'A'$.

In the solutions of the next two examples, we will consider the nine-point circle and the Euler line of a triangle. Please consult Vol. 3, No. 1 of Mathematical Excalibur for discussion if necessary.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***August 9, 2004.***

Problem 201. (*Due to Abderrahim OUARDINI, Talence, France*) Find which nonright triangles ABC satisfy

$$\tan A \tan B \tan C > [\tan A] + [\tan B] + [\tan C],$$

where $[t]$ denotes the greatest integer less than or equal to t. Give a proof.

Problem 202. (*Due to LUK Mee Lin, La Salle College*) For triangle ABC, let D, E, F be the midpoints of sides AB, BC, CA, respectively. Determine which triangles ABC have the property that triangles ADF, BED, CFE can be folded above the plane of triangle DEF to form a tetrahedron with AD coincides with BD; BE coincides with CE; CF coincides with AF.

Problem 203. (*Due to José Luis DÍAZ-BARRERO, Universitat Politecnica de Catalunya, Barcelona, Spain*) Let a, b and c be real numbers such that $a + b + c \neq 0$. Prove that the equation

$$(a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0$$

has only real roots.

Problem 204. Let n be an integer with $n > 4$. Prove that for every n distinct integers taken from 1, 2, …, $2n$, there always exist two numbers whose least common multiple is at most $3n + 6$.

Problem 205. *(Due to HA Duy Hung, Hanoi University of Education, Vietnam)* Let a, n be integers, both greater than 1, such that $a^n - 1$ is divisible by n. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and n is greater than 1.

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Solutions

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Problem 196. (*Due to John PANAGEAS, High School "Kaisari", Athens, Greece*) Let $x_1, x_2, \ldots, x_n$ be positive real numbers with sum equal to 1. Prove that for every positive integer m,

$$n \le n^m (x_1^m + x_2^m + \ldots + x_n^m).$$

Solution. **CHENG Tsz Chung** (La Salle College, Form 5), **Johann Peter Gustav Lejeune DIRICHLET** (Universidade de Sao Paulo – Campus Sao Carlos), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

Applying Jensen's inequality to $f(x) = x^m$ on [0, 1] or the power mean inequality, we have

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^m \le \frac{x_1^m + \cdots + x_n^m}{n}.$$

Using $x_1 + \cdots + x_n = 1$ and multiplying both sides by n^{m+1}, we get the desired inequality.

Other commended solvers: **TONG Yiu Wai** (Queen Elizabeth School, Form 6), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3) and **YEUNG Yuen Chuen** (La Salle College, Form 4).

Problem 197. In a rectangular box, the lengths of the three edges starting at the same vertex are prime numbers. It is also given that the surface area of the box is a power of a prime. Prove that exactly one of the edge lengths is a prime number of the form $2^k - 1$. (*Source: KöMaL Gy.3281*)

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 4), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **John PANAGEAS** (Kaisari High School, Athens, Greece), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5), **TO Ping Leung** (St. Peter's Secondary School), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3), **YEUNG Yuen Chuen** (La Salle College, Form 4) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

Let the prime numbers x, y, z be the lengths of the three edges starting at the same vertex. Then $2(xy + yz + zx) = p^n$ for some prime p and positive integer n. Since the left side is even, we get $p = 2$. So $xy + yz + zx = 2^{n-1}$. Since x, y, z are at least 2, the left side is at least 12, so n is at least 5. If none or exactly one of x, y, z is even, then $xy + yz + zx$ would be odd, a contradiction. So at least two of x, y, z are even and prime, say $x = y = 2$. Then $z = 2^{n-3} - 1$. The result follows.

Other commended solvers: **NGOO Hung Wing** (Valtorta College).

Problem 198. In a triangle ABC, $AC = BC$. Given is a point P on side AB such that $\angle ACP = 30°$. In addition, point Q outside the triangle satisfies $\angle CPQ = \angle CPA + \angle APQ = 78°$. Given that all angles of triangles ABC and QPB, measured in degrees, are integers, determine the angles of these two triangles. (*Source: KöMaL C. 524*)

Solution. **CHAN On Ting Ellen** (True Light Girls' College, Form 4), **CHENG Tsz Chung** (La Salle College, Form 5), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **TONG Yiu Wai** (Queen Elizabeth School, Form 6), **YEUNG Yuen Chuen** (La Salle College, Form 4) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

As $\angle ACB > \angle ACP = 30°$, we get

$$\angle CAB = \angle CBA < (180° - 30°)/2 = 75°.$$

Hence $\angle CAB \le 74°$. Then

$$\angle CPB = \angle CAB + \angle ACP \le 74° + 30° = 104°.$$

Now

$$\angle QPB = 360° - \angle QPC - \angle CPB \ge 360° - 78° - 104° = 178°.$$

Since the angles of triangle QPB are positive integers, we must have

$$\angle QPB = 178°,\ \angle PBQ = 1° = \angle PQB$$

and all less-than-or-equal signs must be equalities so that

$$\angle CAB = \angle CBA = 74° \text{ and } \angle ACB = 32°.$$

Other commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 4), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3), **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 6) and **YIP Kai Shing** (STFA Leung Kau Kui College, Form 4).

Problem 199. Let R^+ denote the positive real numbers. Suppose $f: R^+ \to R^+$ is a strictly decreasing function such that for all $x, y \in R^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$ for every $x > 0$. (*Source: 1997 Iranian Math Olympiad*)

Solution. **Johann Peter Gustav Lejeune DIRICHLET** (Universidade de Sao Paulo – Campus Sao Carlos) and **Achilleas P. PORFYRIADIS** (American College of

Thessaloniki "Anatolia", Thessaloniki, Greece).

Setting $y = x$ gives

$$f(2x) + f(2f(x)) = f(2f(x + f(x))).$$

Setting both x and y to $f(x)$ in the given equation gives

$$\begin{aligned} &f(2f(x)) + f(2f(f(x))) \\ &= f(2f(f(x) + f(f(x)))). \end{aligned}$$

Subtracting this equation from the one above gives

$$\begin{aligned} f(2f(f(x))) - f(2x) = &f(2f(f(x) + f(f(x)))) \\ &- f(2f(x + f(x))). \end{aligned}$$

Assume $f(f(x)) > x$. Then $2f(f(x)) > 2x$. Since f is strictly decreasing , we have $f(2f(f(x))) < f(2x)$. This implies the left side of the last displayed equation is negative. Hence,

$$f(2f(f(x) + f(f(x)))) < f(2f(x + f(x))).$$

Again using f strictly decreasing, this inequality implies

$$2f(f(x) + f(f(x))) > 2f(x + f(x)),$$

which further implies

$$f(x) + f(f(x)) < x + f(x).$$

Canceling $f(x)$ from both sides leads to the contradiction that $f(f(x)) < x$.

Similarly, $f(f(x)) < x$ would also lead to a contradiction as can be seen by reversing all inequality signs above. Therefore, we must have $f(f(x)) = x$.

Problem 200. Aladdin walked all over the equator in such a way that each moment he either was moving to the west or was moving to the east or applied some magic trick to get to the opposite point of the Earth. We know that he travelled a total distance less than half of the length of the equator altogether during his westward moves. Prove that there was a moment when the difference between the distances he had covered moving to the east and moving to the west was at least half of the length of the equator. (*Source: KöMaL F. 3214*)

Solution.

Let us abbreviate Aladdin by A. At every moment let us consider a twin, say $\forall$, of A located at the opposite point of the position of A. *Now draw the equator circle.* Observe that at every moment either both are moving east or both are moving west. Combining the movement swept out by A and $\forall$, we get two continuous paths on the equator. At the same moment, each point in one path will have its opposite point in the other path.

Let N be the initial point of A in his travel and let $P(N)$ denote the path beginning with N. Let W be the westernmost point on $P(N)$. Let N' and W' be the opposite points of N and W respectively. By the westward travel condition on A, W cannot be as far as N'.

Assume the conclusion of the problem is false. Then the easternmost point reached by $P(N)$ cannot be as far as N'. So $P(N)$ will not cover the inside of minor arc WN' and the other path will not cover the inside of minor arc $W'N$. Since A have walked over all points of the equator (and hence A and $\forall$ together walked every point at least twice), $P(N)$ must have covered every point of the minor arc $W'N$ at least twice. Since $P(N)$ cannot cover the entire equator, every point of minor arc $W'N$ must be traveled westward at least once by A or $\forall$. Then A travelled westward at least a distance equal to the sum of lengths of minor arcs $W'N$ and NW, i.e. half of the equator. We got a contradiction.

Other commended solvers: **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same color.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

Problem 4. For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Problem 5. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

for all real numbers $a, b, c > 0$.

Inversion

(continued from page 2)

***Example* 5.** (*1995 Russian Math Olympiad*) Given a semicircle with diameter AB and center O and a line, which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$). Let K be the second point of intersection of the circumcircles of triangles AOC and DOB. Prove that $\angle MKO = 90^\circ$.

Solution. Consider the inversion with center O and radius $r = OA$. By fact (2), A, B, C, D are sent to themselves. By fact (4), the circle through A, O, C is sent to line AC and the circle through D, O, B is sent to line DB. Hence, the point K is sent to the intersection K' of lines AC with DB and the point M is sent to the intersection M' of line AB with the circumcircle of ΔOCD. Then the line MK is sent to the circumcircle of $OM'K'$.

To solve the problem, note by fact (7), $\angle MKO$=90° if and only if $\angle K'M'O$= 90°.

Since $BC \perp AK'$, $AD \perp BK'$ and O is the midpoint of AB, so the circumcircle of ΔOCD is the nine-point circle of $\Delta ABK'$, which intersects side AB again at the foot of perpendicular from K' to AB. This point is M'. So $\angle K'M'O = 90^\circ$ and we are done.

***Example* 6.** (*1995 Iranian Math Olympiad*) Let M, N and P be points of intersection of the incircle of triangle ABC with sides AB, BC and CA respectively. Prove that the orthocenter of ΔMNP, the incenter of ΔABC and the circumcenter of ΔABC are collinear.

Solution. Note the incircle of ΔABC is the circumcircle of ΔMNP. So the first two points are on the Euler line of ΔMNP.

Consider inversion with respect to the incircle of ΔABC with center I. By fact (2), A, B, C are sent to the midpoints A', B', C' of PM, MN, NP, respectively. The circumcenter of $\Delta A'B'C'$ is the center of the nine point circle of ΔMNP, which is on the Euler line of ΔMNP. By fact (3), the circumcircle of ΔABC is also on the Euler line of ΔMNP.

Mathematical Excalibur

Volume 9, Number 3 *August 2004 – September 2004*

Olympiad Corner

The 45th International Mathematical Olympiad took place on July 2004. Here are the problems.

Day 1 Time allowed: 4 hours 30 minutes.

Problem 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of the side BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a\text{-}b)+P(b\text{-}c)+P(c\text{-}a) = 2P(a+b+c)$$

for all real numbers a, b, c such that $ab + bc + ca = 0$.

Problem 3. Define a *hook* to be a figure made up of six unit squares as shown in the diagram

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 20, 2004***.

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IMO 2004

T. W. Leung

The 45th International Mathematical Olympiad (IMO) was held in Greece from July 4 to July 18. Since 1988, we have been participating in the Olympiads. This year our team was composed as follows.

<u>*Members*</u>

Cheung Yun Kuen (Hong Kong Chinese Women's Club College)

Chung Tat Chi (Queen Elizabeth School)

Kwok Tsz Chiu (Yuen Long Merchant Association Secondary School)

Poon Ming Fung (STFA Leung Kau Kui College)

Tang Chiu Fai (HKTA Tang Hin Memorial Secondary School)

Wong Hon Yin (Queen's College)

Cesar Jose Alaban (Deputy Leader)

Leung Tat Wing (Leader)

I arrived at Athens on July 6. After waiting for a couple of hours, leaders were then delivered to Delphi, a hilly town 170 km from the airport, corresponding to 3 more hours of journey. In these days the Greeks were still ecstatic about what they had achieved in the Euro 2004, and were busy preparing for the coming Olympic Games in August. Of course Greece is a small country full of legend and mythology. Throughout the trip, I also heard many times that they were the originators of democracy, their contribution in the development of human body and mind and their emphasis on fair play.

After receiving the short-listed problems leaders were busy studying them on the night of July 8. However obviously some leaders had strong opinions on the beauty and degree of difficulty of the problems, so selections of all six problems were done in one day. Several problems were not even discussed in details of their own merits. The following days were spent on refining the wordings of the questions and translating the problems into different languages.

The opening ceremony was held on July 11. In the early afternoon we were delivered to Athens. After three hours of ceremony we were sent back to Delphi. By the time we were in Delphi it was already midnight. Leaders were not allowed to talk to students in the ceremony.

Contests were held in the next two days. The days following the contests were spent on coordination, i.e. leaders and coordinators discussed how many points should be awarded to the answers of the students. This year the coordinators were in general very careful. I heard several teams spent more than three hours to go over six questions. Luckily coordination was completed on the afternoon of July 15. The final Jury meeting was held that night. In the meeting the cut-off scores were decided, namely 32 points for gold, 24 for silver and 16 for bronze. Our team was therefore able to obtain two silver medals (Kwok and Chung) and two bronze medals (Tang and Cheung). Other members (Poon and Wong) both solved at least one problem completely, thus received honorable mention. Unofficially our team ranked 30 out of 85. The top five teams in order were respectively China, USA, Russia, Vietnam and Bulgaria.

In retrospect I felt that our team was good and balanced, none of the members was particularly weak. In one problem we were as good as any strong team. Every team members solved problem 4 completely. Should we did better in the geometry problems our rank would be much higher. Curiously geometry is in our formal school curriculum while number theory and combinatorics are not. In this Olympiad we had two geometry problems, but fittingly so, after all, it was Greece.

Extending an IMO Problem

Hà Duy Hung

Dept. of Math and Informatics
Hanoi Univ. of Education

In this brief note we give a generalization of a problem in the 41st International Mathematical Olympiad held in Taejon, South Korea in 2000.

IMO 2000/5. Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime divisors, and 2^n+1 is divisible by n.

The answer to the question is positive. This intriguing problem made me recall a well-known theorem due to O. Reutter in [1] as follows.

Theorem 1. If a is a positive integer such that $a+1$ is not a power of 2, then a^n+1 is divisible by n for infinitely many positive integers n.

We frequently encounter the theorem in the case $a=2$. The theorem and the IMO problem prompted me to think of more general problem. Can we replace the number 2 in the IMO problem by other positive integers? The difficulty partly lies in the fact that the two original problems are solved independently. After a long time, I finally managed to prove a generalization as follows.

Theorem 2. Let s, a, b be given positive integers, such that a, b are relatively prime and $a+b$ is not a power of 2. Then there exist infinitely many positive integers n such that

- n has exactly s different prime divisors; and
- a^n+b^n is divisible by n.

We give a proof of Theorem 2 below. We shall make use of two familiar lemmas.

Lemma 1. Let n be an odd positive integer, and a, b be relative prime positive integers. Then

$$\frac{a^n+b^n}{a+b}$$

is an odd integer ≥ 1, equality if and only if $n=1$ or $a=b=1$.

The proof of Lemma 1 is simple and is left for the reader.

Also, we remind readers the usual notations $r \mid s$ means s is divisible by r and $u \equiv v \pmod{m}$ means $u-v$ is divisible by m.

Lemma 2. Let a, b be distinct and relatively prime positive integers, and p an odd prime number which divides $a+b$. Then for any non-negative integer k,

$$p^{k+1} \mid a^m+b^m,$$

where $m=p^k$.

Proof. We prove the lemma by induction. It is clear that the lemma holds for $k=0$. Suppose the lemma holds for some non-negative integer k, and we proceed to the case $k+1$.

Let $x=a^{p^k}$ and $y=b^{p^k}$. Since

$$x^p+y^p=(x+y)\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}y^i,$$

it suffices to show that the whole summation is divisible by p. Since $x\equiv -y \pmod{p^{k+1}}$, we have

$$\begin{aligned}&\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}y^i\\ &\equiv \sum_{i=0}^{p-1}(-1)^{2i}x^{p-1}\\ &\equiv px^{p-1} \qquad \pmod{p^{k+1}}\end{aligned}$$

completing the proof.

In the rest of this note we shall complete the proof of Theorem 2.

Proof of Theorem 2. Without loss of generality, let $a>b$. Since $a+b$ is not a power of 2, it has an odd prime factor p. For natural number k, set

$$x_k=a^{p^k}+b^{p^k},\ y_k=\frac{x_{k+1}}{x_k}.$$

Then y_k is a positive integer and

$$\begin{aligned}y_k&=\sum_{i=0}^{p-1}(-1)^i(a^{p^k})^{p-1-i}(b^{p^k})^i\\ &\equiv \sum_{i=0}^{p-1}(-1)^{2i}(a^{p^k})^{p-1}\\ &\equiv px^{p-1} \qquad \pmod{p^{k+1}}\end{aligned}$$

which implies that $\frac{y_k}{p}$ is a positive integer. Also, we have

$$\frac{y_k}{p}\equiv b^{p^k(p-1)} \quad \left(\bmod \frac{x_k}{p}\right),$$

so that

$$\gcd\left(\frac{x_k}{p},\frac{y_k}{p}\right)=1$$

for $k=1, 2, \ldots$ By Lemma 2, we also have

$$\gcd\left(\frac{y_k}{p},p^k\right)=1$$

for $k=1, 2, \ldots$. Moreover, we have $x_k\geq p^k$. This leads us to

$$\begin{aligned}y_k&=b^{p^k(p-1)}+\sum_{i=1}^{\frac{p-1}{2}}[(a^{p^k})^{2i}(b^{p^k})^{p-1-2i}\\ &\qquad -(a^{p^k})^{2i-1}(b^{p^k})^{p-2i}]\\ &>b^{p^k}+a^{p^k}\\ &=x_k\\ &\geq p^{k+1}\end{aligned}$$

It follows that

$$\frac{y_k}{p}\geq p^k>1.$$

By Lemma 1, $\frac{y_k}{p}$ is an odd positive integer, so we can choose an odd prime divisor q_k of $\frac{y_k}{p}$.

We now have a sequence of odd prime numbers $\{q_k\}_{k=1}^{+\infty}$ satisfying the following properties

(i) $\gcd(x_k,q_k)=1$

(ii) $\gcd(p,q_k)=1$

(iii) $q_k \mid x_{k+1}$

(iv) $x_k \mid x_{k+1}$.

We shall now show that the sequence $\{q_k\}_{k=1}^{+\infty}$ consists of distinct prime numbers and is thus infinite. Indeed, if $k_0<k_1$ are positive integers and $q_{k_0}=q_{k_1}$, then

$$q_{k_1}=q_{k_0}\mid x_{k_0+1}\mid\cdots\mid x_{k_1}$$

by properties (iii) and (iv). But this contradicts property (i).

Next, set $n_0=p^s q_1\ldots q_{s-1}$ and $n_{k+1}=pn_k$ for $k=0, 1, 2, \ldots$. It is evident that $\{n_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***October 20, 2004.***

Problem 206. (*Due to Zdravko F. Starc, Vršac, Serbia and Montenegro*) Prove that if a, b are the legs and c is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a}+(a-b)\sqrt{b}<\sqrt{2\sqrt{2}c\sqrt{c}}.$$

Problem 207. Let $A = \{ 0, 1, 2, ..., 9\}$ and $B_1, B_2, ..., B_k$ be nonempty subsets of A such that B_i and B_j have at most 2 common elements whenever $i \neq j$. Find the maximum possible value of k.

Problem 208. In ΔABC, $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle of ΔABC. Let O be the circumcenter of ΔABC. Let E, F be the intersection points of line AD with the perpendiculars from O to AB, AC, respectively. Let P be the intersection of lines BE and CF. If $PB = PC + PO$, then find $\angle BAC$ with proof.

Problem 209. Prove that there are infinitely many positive integers n such that $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$.

Problem 210. Let $a_1 = 1$ and

$$a_{n+1}=\frac{a_n}{2}+\frac{1}{a_n}$$

for $n = 1, 2, 3, ...$. Prove that for every integer $n > 1$,

$$\frac{2}{\sqrt{a_n^2-2}}$$

is an integer.

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Solutions

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Problem 201. (*Due to Abderrahim OUARDINI, Talence, France*) Find which nonright triangles ABC satisfy

$$\tan A \tan B \tan C > [\tan A] + [\tan B] + [\tan C],$$

where $[t]$ denotes the greatest integer less than or equal to t. Give a proof.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4). , **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece),

From

$$\begin{aligned}\tan C &= \tan (180^\circ - A - B)\\ &= -\tan (A+B)\\ &= -(\tan A + \tan B)/(1-\tan A \tan B),\end{aligned}$$

we get

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Let $x = \tan A$, $y = \tan B$ and $z = \tan C$. If $xyz \le [x]+[y]+[z]$, then $x+y+z \le [x] + [y] + [z]$. As $[t] \le t$, x, y, z must be integers.

If triangle ABC is obtuse, say $A > 90^\circ$, then $x < 0 < 1 \le y \le z$. This implies $1 \le yz = (x + y + z)/x = 1 + (y + z)/x < 1$, a contradiction. If triangle ABC is acute, then we may assume $1 \le x \le y \le z$. Now $xy = (x+y+z)/z \le (3z)/z = 3$. Checking the cases $xy = 1, 2, 3$, we see $x+y+z = xyz$ can only happen when $x=1$, $y=2$ and $z=3$. This corresponds to $A = \tan^{-1} 1$, $B = \tan^{-1} 2$ and $C = \tan^{-1} 3$. Reversing the steps, we see among nonright triangles, the inequality in the problem holds except only for triangles with angles equal $45^\circ = \tan^{-1} 1$, $\tan^{-1} 2$ and $\tan^{-1} 3$.

Problem 202. (*Due to LUK Mee Lin, La Salle College*) For triangle ABC, let D, E, F be the midpoints of sides AB, BC, CA, respectively. Determine which triangles ABC have the property that triangles ADF, BED, CFE can be folded above the plane of triangle DEF to form a tetrahedron with AD coincides with BD; BE coincides with CE; CF coincides with AF.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Observe that $ADEF$, $BEFD$ and $CFDE$ are parallelograms. Hence $\angle BDE = \angle BAC$, $\angle ADF = \angle ABC$ and $\angle EDF = \angle BCA$. In order for AD to coincide with BD in folding, we need to have $\angle BDE + \angle ADF > \angle EDF$. So we need $\angle BAC + \angle ABC > \angle BCA$. Similarly, for BE to coincide with CE and for CF to coincide with AF, we need $\angle ABC + \angle BCA > \angle BAC$ and $\angle BCA + \angle BAC > \angle ABC$. So no angle of ΔABC is 90 or more. Therefore, ΔABC is acute.

Conversely, if ΔABC is acute, then reversing the steps, we can see that the required tetrahedron can be obtained.

Problem 203. (*Due to José Luis DÍAZ-BARRERO, Universitat Politecnica de Catalunya, Barcelona, Spain*) Let a, b and c be real numbers such that $a + b + c \neq 0$. Prove that the equation

$$(a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0$$

has only real roots.

Solution. **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 6), **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Hoi Kit** (SKH Lam Kau Mow Secondary School, Form 7), **CHEUNG Yun Kuen** (HKUST, Math, Year 1), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

The quadratic has real roots if and only if its discriminant

$$\begin{aligned}D &= 4(ab+bc+ca)^2 - 12(a+b+c)abc\\ &= 4[(ab)^2+(bc)^2+(ca)^2-(a+b+c)abc]\\ &= 4[(ab-bc)^2 +(bc-ca)^2 +(ca-ab)^2]\end{aligned}$$

is nonnegative, which is clear.

Other commended solvers: **Jason CHENG Hoi Sing** (SKH Lam Kau Mow Secondary School, Form 7), **POON Ho Yin** (Munsang College (Hong Kong Island), Form 4) and **Anderson TORRES** (Universidade de Sao Paulo – Campus Sao Carlos).

Problem 204. Let n be an integer with $n > 4$. Prove that for every n distinct integers taken from 1, 2, …, $2n$, there always exist two numbers whose least common multiple is at most $3n + 6$.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Let S be the set of n integers taken and k be the minimum of these integers. If $k \le n$, then either $2k$ is also in S or $2k$ is not in S. In the former case, $\text{lcm}(k,2k) = 2k \le 2n < 3n+6$. In the latter case, we replace k in S by $2k$. Note this will not decrease the least common multiple of any pair of numbers. So if the new S satisfies the problem, then the original S will also satisfy the problem. As we repeat this, the new minimum will increase strictly so that we eventually reach either k and $2k$ both in S, in which case we are done, or the new S will consist of $n+1$, $n+2$, …, $2n$. So we need to consider the latter case only.

If $n > 4$ is even, then $3(n+2)/2$ is an integer at most $2n$ and lcm$(n+2, 3(n+2)/2) = 3n+6$. If $n > 4$ is odd, then $3(n+1)/2$ is an integer at most $2n$ and lcm$(n+1, 3(n+1)/2) = 3n+3$.

Problem 205. *(Due to HA Duy Hung, Hanoi University of Education, Vietnam)* Let a, n be integers, both greater than 1, such that $a^n - 1$ is divisible by n. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and n is greater than 1.

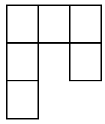
Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Let p be the smallest prime divisor of n. Then $a^n - 1$ is divisibly by p so that $a^n \equiv 1 \pmod p$. In particular, a is not divisible by p. Then, by Fermat's little theorem, $a^{p-1} \equiv 1 \pmod p$.

Let d be the smallest positive integer such that $a^d \equiv 1 \pmod p$. Dividing n by d, we get $n = dq + r$ for some integers q, r with $0 \le r < d$. Then $a^r \equiv (a^d)^q a^r = a^n \equiv 1 \pmod p$. By the definition of d, we get $r = 0$. Then n is divisible by d. Similarly, dividing $p - 1$ by d, we see $a^{p-1} \equiv 1 \pmod p$ implies $p - 1$ is divisible by d. Hence, gcd$(n, p - 1)$ is divisible by d. Since p is the smallest prime dividing n, we must have gcd$(n, p - 1) = 1$. So $d = 1$. By the definition of d, we get $a - 1$ is divisible by p. Therefore, gcd$(a - 1, n) \ge p > 1$.

Olympiad Corner

(continued from page 1)



or any of the figures obtained by applying rotations and reflections to this figure.

Determine all $m \times n$ rectangles that can be covered with hooks so that

- the rectangle is covered without gaps and without overlaps;
- no part of a hook covers area outside the rectangle.

Day 2 Time allowed: 4 hours 30 minutes.

Problem 4. Let $n \ge 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \times \left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \le i < j < k \le n$.

Problem 5. In a convex quadrilateral $ABCD$ the diagonal BD bisects neither the angle ABC not the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

Extending an IMO Problem

(continued from page 2)

of positive integers and each term of the sequence has exactly s distinct prime divisors.

It remains to show that

$$n_k \mid a^{n_k} + b^{n_k}$$

for $k = 0, 1, 2, \ldots$ Note that for odd positive integers m, n with $m \mid n$, we have $a^m + b^m \mid a^n + b^n$. By property (iii), we have, for $0 \le k < s$,

$$q_k \mid x_{k+1} \mid x_s \mid a^{n_0} + b^{n_0} \mid a^{n_j} + b^{n_j}$$

for $j = 0, 1, 2, \ldots$. Now it suffices to show that

$$p^{k+s} \mid a^{n_k} + b^{n_k}$$

for $k = 0, 1, 2, \ldots$. But this follows easily from Lemma 2 since

$$p^{s+k} \mid x_{k+s} \mid a^{n_k} + b^{n_k}.$$

This completes the proof of Theorem 2.

References:

[1] O. Reutter, *Elemente der Math.*, 18 (1963), 89.

[2] W. Sierpinski, *Elementary Theory of Numbers*, English translation, Warsaw, 1964.

2004 Hong Kong team to IMO: From left to right, Cheung Yun Kuen, Poon Ming Fung, Tang Chiu Fai, Cesar Jose Alaban (Deputy Leader), Leung Tat Wing (Leader), Chung Tat Chi, Kwok Tsz Chiu & Wong Hon Yin.

Mathematical Excalibur

Volume 9, Number 4 — *October 2004 – December 2004*

Olympiad Corner

The Czech-Slovak-Polish Match this year took place in Bilovec on June 21-22, 2004.Here are the problems.

Problem 1. Show that real numbers p, q, r satisfy the condition

$$p^4(q-r)^2 + 2p^2(q+r) + 1 = p^4$$

if and only if the quadratic equations

$$x^2 + px + q = 0 \text{ and } y^2 - py + r = 0$$

have real roots (not necessarily distinct) which can be labeled by x_1, x_2 and y_1, y_2, respectively, in such way that the equality $x_1y_1 - x_2y_2 = 1$ holds.

Problem 2. Show that for each natural number k there exist at most finitely many triples of mutually distinct primes p, q, r for which the number $qr - k$ is a multiple of p, the number $pr - k$ is a multiple of q, and the number $pq - k$ is a multiple of r.

Problem 3. In the interior of a cyclic quadrilateral $ABCD$, a point P is given such that $|\angle BPC| = |\angle BAP| + |\angle PDC|$. Denote by E, F and G the feet of the perpendiculars from the point P to the lines AB, AD and DC, respectively. Show that the triangles FEG and PBC are similar.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 20, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Homothety

Kin Y. Li

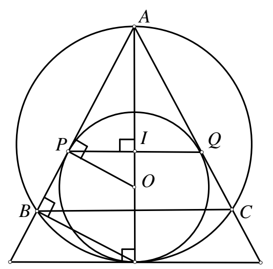
A <u>*geometric transformation*</u> of the plane is a function that sends every point on the plane to a point in the same plane. Here we will like to discuss one type of geometric transformations, called *homothety*, which can be used to solve quite a few geometry problems in some international math competitions.

A <u>*homothety with center O and ratio k*</u> is a function that sends every point X on the plane to the point X' such that

$$\overrightarrow{OX'} = k\overrightarrow{OX}.$$

So if $|k| > 1$, then the homothety is a magnification with center O. If $|k| < 1$, it is a reduction with center O. *A homothety sends a figure to a similar figure.* For instance, let D, E, F be the midpoints of sides BC, CA, AB respectively of ΔABC. The homothety with center A and ratio 2 sends ΔAFE to ΔABC. The homothety with center at the centroid G and ratio $-1/2$ sends ΔABC to ΔDEF.

<u>Example 1</u>. (*1978 IMO*) In ΔABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.

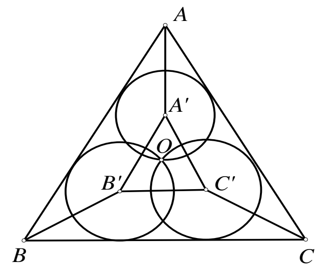


Solution. Let O be the center of the circle. Let the circle be tangent to the circumcircle of ΔABC at D. Let I be the midpoint of PQ. Then A, I, O, D are collinear by symmetry. Consider the homothety with center A that sends ΔABC to $\Delta AB'C'$ such that D is on $B'C'$. Thus, $k=AB'/AB$. As right triangles AIP, ADB', ABD, APO are similar, we have

$$AI/AO = (AI/AP)(AP/AO) = (AD/AB')(AB/AD) = AB/AB' = 1/k.$$

Hence the homothety sends I to O. Then O being the incenter of $\Delta AB'C'$ implies I is the incenter of ΔABC.

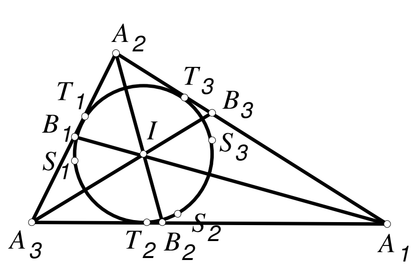
<u>Example 2</u>. (*1981 IMO*) Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.



Solution. Consider the figure shown. Let A', B', C' be the centers of the circles. Since the radii are the same, so $A'B'$ is parallel to AB, $B'C'$ is parallel to BC, $C'A'$ is parallel to CA. Since AA', BB' CC' bisect $\angle A$, $\angle B$, $\angle C$ respectively, they concur at the incenter I of ΔABC. Note O is the circumcenter of $\Delta A'B'C'$ as it is equidistant from A', B', C'. Then the homothety with center I sending $\Delta A'B'C'$ to ΔABC will send O to the circumcenter P of ΔABC. Therefore, I, O, P are collinear.

<u>Example 3</u>. (*1982 IMO*) A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all i=1, 2, 3, M_i is the midpoint of side a_i, and T_i is the point where the incircle touchs side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i.

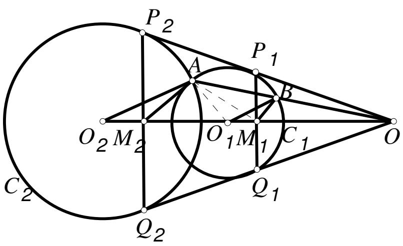
Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.



Solution. Let I be the incenter of $\Delta A_1A_2A_3$. Let B_1, B_2, B_3 be the points where the internal angle bisectors of $\angle A_1$, $\angle A_2$, $\angle A_3$ meet a_1, a_2, a_3 respectively. We will show S_iS_j is parallel to M_iM_j. With respect to A_1B_1, the reflection of T_1 is S_1 and the reflection of T_2 is T_3. So $\angle T_3IS_1 = \angle T_2IT_1$. With respect to A_2B_2, the reflection of T_2 is S_2 and the reflection of T_1 is S_3. So $\angle T_3IS_2 = \angle T_1IT_2$. Then $\angle T_3IS_1 = \angle T_3IS_2$. Since IT_3 is perpendicular to A_1A_2, we get S_2S_1 is parallel to A_1A_2. Since A_1A_2 is parallel to M_2M_1, we get S_2S_1 is parallel to M_2M_1. Similarly, S_3S_2 is parallel to M_3M_2 and S_1S_3 is parallel to M_1M_3.

Now the circumcircle of $\Delta S_1S_2S_3$ is the incircle of $\Delta A_1A_2A_3$ and the circumcircle of $\Delta M_1M_2M_3$ is the nine point circle of $\Delta A_1A_2A_3$. Since $\Delta A_1A_2A_3$ is not equilateral, these circles have different radii. Hence $\Delta S_1S_2S_3$ is not congruent to $\Delta M_1M_2M_3$ and there is a homothety sending $\Delta S_1S_2S_3$ to $\Delta M_1M_2M_3$. Then M_1S_1, M_2S_2 and M_3S_3 concur at the center of the homothety.

Example 4. (*1983 IMO*) Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.



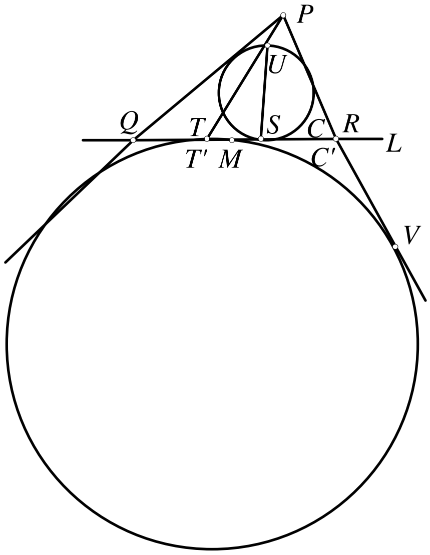
Solution. By symmetry, lines O_2O_1, P_2P_1, Q_2Q_1 concur at a point O. Consider the homothety with center O which sends C_1 to C_2. Let OA meet C_1 at B, then A is the image of B under the homothety. Since ΔBM_1O_1 is sent to ΔAM_2O_2, so $\angle M_1BO_1 = \angle M_2AO_2$.

Now ΔOP_1O_1 similar to ΔOM_1P_1 implies $OO_1/OP_1 = OP_1/OM_1$. Then

$$OO_1 \cdot OM_1 = OP_1^2 = OA \cdot OB,$$

which implies points A, B, M_1, O_1 are concyclic. Then $\angle M_1BO_1 = \angle M_1AO_1$. Hence $\angle M_1AO_1 = \angle M_2AO_2$. Adding $\angle O_1AM_2$ to both sides, we have $\angle O_1AO_2 = \angle M_1AM_2$.

Example 5. (*1992 IMO*) In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of ΔPQR.



Solution. Let L be the tangent to C at S. Let T be the reflection of S with respect to M. Let U be the point on C diametrically opposite S. Take a point P on the locus. The homothety with center P that sends C to the excircle C' will send U to T', the point where QR touches C'. Let line PR touch C' at V. Let s be the semiperimeter of ΔPQR, then

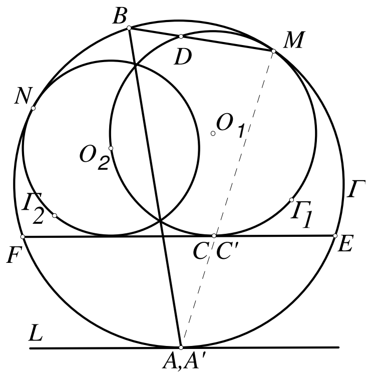
$$TR = QS = s - PR = PV - PR = VR = T'R$$

so that P, U, T are collinear. Then the locus is on the part of line UT, opposite the ray $\overrightarrow{UT}$.

Conversely, for any point P on the part of line UT, opposite the ray $\overrightarrow{UT}$, the homothety sends U to T and T', so $T = T'$. Then $QS = s - PR = PV - PR = VR = T'R = TR$ and $QM = QS - MS = TR - MT = RM$. Therefore, P is on the locus.

For the next example, the solution involves the concepts of power of a point with respect to a circle and the radical axis. We will refer the reader to the article "*Power of Points Respect to Circles*," in *Math Excalibur*, *vol.* 4, *no.* 3, *pp.* 2, 4.

Example 6. (*1999 IMO*) Two circles Γ_1 and Γ_2 are inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.



Solution. (*Official Solution*) Let EF be the chord of Γ which is the common tangent to Γ_1 and Γ_2 on the same side of line O_1O_2 as A. Let EF touch Γ_1 at C'. The homothety with center M that sends Γ_1 to Γ will send C' to some point A' and line EF to the tangent line L of Γ at A'. Since lines EF and L are parallel, A' must be the midpoint of arc $FA'E$. Then $\angle A'EC' = \angle A'FC' = \angle A'ME$. So $\Delta A'EC$ is similar to $\Delta A'ME$. Then the power of A' with respect to Γ_1 is $A'C' \cdot A'M = A'E^2$. Similar, the power of A' with respect to Γ_2 is $A'F^2$. Since $A'E = A'F$, A' has the same power with respect to Γ_1 and Γ_2. So A' is on the radical axis AB. Hence, $A' = A$. Then $C' = C$ and C is on EF.

Similarly, the other common tangent to Γ_1 and Γ_2 passes through D. Let O_i be the center of Γ_i. By symmetry with respect to O_1O_2, we see that O_2 is the midpoint of arc CO_2D. Then

$$\angle DCO_2 = \angle CDO_2 = \angle FCO_2.$$

This implies O_2 is on the angle bisector of $\angle FCD$. Since CF is tangent to Γ_2, therefore CD is tangent to Γ_2.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***January 20, 2005.***

Problem 211. For every a, b, c, d in [1,2], prove that

$$\frac{a+b}{b+c}+\frac{c+d}{d+a}\le 4\,\frac{a+c}{b+d}.$$

(*Source: 32nd Ukranian Math Olympiad*)

Problem 212. Find the largest positive integer N such that if S is any set of 21 points on a circle C, then there exist N arcs of C whose endpoints lie in S and each of the arcs has measure not exceeding 120°.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99\,b = c$, then at least two of them belong to the same subset.

Problem 214. Let the inscribed circle of triangle ABC be tangent to sides AB, BC at E and F respectively. Let the angle bisector of $\angle CAB$ intersect segment EF at K. Prove that $\angle CKA$ is a right angle.

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 3×1 rectangles.

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Solutions

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Problem 206. (*Due to Zdravko F. Starc, Vršac, Serbia and Montenegro*) Prove that if a, b are the legs and c is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a}+(a-b)\sqrt{b}<\sqrt{2\sqrt{2}}\,c\sqrt{c}.$$

Solution. **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Problem Group Discussion Euler-Teorema**(Fortaleza, Brazil), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **TO Ping Leung** (St. Peter's Secondary School) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

By Pythagoras' theorem,

$$a+b\le\sqrt{(a+b)^2+(a-b)^2}=\sqrt{2}c.$$

Equality if and only if $a = b$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&(a+b)\sqrt{a}+(a-b)\sqrt{b}\\ \le\ &\sqrt{(a+b)^2+(a-b)^2}\sqrt{a+b}\\ \le\ &\sqrt{2}c\sqrt{\sqrt{2}c}.\end{aligned}$$

For equality to hold throughout, we need $a+b : a-b = \sqrt{a} : \sqrt{b} = 1:1$, which is not possible for legs of a triangle. So we must have strict inequality.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **TONG Yiu Wai** (Queen Elizabeth School, Form 7).

Problem 207. Let A = { 0, 1, 2, ..., 9} and B_1, B_2, ..., B_k be nonempty subsets of A such that B_i and B_j have at most 2 common elements whenever $i \ne j$. Find the maximum possible value of k.

Solution. **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

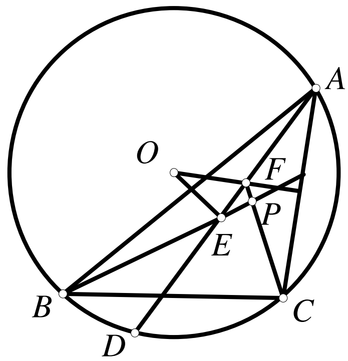
If we take all subsets of A with 1, 2 or 3 elements, then these 10 + 45 + 120 = 175 subsets satisfy the condition. So $k \ge 175$.

Let B_1, B_2, ..., B_k satisfying the condition with k maximum. If there exists a B_i with at least 4 elements, then every 3 element subset of B_i cannot be one of the B_j, $j \ne i$, since B_i and B_j can have at most 2 common elements. So adding these 3 element subsets to B_1, B_2, ..., B_k will still satisfy the conditions. Since B_i has at least four 3 element subsets, this will increase k, which contradicts maximality of k. Then every B_i has at most 3 elements. Hence, $k \le 175$. Therefore, the maximum k is 175.

Other commended solvers: **CHAN Wai Hung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LI Sai Ki** (Carmel Divine Grace Foundation Secondary School, Form 6), **LING Shu Dung**, **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 208. In ΔABC, $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle of ΔABC. Let O be the circumcenter of ΔABC. Let E, F be the intersection points of line AD with the perpendiculars from O to AB, AC, respectively. Let P be the intersection of lines BE and CF. If $PB = PC + PO$, then find $\angle BAC$ with proof.

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Problem Group Discussion Euler - Teorema** (Fortaleza, Brazil) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).



Since E is on the perpendicular bisector of chord AB and F is on the perpendicular bisector of chord AC, $AE = BE$ and $AF = CF$. Applying exterior angle theorem,

$$\begin{aligned}\angle BPC &= \angle AEP + \angle CFD\\ &= 2\,(\angle BAD + \angle CAD)\\ &= 2\angle BAC = \angle BOC.\end{aligned}$$

Hence, B, C, P, O are concyclic. By Ptolemy's theorem,

$$PB\cdot OC = PC\cdot OB + PO\cdot BC.$$

Then $(PB - PC)\cdot OC = PO\cdot BC$. Since $PB - PC = PO$, we get $OC = BC$ and so ΔOBC is equilateral. Then

$$\angle BAC = \tfrac{1}{2}\,\angle BOC = 30°$$

Other commended solvers: **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **TONG Yiu Wai**

(Queen Elizabeth School, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 209. Prove that there are infinitely many positive integers n such that $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$.

Solution. **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7) and **Problem Group Discussion Euler-Teorema**(Fortaleza, Brazil).

As $2^2 + 2 = 6$ is divisible by 2 and $2^2 + 1 = 5$ is divisible by 1, $n = 2$ is one such number.

Next, suppose $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$. We will prove $N = 2^n + 2$ is another such number. Since $N - 1 = 2^n + 1= (n - 1)k$ is odd, so k is odd and n is even. Since $N = 2^n + 2 = 2(2^{n-1} + 1) = nm$ and n is even, so m must be odd. Recall the factorization

$$x^i + 1 = (x + 1)(x^{i-1} - x^{i-3} + \cdots + 1)$$

for odd positive integer i. Since k is odd, $2^N + 2 = 2(2^{N-1} + 1) = 2(2^{(n-1)k} + 1)$ is divisible by $2(2^{n-1} + 1) = 2^n + 2 = N$ using the factorization above. Since m is odd, $2^N + 1 = 2^{nm} + 1$ is divisible by $2^n + 1 = N - 1$. Hence, N is also such a number. As $N > n$, there will be infinitely many such numbers.

Problem 210. Let $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

for $n = 1, 2, 3, \ldots$. Prove that for every integer $n > 1$,

$$\frac{2}{\sqrt{a_n^2 - 2}}$$

is an integer.

Solution. **G.R.A. 20 Problem Group** (Roma, Italy), **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **Problem Group Discussion Euler – Teorema** (Fortaleza, Brazil), **TO Ping Leung** (St. Peter's Secondary School) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Note $a_n = p_n / q_n$, where $p_1 = q_1 = 1$, $p_{n+1} = p_n^2 + 2q_n^2$, $q_{n+1} = 2p_nq_n$ for $n = 1, 2, 3, \ldots$. Then

$$\frac{2}{\sqrt{a_n^2 - 2}} = \frac{2q_n}{\sqrt{p_n^2 - 2q_n^2}}.$$

It suffices to show by mathematical induction that $p_n^2 - 2q_n^2 = 1$ for $n > 1$. We have $p_2^2 - 2q_2^2 = 3^2 - 2 \cdot 2^2 = 1$. Assuming case n is true, we get

$$p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 2(2p_nq_n)$$
$$= (p_n^2 - 2q_n^2)^2 = 1.$$

Other commended solvers: **Ellen CHAN On Ting** (True Light Girls' College, Form 5), **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **LAW Yau Pui** (Carmel Divine Grace Foundation Secondary School, Form 6), **Asger OLESEN** (Toender Gymnasium (grammar school), Denmark), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **Steve ROFFE**, **TONG Yiu Wai** (Queen Elizabeth School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 4).

Olympiad Corner

(continued from page 1)

Problem 4. Solve the system of equations

$$\frac{1}{xy} = \frac{x}{z} + 1,\ \frac{1}{yz} = \frac{y}{x} + 1,\ \frac{1}{zx} = \frac{z}{y} + 1$$

in the domain of real numbers.

Problem 5. In the interiors of the sides AB, BC and CA of a given triangle ABC, points K, L and M, respectively, are given such that

$$\frac{|AK|}{|KB|} = \frac{|BL|}{|LC|} = \frac{|CM|}{|MA|}.$$

Show that the triangles ABC and KLM have a common orthocenter if and only if the triangle ABC is equilateral.

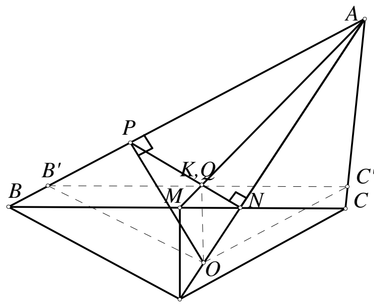
Problem 6. On the table there are k heaps of 1, 2, …, k stones, where $k \geq 3$. In the first step, we choose any three of the heaps on the table, merge them into a single new heap, and remove 1 stone (throw it away from the table) from this new heap. In the second step, we again merge some three of the heaps together into a single new heap, and then remove 2 stones from this new heap. In general, in the i-th step we choose any three of the heaps, which contain more than i stones when combined, we merge them into a single new heap, then remove i stones from this new heap. Assume that after a number of steps, there is a single heap left on the table, containing p stones. Show that the number p is a perfect square if and only if the numbers $2k+2$ and $3k+1$ are perfect squares. Further, find the least number k for which p is a perfect square.

Homothety

(continued from page 2)

Example 7. (*2000APMO*) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA respectively and O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.



Solution (*due to Bobby Poon*). The case $AB = AC$ is clear.

Without loss of generality, we may assume $AB > AC$. Let AN intersect the circumcircle of ΔABC at D. Then

$$\angle DBC = \angle DAC = \tfrac{1}{2}\angle BAC$$
$$= \angle DAB = \angle DCB.$$

So $DB = DC$ and MD is perpendicular to BC.

Consider the homothety with center A that sends ΔDBC to $\Delta OB'C'$. Then $OB' = OC'$ and BC is parallel to $B'C'$. Let $B'C'$ intersect PN at K. Then

$$\angle OB'K = \angle DBC = \angle DAB$$
$$= 90° - \angle AOP = \angle OPK.$$

So points P, B', O, K are concyclic. Hence $\angle B'KO = \angle B'PO = 90°$ and $B'K = C'K$. Since $BC \parallel B'C'$, this implies K is on MA. Hence, $K = Q$. Now $\angle B'KO = 90°$ implies $QO=KO \perp B'C'$. Finally, $BC \parallel B'C'$ implies QO is perpendicular to BC.

Mathematical Excalibur

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Olympiad Corner

The 7th China Hong Kong Math Olympiad took place on December 4, 2004. Here are the problems.

Problem 1. For $n \ge 2$, let $a_1, a_2, \ldots, a_n, a_{n+1}$ be positive and $a_2 - a_1 = a_3 - a_2 = \cdots = a_{n+1} - a_n \ge 0$. Prove that

$$\frac{1}{a_2^2} + \frac{1}{a_3^2} + \cdots + \frac{1}{a_n^2} \le \frac{n-1}{2} \cdot \frac{a_1 a_n + a_2 a_{n+1}}{a_1 a_2 a_n a_{n+1}}.$$

Determine when equality holds.

Problem 2. In a school there are b teachers and c students. Suppose that

(i) each teacher teaches exactly k students; and

(ii) for each pair of distinct students, exactly h teachers teach both of them.

Show that $\frac{b}{h} = \frac{c(c-1)}{k(k-1)}$.

Problem 3. On the sides AB and AC of triangle ABC, there are points P and Q respectively such that $\angle APC = \angle AQB = 45°$. Let the perpendicular line to side AB through P intersects line BQ at S. Let the perpendicular line to side AC through Q intersects line CP at R. Let D be on side BC such that $AD \perp BC$.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 31, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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例析數學競賽中的計數問題 (一)

費振鵬 （江蘇省鹽城市城區永豐中學 224054）

組合數學中的計數問題，數學競賽題中的熟面孔，看似司空見慣，不足爲奇．很多同學認爲只要憑藉單純的課內知識就可左右逢源，迎刃而解．其實具體解題時，卻會使你挖空心思，也無所適從．對於這類問題往往首先要通過構造法描繪出對象的簡單數學模型，繼而借助在計數問題中常用的一些數學原理方可得出所求對象的總數或其範圍．

1 運用分類計數原理與分步計數原理

分類計數原理與分步計數原理（即加法原理與乘法原理）是關於計數的兩個基本原理，是解決競賽中計數問題的基礎．下面提出的三個問題，注意結合排列與組合的相關知識，構造出相應的模型再去分析求解．

例 1　已知兩個實數集合 $A=\{a_1,a_2,\cdots,a_{100}\}$ 與 $B=\{b_1,b_2,\cdots,b_{50}\}$，若從$A$到$B$的映射$f$使得$B$中每個元素都有原象，且 $f(a_1) \le f(a_2) \le \ldots \le f(a_{100})$，則這樣的映射共有（　）個．

（A）C_{100}^{50}（B）C_{99}^{48}（C）C_{100}^{49}（D）C_{99}^{49}

解答　設 $b_1,b_2,\cdots,b_{50}$ 按從小到大排列爲 $c_1<c_2<\cdots<c_{50}$（因集合元素具有互異性，故其中不含相等情形）．

將A中元素 $a_1,a_2,\cdots,a_{100}$ 分成 50 組，每組依次與B中元素 $c_1,c_2,\cdots,c_{50}$ 對應．這裏，我們用 $a_1a_2a_3c_1a_4a_5c_2\cdots$，表示 $f(a_1)=f(a_2)=f(a_3)=c_1$，$f(a_4)=f(a_5)=c_2$，…

考慮 $f(a_1)\le f(a_2)\le \cdots \le f(a_{100})$，容易得到 $f(a_{100})=c_{50}$，這就是說 c_{50} 只能寫在 a_{100} 的右邊，故只需在 $a_1□\,a_2□\,a_3□\cdots□\,a_{98}□\,a_{99}□\,a_{100}c_{50}$ 之間的 99 個空位“□”中選擇 49 個位置並按從左到右的順序依次填上 $c_1,c_2,\cdots,c_{49}$．從而構成滿足題設要求的映射共有 C_{99}^{49} 個．因此選 D．

例 2 有人要上樓，此人每步能向上走 1 階或 2 階，如果一層樓有 18 階，他上一層樓有多少種不同的走法？

解答 1　此人上樓最多走 18 步，最少走 9 步．這裏用 $a_{18},a_{17},a_{16},\cdots,a_9$ 分別表示此人上樓走 18 步，17 步，16 步，…，9 步時走法（對於任意前後兩者的步數，因後者少走 2 步 1 階，而多走 1 步 2 階，計後者少走 1 步）的計數結果．考慮步子中的每步 2 階情形，易得 $a_{18}=C_{18}^0$，$a_{17}=C_{17}^1$，$a_{16}=C_{16}^2$，…，$a_9=C_9^9$．

綜上，他上一層樓共有 $C_{18}^0+C_{17}^1+C_{16}^2+\cdots+C_9^9=1+17+120+\cdots+1=4181$ 種不同的走法．

解答 2　設 F_n 表示上n階的走法的計數結果．

顯然，$F_1=1$，$F_2=2$（2 步 1 階；1 步 2 階）．對於 $F_3,F_4,\cdots$，起步只有兩種不同走法：上 1 階或上 2 階．

因此對於 F_3，第 1 步上 1 階的情形，還剩 $3-1=2$ 階，計 F_2 種不同的走法；對於第 1 步上 2 階的情形，還剩 $3-2=1$ 階，計 F_1 種不同的走法．總計 $F_3=F_2+F_1=2+1=3$．

同理，$F_4=F_3+F_2=3+2=5$，$F_5=F_4+F_3=5+3=8$，…，$F_{18}=F_{17}+F_{16}=2584+1597=4181$．

例 3　在世界盃足球賽前，F國教練爲了考察 $A_1,A_2,\ldots,A_7$ 這七名隊員，準備讓他們在三場訓練比賽（每場 90 分鐘）都上場．假設在比賽的任何時刻，這

些隊員中有且僅有一人在場上，並且 A_1,A_2,A_3,A_4 每人上場的總時間（以分鐘爲單位）均被 7 整除，A_5,A_6,A_7 每人上場的總時間（以分鐘爲單位）均被 13 整除．如果每場換人次數不限，那麼按每名隊員上場的總時間計算，共有多少種不同的情況．

解答　設 $A_i\left(i=1,2,\cdots,7\right)$ 上場的總時間分別爲 $a_i\left(i=1,2,\cdots,7\right)$ 分鐘．

根據題意，可設

$a_i=7k_i\left(i=1,2,3,4\right)$，$a_i=13k_i\left(i=5,6,7\right)$，其中 $k_i\left(i=1,2,\cdots,7\right)\in Z^+$．

令 $\sum_{i=1}^{4}k_i=m$，$\sum_{i=5}^{7}k_i=n$，其中 $m\geq 4$，$n\geq 3$，且 $m,n\in Z^+$．則 $7m+13n=270$．易得其一個整數特解爲 $\begin{cases}m=33\\n=3\end{cases}$，又因 $\left(7,13\right)=1$，故其整數通解爲 $\begin{cases}m=33+13t\\n=3-7t\end{cases}$．再由 $\begin{cases}33+13t\geq 4\\3-7t\geq 3\end{cases}$，得 $-\frac{29}{13}\leq t\leq 0$，故整數 $t=0,-1,-2$．

從而其滿足條件的所有整數解爲 $\begin{cases}m=33,\\n=3;\end{cases}\begin{cases}m=20,\\n=10;\end{cases}\begin{cases}m=7,\\n=17.\end{cases}$

對於 $\sum_{i=1}^{4}k_i=33$ 的正整數解，可以寫一橫排共計 33 個 1，在每相鄰兩個 1 之間共 32 個空位中任選 3 個填入“+”號，再把 3 個“+”號分隔開的 4 個部分裏的 1 分別統計，就可得到其一個正整數解，故 $\sum_{i=1}^{4}k_i=33$ 有 C_{32}^3 個正整數解 $\left(k_1,k_2,k_3,k_4\right)$；同理 $\sum_{i=5}^{7}k_i=3$ 有 C_2^2 個正整數解 $\left(k_5,k_6,k_7\right)$；從而此時滿足條件的正整數解 $\left(k_1,k_2,k_3,k_4,k_5,k_6,k_7\right)$ 有 $C_{32}^3\cdot C_2^2$ 個．…

因此滿足條件的所有正整數解 $\left(k_1,k_2,k_3,k_4,k_5,k_6,k_7\right)$ 有

$C_{32}^3\cdot C_2^2+C_{19}^3\cdot C_9^2+C_6^3\cdot C_{16}^2=42244$

個，即按每名隊員上場的總時間計算，共有 42244 種不同的情況．

2 運用容斥原理

容斥原理，又稱包含排斥原理或逐步淘汰原理．顧名思義，即先計算一個較大集合的元素的個數，再把多計算的那一部分去掉．它由英國數學家 J.J.西爾維斯特首先創立．這個原理有多種表達形式，其中最基本的形式爲：

設 $A_1,A_2,\cdots,A_n$ 是任意 n 個有限集合，以 *card* (*S*) 代表 *S* 的元素的個數，則

$$card\left(A_1\cup A_2\cup\cdots\cup A_n\right)$$
$$=\sum_{1\leq i\leq n}card(A_i)-\sum_{1\leq i<j\leq n}card(A_i\cap A_j)$$
$$+\sum_{1\leq i<j<k\leq n}card\left(A_i\cap A_j\cap A_k\right)-\cdots$$
$$+(-1)^{n-1}card(A_1\cap A_2\cap\cdots\cap A_n).$$

例 4　由數字 1，2，3 組成 n 位數，且在這個 n 位數中，1，2，3 的每一個至少出現一次，問這樣的 n 位數有多少個？

解答　設 U 是由 1，2，3 組成的 n 位元數的集合，A_1 是 U 中不含數字 1 的 n 位元數的集合，A_2 是 U 中不含數字 2 的 n 位元數的集合，A_3 是 U 中不含數字 3 的 n 位元數的集合，則

$card\left(U\right)=3^n$,

$card\left(A_1\right)=card\left(A_2\right)=card\left(A_3\right)=2^n$,

$card\left(A_1\cap A_2\right)=card\left(A_2\cap A_3\right)=card\left(A_3\cap A_1\right)=1$,

$card\left(A_1\cap A_2\cap A_3\right)=0.$ ．

因此

$$card(U)-card(A_1\cup A_2\cup A_3)$$
$$=3^n-3\cdot 2^n+3\cdot 1-0=3^n-3\cdot 2^n+3$$ ．

即符合題意的 n 位數的個數爲 $3^n-3\cdot 2^n+3$ ．

下面，我們再來看一個關於容斥原理應用的變異問題．

例 5　參加大型團體表演的學生共 240 名，他們面對教練站成一行，自左至右按 1，2，3，4，5，…依次報數．教練要求全體學生牢記各自所報的數，並做下列動作：先讓報的數是 3 的倍數的全體同學向後轉；接著讓報的數是 5 的倍數的全體同學向後轉；最後讓報的數是 7 的倍數的全體同學向後轉．問：

⑴此時還有多少名同學面對教練？

⑵面對教練的同學中，自左至右第 66 位同學所報的數是幾？

解答　⑴設 $U=\{1,2,3,\cdots,240\}$，A_i 表示由 U 中所有 i 的倍數組成的集合．則

$$card\left(U\right)=240,\quad card\left(A_3\right)=\left[\frac{240}{3}\right]=80,$$

$$card\left(A_5\right)=\left[\frac{240}{5}\right]=48,\ card\left(A_7\right)=\left[\frac{240}{7}\right]=34$$

$$card\left(A_{15}\right)=\left[\frac{240}{15}\right]=16,\ card\left(A_{21}\right)=\left[\frac{240}{21}\right]=11,$$

$$card\left(A_{35}\right)=\left[\frac{240}{35}\right]=6,\ card\left(A_{105}\right)=\left[\frac{240}{105}\right]=2.$$

從而此時有

$card(U)-[card(A_3)+card(A_5)+card(A_7)]$
$+2[card\left(A_{15}\right)+card\left(A_{21}\right)+card\left(A_{35}\right)]$
$-4card\left(A_{105}\right)=136$

名同學面對教練．

如果我們借助威恩圖進行分析，利用上面所得數據分別填入圖 1，注意按從內到外的順序填．

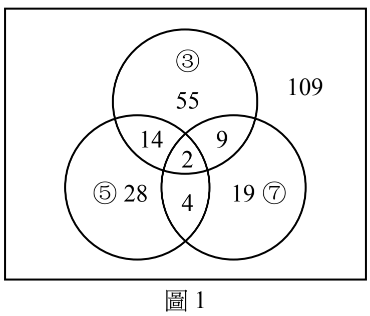


圖 1

如圖 1，此時面對教練的同學一目了然，應有 109+14+4+9=136 名．

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***March 31, 2005.***

Problem 216. (*Due to Alfred Eckstein, Arad, Romania*) Solve the equation

$$4x^6 - 6x^2 + 2\sqrt{2} = 0.$$

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where x_1, x_2, x_3, x_4, x_5 are positive integers. (*Source:* 2002 *Belarussian Mathematical Olympiad, Final Round*)

Problem 218. Let O and P be distinct points on a plane. Let $ABCD$ be a parallelogram on the same plane such that its diagonals intersect at O. Suppose P is not on the reflection of line AB with respect to line CD. Let M and N be the midpoints of segments AP and BP respectively. Let Q be the intersection of lines MC and ND. Prove that P, Q, O are collinear and the point Q does not depend on the choice of parallelogram $ABCD$. (*Source:* 2004 *National Math Olympiad in Slovenia, First Round*)

Problem 219. (*Due to Dorin Mărghidanu, Coleg. Nat. "A.I. Cuza", Corabia, Romania)* The sequences $a_0,a_1,a_2,\ldots$ and $b_0,b_1,b_2,\ldots$ are defined as follows: $a_0,b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1,2,3,\ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Problem 220. (*Due to Cheng HAO, The Second High School Attached to Beijing Normal University*) For $i = 1,2,\ldots,n$, and $k \ge 4$, let $A_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ with $a_{ij} = 0$ or 1 and every A_i has at least 3 of the k coordinates equal 1. Define the distance between A_i and A_j to be

$$\sum_{m=1}^{k} |a_{im} - a_{jm}|.$$

If the distance between any A_i and A_j ($i \ne j$) is greater than 2, then prove that

$$n \le 2^{k-3} - 1.$$

****

Solutions

****

Problem 211. For every a, b, c, d in [1,2], prove that

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \le 4\,\frac{a+c}{b+d}.$$

(*Source: 32nd Ukranian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania).

Since $0 < b + d \le 4$, it suffices to show

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \le a + c.$$

Without loss of generality, we may assume $1 \le a \le c$, say $c = a + e$ with $e \ge 0$. Then

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \le 1 + \left(1 + \frac{e}{d+a}\right)$$

$$\le 2 + e$$

$$\le 2a + e = a + c.$$

In passing, we observe that equality holds if and only if $e = 0$, $a = c = 1$, $b = d = 2$.

Other commended solvers: **CHENG Hei** (Tsuen Wan Government Secondary School, Form 5), **LAW Yau Pui** (Carmel Divine Grace Foundation Secondary School, Form 6) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 5).

Problem 212. Find the largest positive integer N such that if S is any set of 21 points on a circle C, then there exist N arcs of C whose endpoints lie in S and each of the arcs has measure not exceeding 120°.

Solution.

We will $N = 100$. To see that $N \le 100$, consider a diameter AB of C. Place 11 points close to A and 10 points close to B. The number of desired arcs is then

$$\binom{11}{2} + \binom{10}{2} = 100.$$

To see that $N \ge 100$, we need to observe that for every set T of $k = 21$ points on C, there exists a point X in T such that there are at least $[(k-1)/2]$ arcs XY (with Y in T, $Y \ne X$) each having measure not exceeding 120°. This is because we can divide the circle C into three arcs C_1, C_2, C_3 of 120° (only overlapping at endpoints) such that the common endpoint of C_1 and C_2 is a point X of T. If X does not have the required property, then there are $1 + [(k-1)/2]$ points of T lies on C_3 and any of them can serve as X.

Next we remove X and apply the same argument to $k = 20$, then remove that point, and repeat with $k = 19, 18, \ldots, 3$. We get a total of $10 + 9 + 9 + 8 + 8 + \ldots + 1 + 1 = 100$ arcs.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99b = c$, then at least two of them belong to the same subset.

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Let $f(n)$ be the <u>*largest*</u> nonnegative integer k for which n is divisible by 2^k. Then given three positive integers a, b, c satisfying $a + 99\,b = c$ at least two of $f(a)$, $f(b)$, $f(c)$ are equal. To prove this, if $f(a) = f(b)$, then we are done. If $f(a) < f(b)$, then $f(c) = f(a)$. If $f(a) > f(b)$, then $f(c) = f(b)$.

Therefore, the following partition suffices:

$$S_i = \{n \mid f(n) \equiv i \pmod{100}\}$$

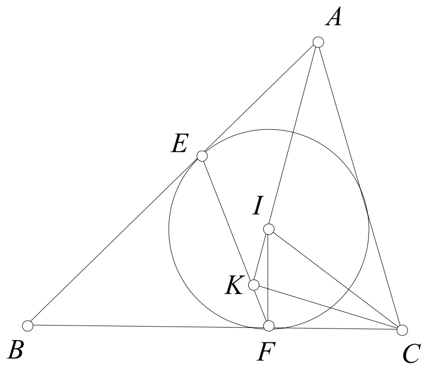
for $1 \le i \le 100$.

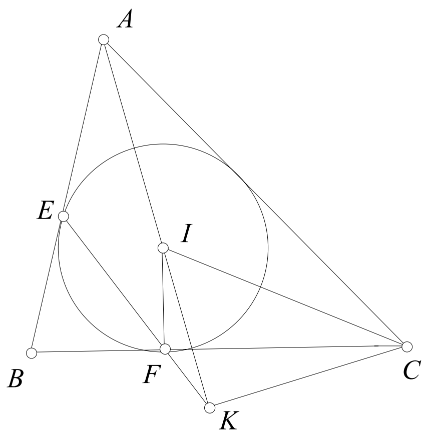
Problem 214. Let the inscribed circle of triangle ABC be tangent to sides AB, BC at E and F respectively. Let the angle bisector of $\angle CAB$ intersect segment EF at K. Prove that $\angle CKA$ is a right angle.

Solution. **CHENG Hei** (Tsuen Wan Government Secondary School, Form 5), **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **YIM Wing Yin** (South Tuen Mun

Government Secondary School, Form 5) and **YUNG Ka Chun** (Carmel Divine Grace Foundation Secondary School, Form 6).

Most of the solvers pointed out that the problem is still true if the angle bisector of $\angle CAB$ intersect line EF at K *outside the segment* EF. So we have two figures.





Let I be the center of the inscribed circle. Then A, I, K are collinear. Now $\angle CIK = \frac{1}{2}(\angle BAC + \angle ACB)$. Next, $BE = BF$ implies that $\angle BFE = 90° - \frac{1}{2}\angle CBA = \frac{1}{2}(\angle BAC + \angle ACB) = \angle CIK$. (In the second figure, we have $\angle CFK = \angle BFE = \angle CIK$.) Hence C, I, K, F are concyclic. Therefore, $\angle CKI = \angle CFI = 90°$.

Other commended solvers: **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1).

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 3×1 rectangles.

Solution. **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1).

Let us number the squares of the board from 1 to 64, with 1 to 8 on the first row, 9 to 16 on the second row and so on.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Using this numbering, a 3×1 rectangle will cover three numbers with a sum divisible by 3. Since $64 \equiv 1 \pmod 3$, only squares with numbers congruent to 1 (mod 3) need to be considered for our problem.

If there is a desired square for the problem, then considering the left-right symmetry of the board and the up-down symmetry of the board, the images of a desired square under these symmetries are also desired squares. Hence they must also have numbers congruent to 1 (mod 3) in them.

However, the only such square and its image squares having this property are the squares with numbers 19, 22, 43 and 46.

Finally square 19 has the required property (and hence also squares 22, 43, 46 by symmetry) by putting 3×1 rectangles as shown in the following figure (those squares having the same letter are covered by the same 3×1 rectangle).

A	A	A	B	B	B	F	G
C	C	C	D	D	D	F	G
H	I		E	E	E	F	G
H	I	J	J	J	K	K	K
H	I	L	L	L	M	M	M
N	O	P	Q	R	S	T	U
N	O	P	Q	R	S	T	U
N	O	P	Q	R	S	T	U

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **NG Siu Hong** (Carmel Divine Grace Foundation Secondary School, Form 6) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Olympiad Corner

(continued from page 1)

Problem 3. *(cont.)* Prove that the lines PS, AD, QR meet at a common point and lines SR and BC are parallel.

Problem 4. Let $S = \{1, 2, \ldots, 100\}$. Determine the number of functions $f: S \to S$ satisfying the following conditions.

(i) $f(1) = 1$;
(ii) f is bijective (i.e. for every y in S, the equation $f(x) = y$ has exactly one solution);
(iii) $f(n) = f(g(n))\, f(h(n))$ for every n in S.

Here $g(n)$ and $h(n)$ denote the uniquely determined positive integers such that $g(n) \le h(n)$, $g(n)\, h(n) = n$ and $h(n) - g(n)$ is as small as possible. (For instance, $g(80) = 8$, $h(80) = 10$ and $g(81) = h(81) = 9$.)

例析數學競賽中的計數問題 (一)

(continued from page 2)

⑵用上面類似的方法可算得自左至右第 1 號至第 105 號同學中面對教練的有 60 名．

考慮所報的數不是 3，5，7 的倍數的同學沒有轉動，他們面對教練；所報的數是 3，5，7 中的兩個數的倍數的同學經過兩次轉動，他們仍面對教練；其餘同學轉動了一次或三次，都背對教練．

作如下分析：106，107，~~108~~（3 的倍數），109，~~110~~（5 的倍數），~~111~~（3 的倍數），~~112~~（7 的倍數），113，~~114~~（3 的倍數），~~115~~（5 的倍數），116，~~117~~（3 的倍數），118，~~119~~（7 的倍數），120（3、5 的倍數），……，可知面對教練的第 66 位同學所報的數是 118.

(*to be continued*)

Mathematical Excalibur

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Olympiad Corner

Following are the problems of 2004 Estonian IMO team selection contest.

Problem 1. Let $k > 1$ be a fixed natural number. Find all polynomials $P(x)$ satisfying the condition $P(x^k) = (P(x))^k$ for all real number x.

Problem 2. Let O be the circumcentre of the acute triangle ABC and let lines AO and BC intersect at a point K. On sides AB and AC, points L and M are chosen such that $KL = KB$ and $KM = KC$. Prove that segments LM and BC are parallel.

Problem 3. For which natural number n is it possible to draw n line segments between vertices of a regular $2n$-gon so that every vertex is an endpoint for exactly one segment and these segments have pairwise different lengths?

Problem 4. Denote

$$f(m) = \sum_{k=1}^{m} (-1)^k \cos \frac{k\pi}{2m+1}.$$

For which positive integers m is $f(m)$ rational?

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 7, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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例析數學競賽中的計數問題(二)

費振鵬 （江蘇省鹽城市城區永豐中學 224054）

3　運用算兩次原理與抽屜原理

算兩次原理，就是把一個量從兩個（或更多）方面去考慮它，然後綜合起來得到一個關係式（可以是等式或不等式），或者導出一個矛盾的結論．具體表示爲三步：“一方面（利用一部分條件）……，另一方面（利用另一部分條件）……，綜合這兩個方面 ……”．義大利數學家富比尼（Fubini）首先應用這個思想方法，因此今天我們也稱它爲富比尼原理．

在解這些問題時，要根據問題的特點選擇一個適當的量，再將這個量用兩（或幾）種不同的方法表達出來．

抽屜原理，德國數學家狄利克雷（Dirichlet）提出．對於這個原理的具體解釋，想必很多同學早就知道了，在此不再贅述．

例 6　有 26 個不同國家的集郵愛好者，想通過互相通信的方法交換各國最新發行的紀念郵票，爲了使這 26 人每人都擁有這 26 個國家的一套最新紀念郵票，他們至少要通多少封信？

解答　不妨設這 26 個集郵愛好者中的某一個人爲組長．

一方面，對於組長，要接收到其他 25 個國家的最新紀念郵票，必須從這 25 個集郵愛好者的手中發出（不管他們是否直接發給組長），至少要通 25 封信；同樣地，其他 25 個集郵愛好者分別要接收到組長的一套紀念郵票，必須由組長發出（不管組長是否直接發給這 25 個集郵愛好者），至少要通 25 封信．總計至少要通 50 封信．

另一方面，其餘 25 個集郵愛好者每人將本國的一套最新的紀念郵票 25 份或 26 份發給組長，計 25 封信；組長收到這 25 封信後，再分別給這 25 個集郵愛好者各發去一封信，每封信中含有 25 套郵票（發給某人的信中不含其本國的郵票）或 26 套郵票（發給某人的信中包含其本國的郵票），計 25 封信．總計 50 封信．這就是說通 50 封信可以使這 26 人每人都擁有這 26 個國家的一套最新紀念郵票．

因此他們至少要通 50 封信．

例 7　從 1，2，3，…，1997 這 1997 個數中至多能選出多少個數，使得選出的數中沒有一個是另一個的 19 倍？

解答　因爲 1997÷19=105…2，所以 106，107，…，1997 這 1892 個數中沒有一個是另一個的 19 倍．

又因 106÷19=5…11，故 1，2，3，4，5，106，107，…，1997 這 1897 個數中沒有一個是另一個的 19 倍．

另一方面，從（6，6×19），（7，7×19），…，（105，105×19）這 100 對互異的數中最多可選出 100 個數（每對中至多選 1 個），即滿足題意的數至少剔除 100 個數．

綜上所述，從 1，2，3，…，1997 中至多選出 1897 個數，使得選出的數中沒有一個是另一個的 19 倍．

例 8　在正整數 1，2，3，…，1995，1996，1997 裏，最多能選出多少數，使其中任意兩個數的和不能被這兩個數的差整除．

解答　在所選的數中，不能出現連續自然數、連續奇數或連續偶數，這是由於連續自然數之和必能被其差 1 整除；連續奇數或連續偶數之和是偶數，必能被其差 2 整除．再考慮差值爲 3 的兩數，不能是 3 的倍數，否則其和仍是 3 的倍數，必能被其差 3 整除；而選擇全是 3 除餘 1，或全是 3 除餘 2 的數，注意到各自中任意兩數之和非 3 的倍數，不能被其差 3 的倍

數整除，滿足題意．

另一方面，從(1,2,3)，(4,5,6)，…，(1993,1993,1995)，(1996,1997)中，最多可選出 666 個(每組至多可選一個)，否則會出現連續自然數、連續奇數或連續偶數，而不滿足題意．又間隔 4 的所有數的個數較上述滿足題意的所有數的個數少．

綜上可知，1，4，7，…，1990，1993，1996（666 個）或 2，5，8，…，1991，1994，1997（666 個）均滿足題意．

即最多可選出 666 個，使其中任意兩數之和不能被這兩數之差整除．

例 9　設自然數n有以下性質：從 1，2，…，n中任取 50 個不同的數，這 50 個數中必有兩個數之差等於 7，這樣的n最大的一個是多少？

解答　n的最大值是 98．說明如下：

⑴一方面當自然數從 1，2，…，98 中任取 50 個不同的數，必有兩個數之差等於 7．這是因為：

首先將自然數 1，2，…，98 分成 7 組：（1，2，3，4，5，6，7，8，9，10，11，12，13，14），（15，16，17，18，19，20，21，22，23，24，25，26，27，28），…，（85，86，87，88，89，90，91，92，93，94，95，96，97，98）．

考慮取出的數中不出現某兩個數之差等於 7 的情形：由於每組中含有差為 7 的兩數，故每組最多可取出 7 個數（即每組中屬於 7 的同一個剩餘類的兩個數只能取其中的任意一個）．並且如果在第 1 組中取出了 m（m=1，2，…，14），那麼後面的每組分別取出 $m+14n$（n=1，2，…，6），可使所取數中的任意兩個數之差都不是 7．這樣從上述 7 組數中最多只能取出 7×7=49 個數．

根據抽屜原理，知從 1，2，…，98 中任取 50 個不同的數，必有兩個數之差等於 7．

⑵另一方面當自然數從 1，2，…，99 中任取 50 個不同的數，不能保證必有兩個數之差等於 7．這是因為：

首先將自然數 1，2，…，99 分成 8 組：（1，2，3，4，5，6，7，8，9，10，11，12，13，14），（15，16，17，18，19，20，21，22，23，24，25，26，27，28），…，（85，86，87，88，89，90，91，92，93，94，95，96，97，98），（99）．

比如，取出前 7 組中每組的前 7 個數，第 8 組的 99 這 50 個數，就不含有兩個數之差等於 7．

綜合⑴、⑵，可得 n 的最大值是 98．

例 10　某校組織了 20 次天文觀測活動，每次有 5 名學生參加，任何 2 名學生都至多同時參加過一次觀測．證明：至少有 21 名學生參加過這些觀測活動．

證法 1　（反證法）假設至多有 20 名學生參加過這些觀測活動．

每次觀測活動中的 5 名學生中有 $C_5^2=\frac{5\times4}{2\times1}=10$ 個 2 人小組，又由題意知 20 次觀測中 2 人小組各不相同，所以 20 次觀測中 2 人小組總共有 $20\times10=200$ 個．

而另一方面，20 名學生中的 2 人小組最多有 $C_{20}^2=\frac{20\times19}{2\times1}=190$ 個．

兩者自相矛盾．故至少有 21 名學生參加過這些觀測活動．

稍作簡化，即可證明如下：

證法 2　（反證法）假設至多有 20 名學生參加過這些觀測活動．

由題意知：⑴共有 20 次觀測；⑵最多有 $\frac{C_{20}^2}{C_5^2}=19$ 次觀測．

兩者自相矛盾．故至少有 21 名學生參加過這些觀測活動．

對於低年級學生，還可作出如下證明：

證法 3　設參加觀測活動次數最多的學生A參加了a次觀測，共有x名學生參加過天文觀測活動．

由於有 A 參加的每次觀測活動中，除了 A，其他學生各不相同（這是因為任何 2 名學生都至多同時參加過一次觀測），故 $x\geq 4a+1$．(I)

另一方面，學生 A 參加觀測的次數不小於每名學生平均觀測次數．即

$a\geq\frac{20\times5}{x}$．(II)

綜合(I)、(II)，得 $x\geq\frac{400}{x}+1$，

$x^2-x-400\geq 0$．從而 $x\geq 21$．

即至少有 21 名學生參加過這些觀測活動．

例 11　$2n$名選手參加象棋循環賽，每一輪中每個選手與其他 $2n-1$ 人各賽一場，勝得 1 分，平各得 $\frac{1}{2}$ 分，負得 0 分．證明：如果每個選手第一輪總分與第二輪總分至少相差n分，那麼每個選手兩輪總分恰好相差n分．

證明　令集A={第二輪總分>第一輪總分的人}，集B={第二輪總分<第一輪總分的人}，並且$|A|=k$，$|B|=h$，$k+h=2n$．

不妨設 $k\geq n\geq h$．考慮 A 中選手第二輪總分之和 S（若 $h\geq n\geq k$，則考慮 B 中選手第一輪總分之和 T）．另一方面，對於每輪 A 中選手和 B 中選手的 kh 場比賽中，所得總分之和為 kh，充其量全為 A 中選手取勝，則 $S\leq C_k^2+kh$．如 A 中選手第一輪總分之和為 S'，那麼 $S-S'\geq kn$，$C_k^2+kh-kn\geq S-kn\geq S'\geq C_k^2$．從而得 $h\geq n$，所以 $n=h=k$，並且以上不等式均為等式．

所以 A 中每個選手第二輪總分恰比第一輪總分多 n 分，B 中每個選手第一輪總分恰比第二輪總分多 n 分．因此，原命題成立．

(to be continued)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 7, 2005.***

Problem 221. (*Due to Alfred Eckstein, Arad, Romania*) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1}+F_{n-2}$ for $n \ge 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

Problem 222. All vertices of a convex quadrilateral $ABCD$ lie on a circle ω. The rays AD, BC intersect in point K and the rays AB, DC intersect in point L.

Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.
(*Source: 2001-2002 Estonian Math Olympiad, Final Round*)

Problem 223. Let $n \ge 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.

Problem 224. (*Due to Abderrahim Ouardini*) Let a, b, c be the sides of triangle ABC and I be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \le \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size?
(*Source: 2003-2004 Iranian Math Olympiad, Second Round*)

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Solutions

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Problem 216. (*Due to Alfred Eckstein, Arad, Romania*) Solve the equation

$$4x^6 - 6x^2 + 2\sqrt{2} = 0.$$

Solution. **Kwok Sze CHAI Charles** (HKU, Math Major, Year 1), **CHAN Tsz Lung**, **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **Badr SBAI** (Morocco), **TAM Yat Fung** (Valtorta College, Form 5), **WANG Wei Hua** and **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 6).

We have $8x^6 - 12x^2 + 4\sqrt{2} = 0$. Let $t = 2x^2$. We get

$$\begin{aligned} 0 &= t^3 - 6t + 4\sqrt{2} \\ &= t^3 - (\sqrt{2})^3 - (6t - 6\sqrt{2}) \\ &= (t - \sqrt{2})(t^2 + \sqrt{2}t - 4) \\ &= (t - \sqrt{2})(t - \sqrt{2})(t + 2\sqrt{2}). \end{aligned}$$

Solving $2x^2 = \sqrt{2}$ and $2x^2 = -2\sqrt{2}$, we get $x = \pm 1/\sqrt[4]{2}$ or $\pm i\sqrt[4]{2}$.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **Kin-Chit O** (STFA Cheng Yu Tung Secondary School) and **WONG Sze Wai** (True Light Girls' College, Form 4).

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where x_1, x_2, x_3, x_4, x_5 are positive integers. (*Source:* 2002 *Belarussian Mathematical Olympiad, Final Round*)

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **Tak Wai Alan WONG** (Markham, ON, Canada).

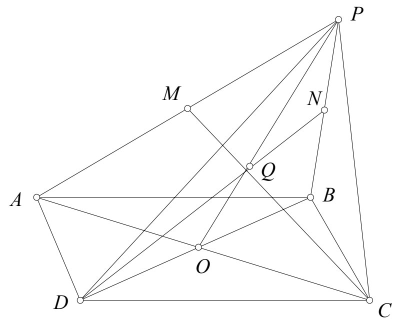
On the interval $[1, n]$, if there is such an integer, then

$$x_1 \le [n^{1/3}], x_2 \le [n^{1/5}], \cdots, x_5 \le [n^{1/11}].$$

So the number of integers in $[1, n]$ of the required form is at most $n^{1/3}n^{1/5}n^{1/7}n^{1/9}n^{1/11} = n^{3043/3465}$. Those not of the form is at least $n - n^{3043/3465}$, which goes to infinity as n goes to infinity.

Problem 218. Let O and P be distinct points on a plane. Let $ABCD$ be a parallelogram on the same plane such that its diagonals intersect at O. Suppose P is not on the reflection of line AB with respect to line CD. Let M and N be the midpoints of segments AP and BP respectively. Let Q be the intersection of lines MC and ND. Prove that P, Q, O are collinear and the point Q does not depend on the choice of parallelogram $ABCD$. (*Source: 2004 National Math Olympiad in Slovenia, First Round*)

Solution. **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).



Let G_1 be the intersection of OP and MC. Since OP and MC are medians of triangle APC, G_1 is the centroid of triangle APC. Hence $OG_1=⅓OP$. Similarly, let G_2 be the intersection of OP and ND. Since OP and ND are medians of triangle BPD, G_2 is the centroid of triangle BPD. Hence $OG_2=⅓OP$. So $G_1=G_2$ and it is on both MC and ND. Hence it is Q. This implies P, Q, O are collinear and Q is the unique point such that $OQ=⅓OP$, which does not depend on the choice of the parallelogram $ABCD$.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7) and **CHAN Tsz Lung**, **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6) and **WONG Tsun Yu** (St. Mark's School, Form 5).

Problem 219. (*Due to Dorin Mărghidanu, Coleg. Nat. "A.I. Cuza", Corabia, Romania)* The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined as follows: $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1,2,3,\ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Solution. **CHAN Tsz Lung**, **Kin-Chit**

O (STFA Cheng Yu Tung Secondary School), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

We have

$$a_{n+1}b_{n+1} = (a_n + \frac{1}{2b_n})(b_n + \frac{1}{2a_n})$$
$$= a_n b_n + \frac{1}{4a_n b_n} + 1$$
$$= a_{n-1}b_{n-1} + \frac{1}{4a_{n-1}b_{n-1}} + \frac{1}{4a_n b_n} + 2$$
$$= \cdots$$
$$= a_0 b_0 + \sum_{i=0}^{n} \frac{1}{4a_i b_i} + n + 1.$$

Then

$$(\max\{a_{2004}, b_{2004}\})^2 \geq a_{2004} \cdot b_{2004}$$
$$> a_0 b_0 + \frac{1}{4a_0 b_0} + 2004$$
$$\geq 2\sqrt{a_0 b_0 \frac{1}{4a_0 b_0}} + 2004$$
$$= 2005$$

and the result follows.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania).

Problem 220. (*Due to Cheng HAO, The Second High School Attached to Beijing Normal University*) For $i = 1, 2, \ldots, n$, and $k \geq 4$, let $A_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ with $a_{ij} = 0$ or 1 and every A_i has at least 3 of the k coordinates equal 1. Define the distance between A_i and A_j to be

$$\sum_{m=1}^{k} | a_{im} - a_{jm} |.$$

If the distance between any A_i and A_j ($i \neq j$) is greater than 2, then prove that

$$n \leq 2^{k-3} - 1.$$

Solution.

Let $|A_i-A_j|$ denote the distance between A_i and A_j. We add $A_0 = (0,\ldots,0)$ to the n A_m's. Then $|A_i-A_j| \geq 3$ still holds for $A_0, A_1, \ldots, A_n$.

Next we put the coordinates of A_0 to A_n into a $(n + 1) \times k$ table with the coordinates of A_i in the $(i + 1)$-st row.

Note if we take any of the k columns and switch all the 0's to 1's and 1's to 0's, then we get $n + 1$ new ordered k-tuples that still satisfy the condition $|A_i-A_j| \geq 3$. Thus, we may change A_0 to any combination with 0 or 1 coordinates. Then the problem is equivalent to showing $n + 1 \leq 2^{k-3}$ for $n + 1$ sets satisfying $|A_i-A_j| \geq 3$, but removing the condition each A_i has at least 3 coordinates equal 1.

For $k = 4$, we have $n + 1 \leq 2$. Next, suppose $k > 4$ and the inequality is true for the case $k-1$.

In column k of the table, there are at least $[(n + 2)/2]$ of the numbers which are the same (all 0's or all 1's). Next we keep only $[(n+2)/2]$ rows whose k-th coordinates are the same and we remove column k. The condition $|A_i-A_j| \geq 3$ still holds for these new ordered $(k-1)$-tuples. By the case $k - 1$, we get $[(n + 2)/2] + 1 \leq 2^{k-4}$. Since $(n + 1)/2 < [(n + 2)/2] + 1$, we get $n + 1 \leq 2^{k-3}$ and case k is true.

Generalization of Problem 203

Naoki Sato

We prove the following generalization of problem 203:

Let $a_1, a_2, \ldots, a_n$ be real numbers, and let s_i be the sum of the products of the a_i taken i at a time. If $s_1 \neq 0$, then the equation

$$s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n = 0$$

has only real roots.

<u>*Proof.*</u> Let

$$f(x) = s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n.$$

We can assume that none of the a_i are equal to 0, for if some of the a_i are equal to 0, then rearrange them so that $a_1, a_2, \ldots, a_k$ are nonzero and $a_{k+1}, a_{k+2}, \ldots, a_n$ are 0. Then $s_{k+1} = s_{k+2} = \ldots = s_n = 0$, so

$$f(x) = s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n$$
$$= s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ks_k x^{n-k}$$
$$= x^{n-k}(s_1 x^{k-1} + 2s_2 x^{k-2} + \cdots + ks_k).$$

Thus, the problem reduces to proving the same result on the numbers $a_1, a_2, \ldots, a_k$.

Let $g(x) = (a_1x+1)(a_2x+1)\ldots(a_nx+1)$. The roots of $g(x) = 0$ are clearly real, namely $-1/a_1, -1/a_2, \ldots, -1/a_n$. We claim that the roots of $g'(x)=0$ are all real.

Suppose the roots of $g(x) = 0$ are distinct. Let $r_1 < r_2 < \ldots < r_n$ be these roots. Then by Rolle's theorem, the equation $g'(x) = 0$ has a root in each of the intervals $(r_1,r_2), (r_2,r_3), \ldots, (r_{n-1},r_n)$, so it has $n - 1$ real roots.

Now, suppose the equation $g(x) = 0$ has j distinct roots $r_1 < r_2 < \ldots < r_j$, and root r_i has multiplicity m_i so $m_1+m_2+\ldots+m_j = n$. Then r_i is a root of the equation $g'(x) = 0$ having multiplicity m_i-1. In addition, again by Rolle's theorem, the equation has a root in each of the interval $(r_1,r_2), (r_2,r_3), \ldots, (r_{j-1},r_j)$, so the equation $g'(x) = 0$ has the requisite

$$(m_1-1)+(m_2-1)+\ldots+(m_j-1)+j-1 = n-1$$

real roots.

Expanding, we have that

$$g(x) = (a_1x+1)(a_2x+1)\ldots(a_nx+1)$$
$$= s_n x^n + s_{n-1}x^{n-1} + \ldots + 1,$$

So $g'(x) = ns_n x^{n-1} + (n-1)s_{n-1}x^{n-2} + \ldots + s_1$. Since $s_1 \neq 0$, 0 is not a root of $g'(x) = 0$. Finally, we get that the polynomial

$$x^{n-1}g'(\frac{1}{x}) = s_1 x^{n-1} + 2s_2 x^{n-1} + \cdots + ns_n$$

has all real roots.

Olympiad Corner

(continued from page 1)

Problem 5. Find all natural numbers n for which the number of all positive divisors of the number lcm$(1,2,\ldots,n)$ is equal to 2^k for some non-negative integer k.

Problem 6. Call a convex polyhedron a *footballoid* if it has the following properties.

(1) Any face is either a regular pentagon or a regular hexagon.

(2) All neighbours of a pentagonal face are hexagonal (a *neighbour* of a face is a face that has a common edge with it).

Find all possibilities for the number of a pentagonal and hexagonal faces of a footballoid.

Mathematical Excalibur

Volume 10, Number 2 *May 2005 – July 2005*

Olympiad Corner

Following are the problems of 2005 Chinese Mathematical Olympiad.

Problem 1. Let $\theta_i \in (-\pi/2, \pi/2)$, i = 1, 2, 3, 4. Prove that there exists $x \in \mathbb{R}$ satisfying the two inequalities

$$\cos^2\theta_1\cos^2\theta_2 - (\sin\theta_1\sin\theta_2 - x)^2 \ge 0$$
$$\cos^2\theta_3\cos^2\theta_4 - (\sin\theta_3\sin\theta_4 - x)^2 \ge 0$$

if and only if

$$\sum_{i=1}^{4}\sin^2\theta_i \le 2\left(1 + \prod_{i=1}^{4}\sin\theta_i + \prod_{i=1}^{4}\cos\theta_i\right).$$

Problem 2. A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2 and F_1, F_2 in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM and CN are concurrent.

Problem 3. As in the figure, a pond is divided into $2n$ ($n \ge 5$) parts. Two parts are called neighbors if they have a common side or arc. Thus every part has

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 10, 2005***.

For individual subscription for the next issue, please send us a stamped self-addressed envelope. Send all correspondence to:

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例析數學競賽中的計數問題(三)

費振鵬　（江蘇省鹽城市城區永豐中學 224054）

例 12 是否可能將正整數 1，2，3，…，64 分別填入 8×8 的正方形的 64 個小方格內，使得形如圖 1（方向可以任意轉置）的任意四個小方格內數之和總能被 5 整除？試說明理由。

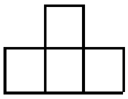

圖 1

解答　不可能。下面用反證法證明：假設圖 2 中 $a,b,c,\cdots,k,l,\cdots$ 就是符合題設填好的數。

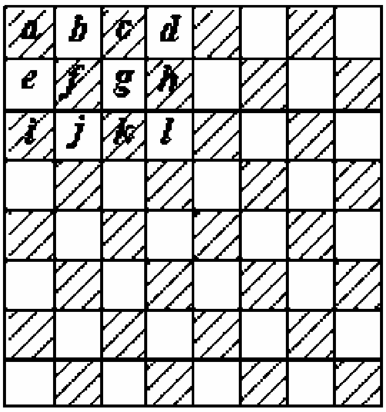

圖 2

因 $5 \mid b+e+f+g$，$5 \mid j+e+f+g$，作差有 $5 \mid b-j$，即 $b \equiv j \pmod 5$，設餘數為 r。

同理因 $5 \mid j+f+b+g$，$5 \mid e+f+b+g$，故 $5 \mid j-e$，即 $j \equiv e \pmod 5$，顯然其餘數也為 r。

將圖中 64 個小方格染成黑白相間的形式，可得 $b,j,e,g,d,l,\cdots$ 即除角上兩白格中的兩數外，其餘白格中的 $\frac{64}{2}-2=30$ 個數被 5 除都同餘 r。

另一方面，由抽屜原理，1～64 這 64 個正整數中最多有 13 個數被 5 除同餘，與前面得出的結論矛盾！因此，不存在滿足題設的填法。

例 13 平面上給定五點 A、B、C、D、E，其中任何三點不在一直線上。試證：任意地用線段連結某些點（這些線段稱為邊），若所得到的圖形中不出現以這五點中的任何三點為頂點的三角形，則這個圖形不可能有 7 條或更多條邊。

證法 1（反證法）假設圖形有 7 條或更多條邊，則各點度數和至少是 14。

(1) 若某點度數是 4，則其餘點的度數和至少是 10，由抽屜原理知其中必有一點度數至少是 $\left[\frac{10}{4}\right]+1=3$（度數是 2 就已足夠），故此時必然出現三角形。

(2) 若每點度數至多是 3，由抽屜原理知至少有 4 點的度數是 3，選其中 2 點，不妨設為 A、B，且 A 與 B、C、D 有連線，此時考慮 B 與 A 已有連線，由抽屜原理知 B 必與 C、D 中某一點有連線，這樣也出現了三角形。

而(1)、(2)所得結論都與題設“圖形中不出現以這五點中的任何三點為頂點的三角形”相矛盾，故原命題成立。

證法 2（反證法）假設圖形有 7 條或更多條邊。

首先我們構造抽屜：每個抽屜裏有三個相異點，共可得 $C_5^3=10$ 個抽屜[注]，又由於同一條邊會在 $5-2=3$ 個抽屜裏出現，則 10 個抽屜裏共有 $7\times3=21$ 條或更多條邊。

由抽屜原理知，至少有一個抽屜裏有 3 條邊，而每條邊在一個三角形中最多出現一次。這 3 條邊恰好與其中不

共線的相異三點構成一個三角形。而這與題設"圖形中不出現以這五點中的任何三點為頂點的三角形"相矛盾，故原命題成立。

注 對於低年級學生計算構造抽屜的個數，我們可以考慮從A、B、C、D、E五點中任取的三個點與剩下的兩點一一對應，而選擇兩點的情形有：AB、AC、AD、AE、BC、BD、BE、CD、CE、DE，共 10 種。

對這個問題稍作引伸，便得下面的問題：

例 14 平面上給定$n(n>3)$個點，其中任何三點不共線。任意地用線段連結某些點 (這些線段稱為邊)，得到x條邊。

(1) 若確保圖形中出現以給定點為頂點的三角形，求證：

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)} 。$$

當$\frac{n(n-1)(n-2)+3}{3(n-2)}$是整數時，求所有$n$連值及對應$x$的最小值；

(2) 若確保圖形中出現以給定點為頂點的m $(m<n)$ 階完全圖(即m點中任何兩點都有邊連接的圖)，求證：

$$x \geq \frac{C_n^m(C_m^2-1)+1}{C_{n-2}^{m-2}} 。$$

證明 (1) Ⅰ．構造抽屜：每個抽屜裏有三個相異點，共可得C_n^3個抽屜。又由於同一條邊會在C_{n-2}^1個抽屜裏出現，根據抽屜原理知，當$x \cdot C_{n-2}^1 \geq 2C_n^3+1$時，才能確保有一個抽屜裏有 3 條邊，而這 3 條邊恰好與其中不共線的相異三點構成一個三角形。

這就是說，確保圖形中出現以給定點為頂點的三角形，則$x \geq \frac{2C_n^3+1}{C_{n-2}^1}$，即

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)} 。$$

Ⅱ．顯然n，$n-1$，$n-2$中有且只有一個是 3 的倍數。

(i) 當n或$n-1$是 3 的倍數時，一方面

$$\frac{n(n-1)(n-2)+3}{3(n-2)} = \frac{n(n-1)}{3} + \frac{1}{n-2}$$

是整數，則$\frac{1}{n-2}$是整數；另一方面$n>3$，$n-2>1$，則$\frac{1}{n-2}$是分數。矛盾！此時n無解。

(ii) 當$n-2$是 3 的倍數時，不妨設$n-2=3k$，考慮

$$\begin{aligned}&\frac{n(n-1)(n-2)+3}{3(n-2)}\\ &= \frac{3k(3k+1)(3k+2)+3}{3\cdot 3k}\\ &= \frac{3k(3k^2+3k+1)+1-k}{3k}\\ &= 3k^2+3k+1+\frac{1-k}{3k}\end{aligned}$$

是整數，則$\frac{1-k}{3k}$是整數。

令$\frac{1-k}{3k}=t$，則$k(3t+1)=1$。從而$k=1$，$3t+1=1$即$k=1$，$t=0$。因此

$$n=3k+2=5 。$$

從而$x \geq 3k^2+3k+1$，即$x \geq 7$。因此x的最小值是 7。

綜合(i)、(ii)可知，當

$$\frac{n(n-1)(n-2)+3}{3(n-2)}$$

是整數時，$n=5$，$x_{\min}=7$。

(2) 構造抽屜：每個抽屜裏有m個相異點，共可得C_n^m個抽屜。又由於同一條邊會在C_{n-2}^{m-2}個抽屜裏出現。根據抽屜原理知，當

$$x \cdot C_{n-2}^{m-2} \geq C_n^m(C_m^2-1)+1$$

時，才能確保有一個抽屜裏有C_m^2條邊，而這C_m^2條邊恰好與其中不共線的相異m點構成一個m階完全圖。

這就是說，確保圖形中出現以給定點為頂點的m階完全圖，則

$$x \geq \frac{C_n^m(C_m^2-1)+1}{C_{n-2}^{m-2}} 。$$

注 題中字母k、m、n、t、x都是指整數。

以上解決數學競賽題的思路與方法告訴我們：見多識廣，可以增強領悟能力；博採眾長，才能減少盲目性。解題中的靈感突現，源自平時的日積月累。只有多鑽研，多探索，做題時便能隨機應變，亦或獨闢蹊徑，以致迎刃而解。

你覺得"數學好玩"嗎？只要你有興趣，數學就會變得迥然不同。你就會感受到數學無盡的魅力，就會具有攻無不克的意志力，就會產生無堅不摧的戰鬥力。如果你根本就沒愛上數學，又怎麼可能碰撞出最為絢爛的火花呢？哪怕非常短暫，瞬間即逝。

有很多同學熱愛數學，都為能在數學奧林匹克的賽場上一試身手、摘金奪銀而默默鑽研，苦苦奮鬥。我想學習中保持長久的數學興趣和培養創造性的思維是成功的關鍵，也是將來可持續發展的保障。而汲取眾家之長是創造性思維的源泉，學會獨立思考是提高創造性思維能力的良策。

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 10, 2005.***

Problem 226. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there is a nonempty subset of $\{z_1, z_2, \ldots, z_n\}$ the sum of whose elements has modulus at least 1/4.

Problem 227. For every integer $n \geq 6$, prove that

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}} \leq \frac{16}{5}.$$

Problem 228. In $\triangle ABC$, M is the foot of the perpendicular from A to the angle bisector of $\angle BCA$. N and L are respectively the feet of perpendiculars from A and C to the bisector of $\angle ABC$. Let F be the intersection of lines MN and AC. Let E be the intersection of lines BF and CL. Let D be the intersection of lines BL and AC.

Prove that lines DE and MN are parallel.

Problem 229. For integer $n \geq 2$, let a_1, a_2, a_3, a_4 be integers satisfying the following two conditions:

(1) for i = 1, 2, 3, 4, the greatest common divisor of n and a_i is 1 and
(2) for every $k = 1, 2, \ldots, n-1$, we have

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n,$$

where $(a)_n$ denotes the remainder when a is divided by n.

Prove that $(a_1)_n$, $(a_2)_n$, $(a_3)_n$, $(a_4)_n$ can be divided into two pairs, each pair having sum equals n.
(*Source: 1992 Japanese Math Olympiad*)

Problem 230. Let k be a positive integer. On the two sides of a river, there are in total at least 3 cities. From each of these cities, there are exactly k routes, each connecting the city to a distinct city on the other side of the river. Via these routes, people in every city can reach any one of the other cities.

Prove that if any one route is removed, people in every city can still reach any one of the other cities via the remaining routes.
(*Source: 1996 Iranian Math Olympiad, Round 2*)

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Solutions

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Due to an editorial mistake in the last issue, solutions to problems 216, 217, 218, 219 by **D. Kipp Johnson** (teacher, Valley Catholic School, Beaverton, Oregon, USA) were overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 221. (*Due to Alfred Eckstein, Arad, Romania*) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

Solution. **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **Kin-Chit O** (STFA Cheng Yu Tung Secondary School).

As $a = 7F_{n+2}^3 + 7F_{n+1}^3$ is divisible by $F_{n+2} + F_{n+1} = F_{n+3}$ and $b = 8F_{n+1}^3 + F_n^3$ is divisible by $2F_{n+1} + F_n = F_{n+2} + F_{n+1} = F_{n+3}$, so $7F_{n+2}^3 - F_n^3 - F_{n+1}^3 = a - b$ is divisible by F_{n+3}.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **CHAN Tsz Lung**, **CHAN Yee Ling** (Carmel Divine Grace Foundation Secondary School, Form 6), **G.R.A. 20 Math Problem Group** (Roma, Italy), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 6) and **WONG Kwok Kit** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 222. All vertices of a convex quadrilateral $ABCD$ lie on a circle ω**.** The rays AD, BC intersect in point K and the rays AB, DC intersect in point L.

Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.
(*Source: 2001-2002 Estonian Math Olympiad, Final Round*)

Solution. **LEE Kai Seng** (HKUST) and **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5).

Let ω_1 and ω_2 be the circumcircles of ΔAKL and ΔCKL respectively. For a point P on a circle Ω, let $\Omega(P)$ denote the tangent line to Ω at P.

Pick D' on $\omega(A)$ so that D and D' are on opposite sides of line BL and pick L' on $\omega_1(A)$ so that L and L' are on opposite sides of line BL.

Next, pick D'' on $\omega(C)$ so that D and D'' are on opposite sides of line BK and pick L'' on $\omega_2(C)$ so that L and L'' are on opposite sides of line BK. Now ω, ω_1 both contain A and ω, ω_2 both contain C. So

$$\begin{aligned} & \omega(A) = \omega_1(A) \\ \Leftrightarrow\ & \angle D'AB = \angle L'AB \\ \Leftrightarrow\ & \angle ADB = \angle ALB \\ \Leftrightarrow\ & BD \parallel LK \\ \Leftrightarrow\ & \angle BDC = \angle KLC \\ \Leftrightarrow\ & \angle BCD'' = \angle KCL'' \\ \Leftrightarrow\ & \omega(C) = \omega_2(C). \end{aligned}$$

Other commended solvers: **CHAN Tsz Lung** and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

Problem 223. Let $n \geq 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.
(*Source: 1997 Romanian Math Olympiad, Final Round*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy)**.**

By hypotheses, there are integers a, b such that $x^2 = x + a$ and $x^n = x + b$. Since x is real, the discriminant $\Delta = 1 + 4a$ of $x^2 - x - a = 0$ is nonnegative. So $a \geq 0$. If $a = 0$, then $x = 0$ or 1.

If $a > 0$, then define integers c_j, d_j so that $x^j = c_jx + d_j$ for $j \geq 2$ by $c_2 = 1$, $d_2 = a > 0$,

$$x^3 = x^2 + ax = (1 + a)x + a$$

leads to $c_3 = 1 + a$, $d_3 = a$ and for $j > 3$, $x^j = (x + a)x^{j-2} = (c_{j-1} + ac_{j-2})x + (d_{j-1} +$

ad_{j-2}) leads to $c_j = c_{j-1} + ac_{j-2} > c_{j-1} > 1$ and $d_j = d_{j-1} + ad_{j-2}$.

Now $c_n x + d_n = x^n = x + b$ with $c_n > 1$ implies $x = (b - d_n)/(c_n - 1)$ is rational. This along with a being an integer and $x^2 - x - a = 0$ imply x is an integer.

Other commended solvers: **CHAN Tsz Lung**, **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

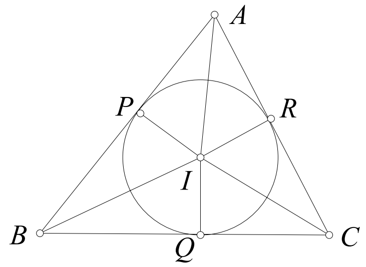
Problem 224. (*Due to Abderrahim Ouardini*) Let a, b, c be the sides of triangle ABC and I be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \le \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Solution. **CHAN Tsz Lung** and **Kin-Chit O** (STFA Cheng Yu Tung Secondary School).



Let r be the radius of the incircle and s be the semiperimeter $(a + b + c)/2$. The area of $\triangle ABC$ is $(a + b + c)r/2 = sr$ and $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. So

$$r^2 = (s-a)(s-b)(s-c)/s. \qquad (*)$$

Let P, Q, R be the feet of perpendiculars from I to AB, BC, CA. Now $s = AP + BQ + CR = AP + BC$, so $AP = s-a$. Similarly, $BQ = s-b$ and $CR = s-c$. By the *AM-GM* inequality,

$$\begin{aligned} s/3 &= [(s-a)+(s-b)+(s-c)]/3 \\ &\ge \sqrt[3]{(s-a)(s-b)(s-c)}. \qquad (**) \end{aligned}$$

Using Pythagoras' theorem, (*) and (**), we have

$$\begin{aligned} &IA^2 \cdot IB^2 \cdot IC^2 \\ &= [r^2+(s-a)^2][r^2+(s-b)^2][r^2+(s-c)^2] \\ &= [(s-a)bc/s][(s-b)ca/s][(s-c)ab/s] \\ &\le (abc)^2/3^3 \end{aligned}$$

with equality if and only if $a = b = c$. The result follows.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 5), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size?
(*Source: 2003-2004 Iranian Math Olympiad, Second Round*)

Official Solution.

Let the luminous point be at the origin. Consider all spheres of radius $r = \sqrt{2}/4$ centered at (i, j, k), where i, j, k are integers (not all zero) and $|i|$, $|j|$, $|k| \le 64$. The spheres are disjoint as the radii are less than 1/2. For any line L through the origin, by the symmetries of the spheres, we may assume L has equations of the form $y = ax$ and $z = bx$ with $|a|$, $|b| \le 1$. It suffices to show L intersects one of the spheres.

We claim that for every positive integer n and every real number c with $|c| \le 1$, there exists a positive integer $m \le n$ such that $|\{mc\}| < 1/n$, where $\{x\} = x - [x]$ is the fractional part of x.

To see this, partition [0,1) into n intervals of length $1/n$. If one of $\{c\}$, $\{2c\}$, …, $\{nc\}$ is in $[0,1/n)$, then the claim is true. Otherwise, by the pigeonhole principle, there are $0 < m' < m'' \le n$ such that $\{m'c\}$ and $\{m''c\}$ are in the same interval. Then $|\{m'c\}-\{m''c\}| < 1/n$ implies $|\{mc\}| < 1/n$ for $m = m'' - m' \le n$.

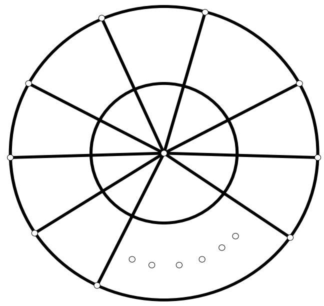
Since $|a| \le 1$, by the claim, there is a positive integer $m \le 16$ such that $|\{ma\}| < 1/16$ and there is a positive integer $n \le 4$ such that $|\{nmb\}| < 1/4$. Now $|\{ma\}| < 1/16$ and $n \le 4$ imply $|\{nma\}| < 1/4$. Then $i = nm \le 64$ and $j = [nma]$, $k = [nmb]$ satisfy $|j-nma| < 1/4$ and $|k-nmb| < 1/4$. So the distance between the point (i, ia, ib) on L and the center (i, j, k) is less than r. Therefore, every line L through the origin will intersect some sphere.

Olympiad Corner

(continued from page 1)

three neighbors. Now there are $4n + 1$ frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighbors respectively.

Prove that at some time later, the frogs at the pond will be uniformly distributed. That is, for any part, either there is at least one frog at the part or there is at least one frog at each of its neighbors.



Problem 4. Given a sequence $\{a_n\}$ satisfying $a_1 = 21/16$ and $2a_n - 3a_{n-1} = 3/2^{n+1}$, $n \ge 2$. Let m be a positive integer, $m \ge 2$.

Prove that if $n \le m$, then

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{1/m}\left(m - \left(\frac{2}{3}\right)^{n(m-1)/m}\right) < \frac{m^2-1}{m-n+1}.$$

Problem 5. Inside and including the boundary of a rectangle $ABCD$ with area 1, there are 5 points, no three of which are collinear.

Find (with proof) the least possible number of triangles having vertices among these 5 points with areas not greater than 1/4.

Problem 6. Find (with proof) all nonnegative integral solutions (x, y, z, w) to the equation

$$2^x \cdot 3^y - 5^z \cdot 7^w = 1.$$

Mathematical Excalibur

Volume 10, Number 3 *August 2005 – September 2005*

Olympiad Corner

The 2005 International Mathematical Olympiad was held in Merida, Mexico on July 13 and 14. Below are the problems.

Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

Problem 3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 30, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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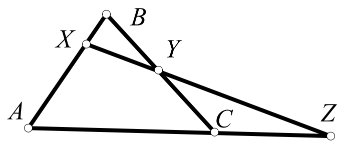
Famous Geometry Theorems

Kin Y. Li

There are many famous geometry theorems. We will look at some of them and some of their applications. Below we will write $P = WX \cap YZ$ to denote P is the point of intersection of lines WX and YZ. If points A, B, C are collinear, we will introduce the <u>*sign convention*</u>: $AB/BC = \overrightarrow{AB}/\overrightarrow{BC}$ (so if B is between A and C, then $AB/BC \geq 0$, *otherwise* $AB/BC \leq 0$).

<u>Menelaus' Theorem</u> Points X, Y, Z are taken from lines AB, BC, CA (which are the sides of $\triangle ABC$ extended) respectively. If there is a line passing through X, Y, Z, then

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=-1.$$



<u>Proof</u> Let L be a line perpendicular to the line through X, Y, Z and intersect it at O. Let A', B', C' be the feet of the perpendiculars from A, B, C to L respectively. Then

$$\frac{AX}{XB}=\frac{A'O}{OB'},\frac{BY}{YC}=\frac{B'O}{OC'},\frac{CZ}{ZA}=\frac{C'O}{OA'}.$$

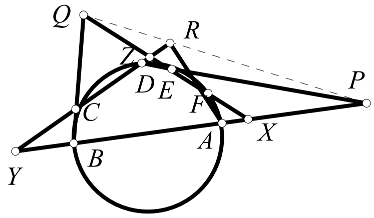
Multiplying these equations together, we get the result.

The converse of Menelaus' Theorem is also true. To see this, let $Z'=XY\cap CA$. Then applying Menelaus theorem to the line through X, Y, Z' and comparing with the equation above, we get $CZ/ZA=CZ'/Z'A$. It follows $Z=Z'$.

<u>Pascal's Theorem</u> Let A, B, C, D, E, F be points on a circle (which are not necessarily in cyclic order). Let

$$P=AB\cap DE,\ Q=BC\cap EF,\ R=CD\cap FA.$$

Then P,Q,R are collinear.



<u>Proof</u> Let $X = EF \cap AB$, $Y = AB \cap CD$, $Z = CD \cap EF$. Applying Menelaus' Theorem respectively to lines BC, DE, FA cutting $\triangle XYZ$ extended , we have

$$\frac{ZQ}{QX}\cdot\frac{XB}{BY}\cdot\frac{YC}{CZ}=-1,$$
$$\frac{XP}{PY}\cdot\frac{YD}{DZ}\cdot\frac{ZE}{EX}=-1,$$
$$\frac{YR}{RZ}\cdot\frac{ZF}{FX}\cdot\frac{XA}{AY}=-1.$$

Multiplying these three equations together, then using the intersecting chord theorem (see *vol* 4, *no*. 3, *p*. 2 of *Mathematical Excalibur*) to get $XA\cdot XB = XE\cdot XF$, $YC\cdot YD = YA\cdot YB$, $ZE\cdot ZF = ZC\cdot ZD$, we arrive at the equation

$$\frac{ZQ}{QX}\cdot\frac{XP}{PY}\cdot\frac{YR}{RZ}=-1.$$

By the converse of Menelaus' Theorem, this implies P, Q, R are collinear.

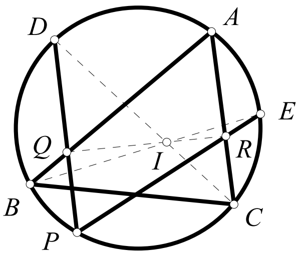
We remark that there are limiting cases of Pascal's Theorem. For example, we may move A to approach B. In the limit, A and B will coincide and the line AB will become the tangent line at B.

Below we will give some examples of using Pascal's Theorem in geometry problems.

<u>Example 1</u> (*2001 Macedonian Math Olympiad*) For the circumcircle of $\triangle ABC$, let D be the intersection of the tangent line at A with line BC, E be the intersection of the tangent line at B with line CA and F be the intersection of the tangent line at C with line AB. Prove that points D, E, F are collinear.

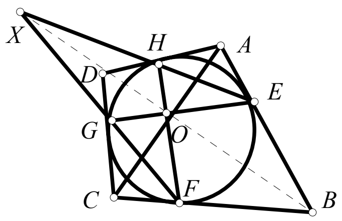
Solution Applying Pascal's Theorem to *A*, *A*, *B*, *B*, *C*, *C* on the circumcircle, we easily get *D*, *E*, *F* are collinear.

Example 2 Let *D* and *E* be the midpoints of the minor arcs *AB* and *AC* on the circumcircle of △ *ABC*, respectively. Let *P* be on the minor arc *BC*, $Q = DP \cap BA$ and $R = PE \cap AC$. Prove that line *QR* passes through the incenter *I* of △*ABC*.



Solution Since *D* is the midpoint of arc *AB*, line *CD* bisects ∠ *ACB*. Similarly, line *EB* bisects ∠*ABC*. So $I = CD \cap EB$. Applying Pascal's Theorem to *C, D, P, E, B, A*, we get *I, Q, R* are collinear.

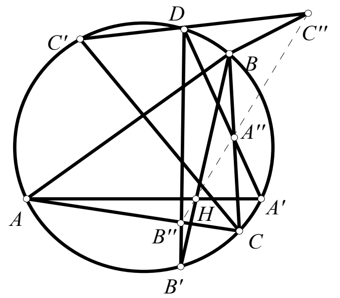
Newton's Theorem A circle is inscribed in a quadrilateral *ABCD* with sides *AB, BC, CD, DA* touch the circle at points *E, F, G, H* respectively. Then lines *AC, EG, BD, FH* are concurrent.



Proof. Let $O = EG \cap FH$ and $X = EH \cap FG$. Since *D* is the intersection of the tangent lines at *G* and at *H* to the circle, applying Pascal's Theorem to *E,G,G,F,H,H*, we get *O, D, X* are collinear. Similarly, applying Pascal's Theorem to *E, E, H, F, F, G*, we get *B, X, O* are collinear.

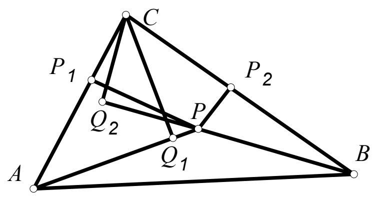
Then *B,O,D* are collinear and so lines *EG, BD, FH* are concurrent at *O*. Similarly, we can also obtain lines *AC, EG, FH* are concurrent at *O*. Then Newton's Theorem follows.

Example 3 (*2001 Australian Math Olympiad*) Let *A, B, C, A', B', C'* be points on a circle such that *AA'* is perpendicular to *BC, BB'* is perpendicular to *CA, CC'* is perpendicular to *AB*. Further, let *D* be a point on that circle and let *DA'* intersect *BC* in *A"*, *DB'* intersect *CA* in *B"*, and *DC'* intersect *AB* in *C"*, all segments being extended where required. Prove that *A", B", C"* and the orthocenter of triangle *ABC* are collinear.



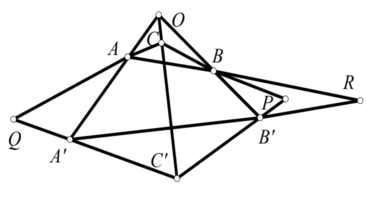
Solution Let *H* be the orthocenter of △ *ABC*. Applying Pascal's theorem to *A, A', D, C', C, B*, we see *H, A", C"* are collinear. Similarly, applying Pascal's theorem to *B', D, C', C, A, B*, we see *B", C", H* are collinear. So *A", B", C", H* are collinear.

Example 4 (*1991 IMO unused problem*) Let *ABC* be any triangle and *P* any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from *P* to the two sides *AC* and *BC*. Draw *AP* and *BP* and from *C* drop perpendiculars to *AP* and *BP*. Let Q_1 and Q_2 be the feet of these perpendiculars. If $Q_2 \neq P_1$ and $Q_1 \neq P_2$, then prove that the lines P_1Q_2, Q_1P_2 and *AB* are concurrent.



Solution Since $\angle CP_1P$, $\angle CP_2P$, $\angle CQ_2P$, $\angle CQ_1P$ are all right angles, we see that the points *C*, Q_1, P_1, *P*, P_2, Q_2 lie on a circle with *CP* as diameter. Note $A = CP_1 \cap PQ_1$ and $B = Q_2P \cap P_2C$. Applying Pascal's theorem to *C*, P_1, Q_2, *P*, Q_1, P_2, we see $X = P_1Q_2 \cap Q_1P_2$ is on line *AB*.

Desargues' Theorem For △*ABC* and △ *A'B'C'*, if lines *AA', BB', CC'* concur at a point *O*, then points *P, Q, R* are collinear, where $P = BC \cap B'C'$, $Q = CA \cap C'A'$, $R = AB \cap A'B'$.



Proof Applying Menelaus' Theorem respectively to line *A'B'* cutting △*OAB* extended, line *B'C'* cutting △ *OBC* extended and the line *C'A'* cutting △*OCA* extended, we have

$$\frac{OA'}{A'A}\cdot\frac{AR}{RB}\cdot\frac{BB'}{B'O}=-1,$$

$$\frac{OB'}{B'B}\cdot\frac{BP}{PC}\cdot\frac{CC'}{C'O}=-1,$$

$$\frac{AA'}{A'O}\cdot\frac{OC'}{C'C}\cdot\frac{CQ}{QA}=-1.$$

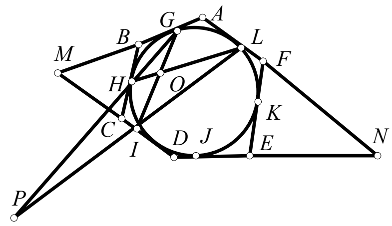
Multiplying these three equations,

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1.$$

By the converse of Menelaus' Theorem, this implies *P, Q, R* are collinear.

We remark that the converse of Desargues' Theorem is also true. We can prove it as follow: let $O = BB' \cap CC'$. Consider △*RBB'* and △*QCC'*. Since lines *RQ, BC, B'C'* concur at *P*, and $A = RB \cap QC$, $O = BB' \cap CC'$, $A' = BR' \cap C'Q$, by Desargues' Theorem, we have *A,O,A'* are collinear. Therefore, lines *AA', BB', CC'* concur at *O*.

Brianchon's Theorem Lines *AB, BC, CD, DE, EF, FA* are tangent to a circle at points *G, H, I, J, K, L* (not necessarily in cyclic order). Then lines *AD, BE, CF* are concurrent.



Proof Let $M = AB \cap CD$, $N = DE \cap FA$. Applying Newton's Theorem to quadrilateral *AMDN*, we see lines *AD, IL, GJ* concur at a point *A'*. Similarly, lines *BE, HK, GJ* concur at a point *B'* and lines *CF, HK, IL* concur at a point *C'*. Note line *IL* coincides with line *A'C'*. Next we apply Pascal's Theorem to *G, G, I, L, L, H* and get points *A, O, P* are collinear, where $O = GI \cap LH$ and $P = IL \cap HG$. Applying Pascal's Theorem again to *H, H, L, I, I, G*, we get *C, O, P* are collinear. Hence *A, C, P* are collinear.

Now $G = AB \cap A'B'$, $H = BC \cap B'C'$, $P = CA \cap IL = CA \cap C'A'$. Applying the converse of Desargues' Theorem to △*ABC* and △*A'B'C'*, we get lines $AA' = AD$, $BB' = BE$, $CC' = CF$ are concurrent.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***October 30, 2005.***

Problem 231. On each planet of a star system, there is an astronomer observing the nearest planet. The number of planets is odd, and pairwise distances between them are different. Prove that at least one planet is not under observation.
(*Source*: *1966 Soviet Union Math Olympiad*)

Problem 232. B and C are points on the segment AD. If $AB = CD$, prove that $PA + PD \ge PB + PC$ for any point P.
(*Source*: *1966 Soviet Union Math Olympiad*)

Problem 233. Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

Problem 234. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

a) The coefficient of the highest power is 200.
b) The coefficient of the lowest power for which it is not equal to zero is 2.
c) The sum of all its coefficients is 4.
d) $P(-1) = 0$, $P(2) = 6$ and $P(3) = 8$.

(*Source: 2002 Austrian National Competition*)

Problem 235. Forty-nine students solve a set of three problems. The score for each problem is an integer from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.

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Solutions

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Problem 226. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there is a nonempty subset of $\{z_1, z_2, \ldots, z_n\}$ the sum of whose elements has modulus at least 1/4.

Solution. **LEE Kai Seng** (HKUST).

Let $z_k = a_k + b_k i$ with a_k, b_k real. Then $|z_k| \le |a_k| + |b_k|$. So

$$1 = \sum_{k=1}^{n} |z_k| \le \sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |b_k|$$

$$= \sum_{a_k \ge 0} a_k + \sum_{a_k < 0} (-a_k) + \sum_{b_k \ge 0} b_k + \sum_{b_k < 0} (-b_k).$$

Hence, one of the four sums is at least 1/4, say $\sum_{a_k \ge 0} a_k \ge \frac{1}{4}$. Then

$$\left| \sum_{a_k \ge 0} z_k \right| \ge \left| \sum_{a_k \ge 0} a_k \right| \ge \frac{1}{4}.$$

Problem 227. For every integer $n \ge 6$, prove that

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}} \le \frac{16}{5}.$$

Comments. In the original statement of the problem, the displayed inequality was stated incorrectly. The < sign should be an $\le$ sign.

Solution. **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **Roger CHAN** (Vancouver, Canada) and **LEE Kai Seng** (HKUST).

For $n = 6, 7, \ldots$, let

$$a_n = \sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}}.$$

Then $a_6 = 16/5$. For $n \ge 6$, if $a_n \le 16/5$, then

$$a_{n+1} = \sum_{k=1}^{n} \frac{n+1}{n+1-k} \cdot \frac{1}{2^{k-1}} = \sum_{j=0}^{n-1} \frac{n+1}{n-j} \cdot \frac{1}{2^j}$$

$$= \frac{n+1}{n} + \frac{n+1}{2n} \sum_{j=1}^{n-1} \frac{n}{n-j} \cdot \frac{1}{2^{j-1}}$$

$$= \frac{n+1}{n}\left(1 + \frac{a_n}{2}\right) \le \frac{7}{6}\left(1 + \frac{8}{5}\right) < \frac{16}{5}.$$

The desired inequality follows by mathematical induction.

Problem 228. In $\triangle ABC$, M is the foot of the perpendicular from A to the angle bisector of $\angle BCA$. N and L are respectively the feet of perpendiculars from A and C to the bisector of $\angle ABC$. Let F be the intersection of lines MN and AC. Let E be the intersection of lines BF and CL. Let D be the intersection of lines BL and AC.

Prove that lines DE and MN are parallel.

Solution. **Roger CHAN** (Vancouver, Canada).

Extend AM to meet BC at G and extend AN to meet BC at I. Then $AM = MG$, $AN = NI$ and so lines MN and BC are parallel.

From $AM = MG$, we get $AF = FC$. Extend CL to meet line AB at J. Then $JL = LC$. So lines LF and AB are parallel.

Let line LF intersect BC at H. Then $BH = HC$. In $\triangle BLC$, segments BE, LH and CD concur at F. By Ceva's theorem (see *vol.* 2, *no.* 5, *pp.* 1-2 of *Mathematical Excalibur*),

$$\frac{BH}{HC} \cdot \frac{CE}{EL} \cdot \frac{LD}{DB} = 1.$$

Since $BH = HC$, we get $CE/EL = DB/LD$, which implies lines DE and BC are parallel. Therefore, lines DE and MN are parallel.

Problem 229. For integer $n \ge 2$, let a_1, a_2, a_3, a_4 be integers satisfying the following two conditions:

(1) for $i = 1, 2, 3, 4$, the greatest common divisor of n and a_i is 1 and
(2) for every $k = 1, 2, \ldots, n-1$, we have

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n,$$

where $(a)_n$ denotes the remainder when a is divided by n.

Prove that $(a_1)_n, (a_2)_n, (a_3)_n, (a_4)_n$ can be divided into two pairs, each pair having sum equals n.
(*Source: 1992 Japanese Math Olympiad*)

Solution. (***Official Solution***)

Since n and a_1 are relatively prime, the remainders $(a_1)_n, (2a_1)_n, \ldots, ((n-1)a_1)_n$ are nonzero and distinct. So there is a k among $1, 2, \ldots, n-1$ such that $(ka_1)_n = 1$. Note that such k is relatively prime to n. If $(ka_1)_n + (ka_j)_n = n$, then $ka_1 + ka_j \equiv 0 \pmod n$ so that $a_1 + a_j \equiv 0 \pmod n$ and $(a_1)_n + (a_j)_n = n$. Thus, to solve the problem, we may replace a_i by $(ka_i)_n$ and assume $1 = a_1 \le a_2 \le a_3 \le$

$a_4 \le n-1$. By condition (2), we have

$$1+a_2+a_3+a_4=2n. \qquad \text{(A)}$$

For $k = 1, 2, \ldots, n-1$, let

$$f_i(k) = [ka_i/n] - [(k-1)a_i/n],$$

then $f_i(k) \le (ka_i/n) + 1 - (k-1)a_i/n = 1 + (a_i/n) < 2$. So $f_i(k) = 0$ or 1. Since $x = [x/n]n + (x)_n$, subtracting the case $x = ka_i$ from the case $x = (k-1)a_i$, then summing $i = 1, 2, 3, 4$, using condition (2) and (A), we get

$$f_1(k) + f_2(k) + f_3(k) + f_4(k) = 2.$$

Since $a_1 = 1$, we see $f_1(k)=0$ and exactly two of $f_2(k), f_3(k), f_4(k)$ equal 1. (B)

Since $a_i<n, f_i(2) = [2a_i/n]$. Since $a_2 \le a_3 \le a_4 < n$, we get $f_2(2) = 0, f_3(2) = f_4(2) = 1$, i.e. $1 = a_1 \le a_2 < n/2 < a_3 \le a_4 \le n-1$.

Let $t_2 = [n/a_2] + 1$, then $f_2(t_2) = [t_2a_2/n] - [(t_2-1)a_2/n] = 1 - 0 = 1$. If $1 \le k < t_2$, then $k < n/a_2, f_2(k) = [ka_2/n] - [(k-1)a_2/n] = 0 - 0 = 0$. Next if $f_2(j) = 1$, then $f_2(k) = 0$ for $j < k < j + t_2 - 1$ and exactly one of $f_2(j + t_2 - 1)$ or $f_2(j + t_2) = 1$. (C)

Similarly, for $i = 3, 4$, let $t_i = [n/(n-a_i)] + 1$, then $f_i(t_i) = 0$ and $f_i(k) = 1$ for $1 \le k < t_i$. Also, if $f_i(j) = 0$, then $f_i(k) = 1$ for $j < k < j + t_i - 1$ and exactly one of $f_i(j + t_i - 1)$ or $f_i(j + t_i) = 0$. (D)

Since $f_3(t_3) = 0$, by (B), $f_2(t_3) = 1$. If $k < t_3 \le t_4$, then by (D), $f_3(k) = f_4(k) = 1$. So by (B), $f_2(k) = 0$. Then by (C), $t_2 = t_3$.

Assume $t_4 < n$. Since $n/2 < a_4 < n$, we get $f_4(n-1) = (a_4-1) - (a_4-2) = 1 \ne 0 = f_4(t_4)$ and so $t_4 \ne n-1$. Also, $f_4(t_4) = 0$ implies $f_2(t_4) = f_3(t_4) = 1$ by (B).

Since $f_3(t_3) = 0 \ne 1 = f_3(t_4)$, $t_3 \ne t_4$. Thus $t_2 = t_3 < t_4$. Let $s < t_4$ be the largest integer such that $f_2(s) = 1$. Since $f_2(t_4) = 1$, we have $t_4 = s + t_2 - 1$ or $t_4 = s + t_2$. Since $f_2(s) = f_4(s) = 1$, we get $f_3(s) = 0$. As $t_2 = t_3$, we have $t_4 = s + t_3 - 1$ or $t_4 = s + t_3$. Since $f_3(s) = 0$ and $f_3(t_4) = 1$, by (D), we get $f_3(t_4 - 1) = 0$ or $f_3(t_4 + 1) = 0$. Since $f_2(s) = 1, f_2(t_4) = 1$ and $t_2 > 2$, by (C), we get $f_2(s + 1) = 0$ and $f_2(t_4 + 1) = 0$. So $s + 1 \ne t_4$, which implies $f_2(t_4 - 1) = 0$ by the definition of s. Then $k = t_4 - 1$ or $t_4 + 1$ contradicts (B).

So $t_4 \ge n$, then $n - a_4 = 1$. We get $a_1 + a_4 = n = a_2 + a_3$.

Problem 230. Let k be a positive integer. On the two sides of a river, there are in total at least 3 cities. From each of these cities, there are exactly k routes, each connecting the city to a distinct city on the other side of the river. Via these routes, people in every city can reach any one of the other cities.

Prove that if any one route is removed, people in every city can still reach any one of the other cities via the remaining routes.

(*Source: 1996 Iranian Math Olympiad, Round 2*)

Solution. **LEE Kai Seng** (HKUST).

Associate each city with a vertex of a graph. Suppose there are X and Y cities to the left and to the right of the river respectively. Then the number of routes (or edges of the graph) in the beginning is $Xk = Yk$ so that $X = Y$. We have $X + Y \ge 3$.

After one route between city A and city B is removed, assume the cities can no longer be connected via the remaining routes. Then each of the other cities can only be connected to exactly one of A or B. Then the original graph decomposes into two connected graphs G_A and G_B, where G_A has A as vertex and G_B has B as vertex.

Let X_A be the number of cities among the X cities on the left sides of the river that can still be connected to A after the route between A and B was removed and similarly for X_B, Y_A, Y_B. Then the number of edges in G_A is $X_Ak-1 = Y_Ak$. Then $(X_A - Y_A)k = 1$. So $k = 1$. Then in the beginning $X = 1$ and $Y = 1$, contradicting $X + Y \ge 3$.

Olympiad Corner

(continued from page 1)

Problem 4. Consider the sequence a_1, a_2, … defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots)$$

Determine all positive integers that are relatively prime to every term of the sequence.

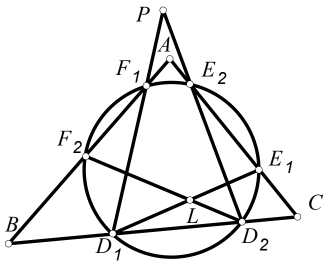
Problem 5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

Problem 6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than 2/5 of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

Famous Geometry Theorems

(continued from page 2)

Example 5 (*2005 Chinese Math Olympiad*) A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2 and F_1, F_2 in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM and CN are concurrent.



Solution. Let $P = D_1F_1 \cap D_2E_2$, $Q = E_1D_1 \cap E_2F_2$, $R = F_1E_1 \cap F_2D_2$. Applying Pascal's Theorem to E_2, E_1, D_1, F_1, F_2, D_2, we get A, L, P are collinear. Applying Pascal's Theorem to F_2, F_1, E_1, D_1, D_2, E_2, we get B, M, Q are collinear. Applying Pascal's Theorem to D_2, D_1, F_1, E_1, E_2, F_2, we get C, N, R are collinear.

Let $X = E_2E_1 \cap D_1F_2 = CA \cap D_1F_2$, $Y = F_2F_1 \cap E_1D_2 = AB \cap E_1D_2$, $Z = D_2D_1 \cap F_1E_2 = BC \cap F_1E_2$. Applying Pascal's Theorem to D_1, F_1, E_1, E_2, D_2, F_2, we get P, R, X are collinear. Applying Pascal's Theorem to E_1, D_1, F_1, F_2, E_2, D_2, we get Q, P, Y are collinear. Applying Pascal's Theorem to F_1, E_1, D_1, D_2, F_2, E_2, we get R, Q, Z are collinear.

For $\triangle ABC$ and $\triangle PQR$, we have $X = CA \cap RP$, $Y = AB \cap PQ$, $Z = BC \cap QR$. By the converse of Desargues' Theorem, lines $AP = AL$, $BQ = BM$, $CR = CN$ are concurrent.

Mathematical Excalibur

Volume 10, Number 4 *October 2005 – November 2005*

Olympiad Corner

Below is the Bulgarian selection test for the 46th IMO given on May 18 – 19, 2005.

Problem 1. An acute triangle ABC is given. Find the locus of points M in the interior of the triangle such that $AB-FG = (MF\cdot AG+MG\cdot BF)/CM$, where F and G are the feet of perpendiculars from M to the lines BC and AC, respectively.

Problem 2. Find the number of subsets B of the set $\{1, 2, \ldots, 2005\}$ such that the sum of the elements of B is congruent to 2006 modulo 2048.

Problem 3. Let R_* be the set of non-zero real numbers. Find all functions $f: R_* \to R_*$ such that

$$f(x^2+y) = f^2(x) + \frac{f(xy)}{f(x)}$$

for all $x, y \in R_*$, $y \neq -x^2$.

Problem 4. Let $a_1, a_2, \ldots, a_{2005}, b_1, b_2, \cdots, b_{2005}$ be real numbers such that

$$(a_i x - b_i)^2 \geq \sum_{j=1, j\neq i}^{2005} (a_j x - b_j)$$

for any real number x and $i = 1, 2, \ldots, 2005$. What is the maximal number of positive a_i's and b_i's?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 10, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Method of Infinite Descent

Leung Tat-Wing

The technique of infinite descent (*descent infini*) was developed by the great amateur mathematician Pierre de Fermat (1601-1665). Besides using the technique to prove negative results such as the equation $x^4 + y^4 = z^2$ has no nontrivial integer solution, he also used the technique to prove positive results.

For instance, he knew that an odd prime p can be expressed as the sum of two integer squares if and only if p is of the form $4k + 1$. To show that a prime of the form $4k + 3$ is not a sum of two squares is not hard. In fact, every square equals 0 or 1 mod 4, thus no matter what possibilities, the sum of two squares cannot be of the form $4k + 3 \equiv 3 \pmod 4$. To prove a prime of the form $4k + 1$ is the sum of two squares, he assumed that if there is a prime of the form $4k + 1$ which is not the sum of two squares, then there will be another (smaller) prime of the same nature, and hence a third one, and so on. Eventually he would come to the number 5, which should not be the sum of two squares. But we know $5 = 1^2 + 2^2$ a sum of two squares, a contradiction!

The idea of infinite descent may be described as follows. Mainly it is because a finite subset of natural numbers must have a smallest member. So if A is a subset of the natural numbers N, and if we need to prove, for every $a \in A$, the statement $P(a)$ is valid. Suppose by contradiction, the statement is not valid for all $a \in A$, i.e. there exists a non-empty subset of A, denoted by B, and such that $P(x)$ is not true for any $x \in B$. Now because B is non-empty, there exists a smallest element of B, denoted by b and such that $P(b)$ is not valid. Using the given conditions, if we can find a still smaller $c \in A$ ($c < b$), and such that $P(c)$ is not valid, then this will contradict the assumption of b. The conclusion is that $P(a)$ must be valid for all $a \in A$.

There are variations of this scenario. For instance, suppose there is a positive integer a_1 such that $P(a_1)$ is valid, and from this, if we can find a smaller positive integer a_2 such that $P(a_2)$ is valid, then we can find a still smaller positive integer a_3 such that $P(a_3)$ is valid, and so on. Hence we can find an infinite and decreasing chain of positive integers (infinite descent) $a_1 > a_2 > a_3 > \cdots$. This is clearly impossible. So the initial hypothesis $P(a_1)$ cannot be valid.

So the method of descent is essentially another form of induction. Recall that in mathematical induction, we start from a smallest element a of a subset of natural numbers, (initial step), and prove the so-called inductive step. So we can go from $P(a)$ to $P(a + 1)$, then $P(a + 2)$ and so on.

Many problems in mathematics competition require the uses of the method of descent. We give a few examples. First we use the method of infinite descent to prove the well-known result that $\sqrt{2}$ is irrational. Of course the classical proof is essentially a descent argument.

Example 1: Show that $\sqrt{2}$ is irrational.

Solution. We need to show that there do not exist positive integers x and y such that $x/y = \sqrt{2}$ or by taking squares, we need to show the equation $x^2 = 2y^2$ has no positive integer solution.

Suppose otherwise, let $x = m$, $y = n$ be a solution of the equation and such that m is the *smallest* possible value of x that satisfies the equation. Then $m^2 = 2n^2$ and this is possible only if m is even, hence $m=2m_1$. Thus, $4m_1^2 = (2m_1)^2=2n^2$, so $n^2 = 2m_1^2$. This implies n is also a possible value of x in the equation $x^2 = 2y^2$. However, $n < m$, contradicting the minimality of m.

Example 2 (Hungarian MO 2000): Find all positive primes p for which there exist positive integers x, y and n such that $p^n = x^3 + y^3$.

Solution. Observe $2^1 = 1^3 + 1^3$ and $3^2 = 2^3 + 1^3$. After many trials we found no more primes with this property. So we suspect the only answers are $p = 2$ or $p = 3$. Thus, we need to prove there exists no prime p ($p > 3$) satisfying $p^n = x^3 + y^3$. Clearly we need to prove by contradiction and one possibility is to make use of the descent method. (In this case we make descent on n and it works.)

So we assume $p^n = x^3 + y^3$ with x, y, n positive integers and n of the smallest possible value. Now $p \geq 5$. Hence at least one of x and y is greater than 1. Also

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2),$$

with $x + y \geq 3$ and

$$x^2 - xy + y^2 = (x-y)^2 + xy \geq 2.$$

Hence both $x + y$ and $x^2 - xy + y^2$ are divisible by p. Therefore

$$(x+y)^2 - (x^2 - xy + y^2) = 3xy$$

is also divisible by p. However, 3 is not divisible by p, so at least one of x or y must be divisible by p. As $x + y$ is divisible by p, both x and y are divisible by p. Then $x^3 + y^3 \geq 2p^3$. So we must have $n > 3$ and

$$p^{n-3} = \frac{p^n}{p^3} = \frac{x^3}{p^3} + \frac{y^3}{p^3} = \left(\frac{x}{p}\right)^3 + \left(\frac{y}{p}\right)^3.$$

This contradicts the minimality of n.

Example 3 (Putnam Exam 1973): Let $a_1, a_2, \cdots, a_{2n+1}$ be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.

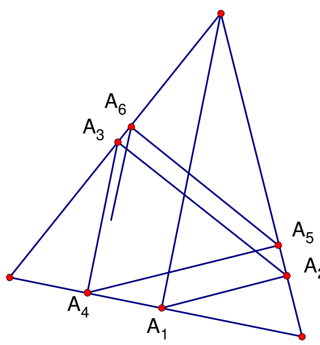
Solution. Assume $a_1 \leq a_2 \leq \cdots \leq a_{2n+1}$. By subtracting the smallest number from the sequence we observe the new sequence still maintain the property. So we may assume $a_1 = 0$. The sum of any $2n$ members equals 0 mod 2, so any two members must be of the same parity, (otherwise we may swap two members to form two groups of $2n$ elements which are of different parity). Therefore

$$0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod 2.$$

Dividing by 2, we note the new sequence will maintain the same property. Using the same reasoning we see that $0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod{2^2}$. We may descent to $0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod{2^m}$ for all $m \geq 1$. This is possible only if the initial numbers are all equal to others.

Example 4: Starting from a vertex of an acute triangle, the perpendicular is drawn, meeting the opposite side (side 1) at A_1. From A_1, a perpendicular is drawn to meet another side (side 2) at A_2. Starting from A_2, the perpendicular is drawn to meet the third side (side 3) at A_3. The perpendicular from A_3 is then drawn to meet side 1 at A_4 and then back to side 2, and so on.

Prove that the points $A_1, A_2, \ldots$ are all distinct.



Solution. First note that because the triangle is acute, all the points A_i, $i \geq 1$ lie on the sides of the triangle, instead of going outside or coincide with the vertices of the triangle. This implies A_i and A_{i+1} will not coincide because they lie on adjacent sides of the triangle. Suppose now A_i coincides with A_j ($i < j$), and i is the smallest index with this property. Then in fact $i = 1$. For otherwise A_{i-1} will coincide with A_{j-1}, contradicting the minimality of i. Finally suppose A_1 coincides with A_j, $j \geq 3$, this happens precisely when A_{j-1} is the vertex of the triangle facing side 1. But we know that no vertices of the triangle are in the list, so again impossible.

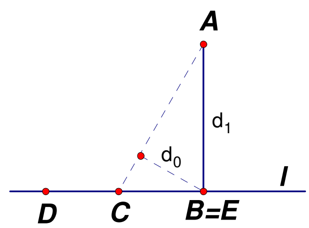
The following example was a problem of Sylvester (1814-1897). Accordingly Sylvester was annoyed to find that he was unable to tackle this deceptively simple problem. It was later solved by the technique of descent. The idea is to consider the smallest possible element with a certain property.

Example 5 (Sylvester's Problem): Given n ($n \geq 3$) points on the plane. If a line passing through any two points also passes through a third point of the set, then prove that all the points lie on the same line.

Solution. We prove an equivalent statement. Namely if there are n ($n \geq 3$) points on the plane and such that they are not on the same line, then there exists a line passing through exactly two points.

Now there are finitely many lines that may be formed by the points of the point set. Given such a line, there is at least one point of the set which does not lie on the line. We then consider the distance between the point and the line. Finally we list all such distances as $d_1 \leq d_2 \leq \cdots \leq d_m$, namely d_1 is the minimum distance between all possible points and all possible lines, say it is the distance between A and the line l. We now proceed to show that l contains exactly two points of the point set.

Suppose not, say points B, C and D of the point set also lie on l. From A, draw the line AE perpendicular to l, with E on l. If E is one of the B, C or D, say E and B coincide, we have the picture



Now $AB = d_1$. However if we draw a perpendicular line from B to AC, then we will get a distance d_0 less than d_1, contradicting its minimality. Similarly if E coincides with C or D, we can also obtain a smaller distance.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***December 10, 2005.***

Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point P, different from the center of the pizza and they do three straight cuts through P, which pairwise intersect at 60° and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara? (*Source: 2002 Slovenian National Math Olympiad*)

Problem 237. Determine (with proof) all polynomials p with real coefficients such that $p(x)\,p(x+1) = p(x^2)$ holds for every real number x.
(*Source*: *2000 Bulgarian Math Olympiad*)

Problem 238. For which positive integers n, does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of the numbers 1, 2, …, n such that the number $x_1 + x_2+ \cdots+x_k$ is divisible by k for every $k\in\{1,2, \ldots, n\}$?
(*Source: 1998 Nordic Mathematics Contest*)

Problem 239. (*Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) In any acute triangle ABC, prove that

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)$$
$$\le\frac{\sqrt{2}}{2}\left(\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}\right).$$

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct.

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Solutions

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Problem 231. On each planet of a star system, there is an astronomer observing the nearest planet. The number of planets is odd, and pairwise distances between them are different. Prove that at least one planet is not under observation.
(*Source*: *1966 Soviet Union Math Olympiad*)

Solution. **CHAN Pak Woon** (HKU Math, Year 1), **LEE Kai Seng** (HKUST), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Let there be n planets. The case of $n = 1$ is clear. For $n \ge 3$, suppose the case $n–2$ is true. For the two closest planets, the astronomers on them observe each other. If any of the remaining $n – 2$ astronomers observes one of these two planets, then we do not have enough astronomers to observe the $n – 2$ remaining planets. Otherwise, we can discard these two closest planets and apply the case $n – 2$.

Commended Solvers: **Roger CHAN** (Vancouver, Canada) and **Anna Ying PUN** (STFA Leung Kau Kui College).

Problem 232. B and C are points on the segment AD. If $AB = CD$, prove that $PA+PD \ge PB+PC$ for any point P.
(*Source*: *1966 Soviet Union Math Olympiad*)

Solution 1. **Anna Ying PUN** (STFA Leung Kau Kui College).

Suppose P is not on line AD. Let P' be such that $PAP'D$ is a parallelogram. Now $AB=CD$ implies $PBP'C$ is a parallelogram. By interchanging B and C, we may assume B is between A and C. Let line PB intersect AP' at F. Then $PA+PD = PA+AP' = PA+AF +FP' > PF + FP' = PB + BF + FP' > PB +BP' = PB + PC$. The case P is on line AD is easy to check.

Solution 2. **LEE Kai Seng** (HKUST).

Consider the complex plane with line AD as the real axis and the origin at the midpoint O of segment AD. Let the complex numbers correspond to A, B, P be a, b, p, respectively. Since $|p \pm a|^2 = |p|^2 \pm 2\text{Re}\, ap + a^2$, so $(PA + PD)^2 = 2(|p|^2 + |p^2–a^2| + a^2)$. Then

$$(PA + PD)^2 – (PB + PC)^2$$
$$= 2(|p^2–a^2| + a^2 – b^2 – |p^2–b^2|) \ge 0$$

by the triangle inequality. So $PA+PD \ge PB+PC$.

Also equality holds if and only if the ratio of $p^2 – a^2$ and $a^2 – b^2$ is a nonnegative number, which is the same as $p \ge a$ or $p \le –a$.

Commended Solvers: **CHAN Wai Hung** (Carmel Divine Grace Foundation Secondary School, Form 7), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Problem 233. Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 6), **G.R.A. 20 Math Problem Group** (Roma, Italy), **LEE Kai Seng** (HKUST) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

We prove by induction on n. The case $n = 1$ is clear. Suppose case $n – 1$ is true. For $n > 1$, let $1 \le k \le n!$ and let q and r be such that $k = qn + r$ with $0 \le r < n$. Then $0 \le q \le (n–1)!$. By the case $n –1$, q can be expressed as $d_1 + d_2 + \cdots + d_m$, where $m \le n – 1$ and d_i is a divisor of $(n – 1)!$ and d_i's are distinct. Omitting r if $r = 0$, we see $d_1n + d_2n + \cdots + d_mn + r$ is a desired expansion of k.

Problem 234. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

a) The coefficient of the highest power is 200.
b) The coefficient of the lowest power for which it is not equal to zero is 2.
c) The sum of all its coefficients is 4.
d) $P(−1) = 0$, $P(2) = 6$ and $P(3) = 8$.

(*Source: 2002 Austrian National Competition*)

Solution. **CHAN Pak Woon** (HKU Math, Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Note c) is the same as $P(1) = 4$. For

$$P(x)=200x(x+1)(x–1)(x–2)(x–3)+2x+2$$
$$=200x^5–1000x^4+1000x^3+1000x^2–1198x+2,$$

all conditions are satisfied. Assume R is another such polynomial with degree at most 5. Then P and R agree at –1, 1, 2, 3. So

$$P(x)–R(x) = (x+1)(x–1)(x–2)(x–3)S(x)$$

with degree of S at most 1. If S is constant, then b) implies $P(0)–R(0)$ is 0 or 2. Then $S(x) = –1/3$ and we get

$$R(x) = P(x)+(x+1)(x–1)(x–2)(x–3)/3$$
$$= 200x^5 + \cdots –1196⅓x,$$

which fails b). If S is of degree 1, then a) and b) imply $S(x)=200x–1/3$ and we will get $R(x)$ =

$$P(x) – (x+1)(x–1)(x–2)(x–3)(200x–1/3)$$
$$= x^4/3 + \cdots,$$

which fails a). So no such R exists and P is the unique answer.

Problem 235. Forty-nine students solve a set of three problems. The score for each problem is an integer from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.
(*Source: 29th IMO Unused Problem*)

Solution. **LEE Kai Seng** (HKUST) and **Anna Ying PUN** (STFA Leung Kau Kui College).

For n = 0,1,2,3, let S_n be the set of ordered pairs $(0,n),(1,n),\cdots,(7–n,n)$ and $(7 – n, n + 1),\cdots, (7 – n, 7)$. Let $S_4 = \{(x,y)\colon x=2 \text{ or } 3;\ y= 4,5,6 \text{ or } 7\}$ and $S_5 = \{(x,y)\colon x=0 \text{ or } 1;\ y=4,5,6 \text{ or } 7\}$.

For each student, let his/her score on the first problem be x and on the second problem be y. Note if two students have both of their (x,y) pairs in one of S_0, S_1, S_2 or S_3, then one of them will score at least as many point as the other in each of the first two problems.

Of the 49 pairs (x,y), there are $[49/6]+1 = 9$ of them belong to the same S_n. If this S_n is S_4 or S_5, which has 8 elements, then two of the 9 pairs are the same and the two students will satisfy the desired condition. If the S_n is S_0, S_1, S_2 or S_3, then two of these 9 students will have the same score on the third problem and they will satisfy the desired condition by the note in the last paragraph.

Commended Solvers: **CHAN Pak Woon** (HKU Math, Year 1), **LAW Yan Pui** (Carmel Divine Grace Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Olympiad Corner

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Problem 5. Let ABC be an acute triangle with orthocenter H, incenter I and $AC \neq BC$. The lines CH and CI meet the circumcircle of $\triangle ABC$ for the second time at points D and L, respectively. Prove that $\angle CIH = 90°$ if and only if $\angle IDL = 90°$.

Problem 6. In a group of nine people there are no four every two of which know each other. Prove that the group can be partitioned into four groups such that the people in every group do not know each other.

The Method of Infinite Descent

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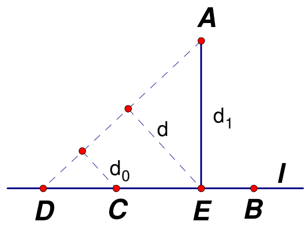
Now if the perpendicular from A to l does not meet any of B, C or D, then by the pigeonhole principle, there are two points (say C and D) which lie on one side of the perpendicular. Again from the diagram



We draw perpendiculars from E and C to AD, and we observe the distances $d_0 < d < d_1$, again contradicting the minimality of d_1. From the above arguments, we conclude that l contains exactly two points.

From the above example, we have

__Example 6 (Polish MO 1967-68)__: Given n ($n \geq 3$) points on the plane and these points are not on the same line. From any two of these points a line is drawn and altogether k distinct lines are formed. Show that $k \geq n$.

__**Solution.**__ We proceed by induction. Clearly three distinct lines may be drawn from three points not on a line. Hence the statement is true for n = 3. Suppose the statement is valid for some $n \geq 3$. Now let $A_1, A_2, \ldots, A_n, A_{n+1}$ be n + 1 distinct points which are not on the same line. By Sylvester's "theorem", there exists a line containing exactly two points of the point set, say A_1A_{n+1}.

Let's consider the sets $Z_1 = \{A_1, A_2, \ldots, A_n\}$ and $Z_2 = \{A_2, A_3, \ldots, A_n, A_{n+1}\}$ Clearly at least one of the point sets does not lie on a line. If $A_1, A_2, \ldots, A_n$ do not lie on a line, by the inductive hypothesis, we can form at least n lines using these points. As A_{n+1} is not one of the members of Z_1, so A_1A_{n+1} will form a new line, (A_1A_{n+1} contains no other points of the set) and we have at least n + 1 lines. If $A_2, A_3, \ldots, A_n, A_{n+1}$ do not lie on a line, then again we can form at least n lines using these points. As A_1 is not one of the members of Z_2, so A_1A_{n+1} will form a new line, (A_1A_{n+1} contains no other points of the set) and we have at least n + 1 lines.

The method of infinite descent was used to prove a hard IMO problem.

__Example 7 (IMO 1988)__: Prove that if positive integers a and b are such that $ab + 1$ divides $a^2 + b^2$, then $(a^2 + b^2)/(ab + 1)$ is a perfect square.

__**Solution.**__ Assume $(a^2 + b^2)/(ab + 1) = k$ and k is not a perfect square. After rearranging we have $a^2 – kab + b^2 = k$, with $a > 0$ and $b > 0$. Assume now (a_0, b_0) is a solution of the Diophantine equation and such that $a_0 + b_0$ is as *small* as possible. By symmetry we may assume $a_0 \geq b_0 > 0$. Fixing b_0 and k, we may assume a_0 is a solution of the quadratic equation

$$x^2 – kb_0x + b_0^2 – k = 0.$$

Now let the other root of the equation be a'. Using sum and product of roots, we have $a_0 + a' = kb_0$ and $a_0a' = b_0^2 – k$. The first equation implies a' is an integer. The second equation implies $a' \neq 0$, otherwise k is a perfect square, contradicting our hypothesis. Now a' also cannot be negative, otherwise

$$a'^2 – ka'b_0 + b_0^2 \geq a'^2 + k + b_0^2 > k.$$

Hence $a' > 0$. Finally

$$a' = \frac{b_0^2 - k}{a_0} \leq \frac{b_0^2 - 1}{a_0} \leq \frac{a_0^2 - 1}{a_0} < a_0.$$

This implies (a', b_0) is a positive integer solution of $a^2 – kab + b^2 = k$, and $a' + b_0 < a_0 + b_0$, contradicting the minimality of $a_0 + b_0$. Therefore k must be a perfect square.

Mathematical Excalibur

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Olympiad Corner

Below is the Czech-Polish-Slovak Match held in Zwardon on June 20-21, 2005.

Problem 1. Let n be a given positive integer. Solve the system of equations

$$x_1 + x_2^2 + x_3^3 + \cdots + x_n^n = n,$$
$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = \frac{n(n+1)}{2}$$

in the set of nonnegative real numbers $x_1, x_2, \ldots, x_n$.

Problem 2. Let a convex quadrilateral $ABCD$ be inscribed in a circle with center O and circumscribed to a circle with center I, and let its diagonals AC and BD meet at a point P. Prove that the points O, I and P are collinear.

Problem 3. Determine all integers $n \geq 3$ such that the polynomial $W(x) = x^n - 3x^{n-1} + 2x^{n-2} + 6$ can be expressed as a product of two polynomials with positive degrees and integer coefficients.

Problem 4. We distribute $n \geq 1$ labelled balls among nine persons $A, B, C, D, E, F, G, H, I$. Determine in how many ways

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 12, 2006***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Using Tangent Lines to Prove Inequalities

Kin-Yin Li

For students who know calculus, sometimes they become frustrated in solving inequality problems when they do not see any way of using calculus. Below we will give some examples, where finding the equation of a tangent line is the critical step to solving the problems.

Example 1. Let a,b,c,d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$6(a^3+b^3+c^3+d^3) \geq (a^2+b^2+c^2+d^2) + 1/8.$$

Solution. We have $0 < a, b, c, d < 1$. Let $f(x) = 6x^3 - x^2$. (Note: Since there is equality when $a = b = c = d = 1/4$, we consider the graph of $f(x)$ and its tangent line at $x = 1/4$. By a simple sketch, it seems the tangent line is below the graph of $f(x)$ on the interval (0,1). Now the equation of the tangent line at $x = 1/4$ is $y = (5x - 1)/8$.) So we claim that for $0 < x < 1$, $f(x) = 6x^3 - x^2 \geq (5x - 1)/8$. This is equivalent to $48x^3 - 8x^2 - 5x + 1 \geq 0$. (Note: Since the graphs intersect at $x = 1/4$, we expect $4x - 1$ is a factor.) Indeed, $48x^3 - 8x^2 - 5x + 1 = (4x - 1)^2(3x + 1) \geq 0$ for $0 < x < 1$. So the claim is true. Then $f(a) + f(b) + f(c) + f(d) \geq 5(a + b + c + d)/8 - 4/8 = 1/8$, which is equivalent to the required inequality.

Example 2. (*2003 USA Math Olympiad*) Let a,b,c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8.$$

Solution. Setting $a' = a/(a + b + c)$, $b' = b/(a + b + c)$, $c' = c/(a + b + c)$ if necessary, we may assume $0 < a, b, c < 1$ and $a + b + c = 1$. Then the first term on the left side of the inequality is equal to

$$f(a)=\frac{(a+1)^2}{2a^2+(1-a)^2}=\frac{a^2+2a+1}{3a^2-2a+1}.$$

(Note: When $a = b = c = 1/3$, there is equality. A simple sketch of $f(x)$ on [0,1] shows the curve is below the tangent line at $x = 1/3$, which has the equation $y = (12x + 4)/3$.) So we claim that

$$\frac{a^2+2a+1}{3a^2-2a+1}\leq\frac{12a+4}{3}$$

for $0 < a < 1$. Multiplying out, we see this is equivalent to $36a^3 - 15a^2 - 2a + 1 \geq 0$ for $0 < a < 1$. (Note: Since the curve and the line intersect at $a = 1/3$, we expect $3a-1$ is a factor.) Indeed, $36a^3 - 15a^2 - 2a + 1 = (3a - 1)^2(4a + 1) \geq 0$ for $0 < a < 1$. Finally adding the similar inequality for b and c, we get the desired inequality.

The next example looks like the last example. However, it is much more sophisticated, especially without using tangent lines. The solution below is due to Titu Andreescu and Gabriel Dospinescu.

Example 3. (*1997 Japanese Math Olympiad*) Let a,b,c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Solution. As in the last example, we may assume $0 < a, b, c < 1$ and $a + b + c = 1$. Then the first term on the left become $\frac{(1-2a)^2}{(1-a)^2+a^2}=2-\frac{2}{1+(1-2a)^2}$.

Next, let $x_1 = 1 - 2a$, $x_2 = 1 - 2b$, $x_3 = 1 - 2c$, then $x_1 + x_2 + x_3 = 1$, but $-1 < x_1, x_2, x_3 < 1$. In terms of x_1, x_2, x_3, the desired inequality is

$$\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\frac{1}{1+x_3^2}\leq\frac{27}{10}.$$

(Note: As in the last example, we consider the equation of the tangent line to $f(x) = 1/(1 + x^2)$ at $x = 1/3$, which is $y = 27(-x + 2)/50$.) So we claim that $f(x) \leq 27(-x + 2)/50$ for $-1 < x < 1$. This is equivalent to $(3x - 1)^2(4 - 3x) \geq 0$. Hence the claim is true for $-1 < x < 1$. Then $f(x_1) + f(x_2) + f(x_3) \leq 27/10$ and the desired inequality follows.

Schur's Inequality

Kin Yin Li

Sometimes in proving an inequality, we do not see any easy way. It will be good to know some brute force methods in such situation. In this article, we introduce a simple inequality that turns out to be very critical in proving inequalities by brute force.

Schur's Inequality. For any $x, y, z \geq 0$ and any real number r,

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \geq 0.$$

Equality holds if and only if $x = y = z$ or two of x, y, z are equal and the third is zero.

Proof. Observe that the inequality is symmetric in x, y, z. So without loss of generality, we may assume $x \geq y \geq z$. We can rewrite the left hand side as $x^r(x-y)^2+(x^r-y^r+z^r)(x-y)(y-z)+z^r(y-z)^2$. The first and third terms are clearly nonnegative. For the second term, if $r \geq 0$, then $x^r \geq y^r$. If $r < 0$, then $z^r \geq y^r$. Hence, $x^r-y^r+z^r \geq 0$ and the second term is nonnegative. So the sum of all three terms is nonnegative. In case $x \geq y \geq z$, equality holds if and only if $x = y$ first and z equals to them or zero.

In using the Schur's inequality, we often expand out expressions. So to simplify writing, we introduce the *symmetric sum* notation $\sum_{sym} f(x,y,z)$ to denote the sum of the six terms $f(x,y,z)$, $f(x,z,y)$, $f(y,z,x)$, $f(y,x,z)$, $f(z,x,y)$ and $f(z,y,x)$. In particular,

$$\sum_{sym} x^3 = 2x^3 + 2y^3 + 2z^3,$$

$$\sum_{sym} x^2y = x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \text{ and}$$

$$\sum_{sym} xyz = 6xyz.$$

Similarly, for a function of n variables, the symmetric sum is the sum of all $n!$ terms, where we take all possible permutations of the n variables.

The $r = 1$ case of Schur's inequality is $x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) = x^3 + y^3 + z^3 - (x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 3xyz \geq 0$. In symmetric sum notation, it is

$$\sum_{sym}(x^3 - 2x^2y + xyz) \geq 0.$$

By expanding both sides and rearranging terms, each of the following inequalities is equivalent to the $r = 1$ case of Schur's inequality. These are common disguises.

a) $x^3+y^3+z^3+3xyz \geq xy(x+y)+yz(y+z)+zx(z+x)$,

b) $xyz \geq (x+y-z)(y+z-x)(z+x-y)$,

c) $4(x+y+z)(xy+yz+zx) \leq (x+y+z)^3+9xyz$.

Example 1. (*2000 IMO*) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$(a-1+\frac{1}{b})(b-1+\frac{1}{c})(c-1+\frac{1}{a}) \leq 1.$$

Solution. Let $x = a$, $y = 1$, $z = 1/b = ac$. Then $a = x/y$, $b = y/z$ and $c = z/x$. Substituting these into the desired inequality, we get

$$\frac{(x-y+z)}{y}\frac{(y-z+x)}{z}\frac{(z-x+y)}{x} \leq 1,$$

which is disguise *b*) of the $r = 1$ case of Schur's inequality.

Example 2. (*1984 IMO*) Prove that

$$0 \leq yz + zx + xy - 2xyz \leq 7/27,$$

where x, y, z are nonnegative real numbers such that $x + y + z = 1$.

Solution. In Schur's inequality, all terms are of the same degree. So we first change the desired inequality to one where all terms are of the same degree. Since $x + y + z = 1$, the desired inequality is the same as

$$0 \leq (x+y+z)(yz+zx+xy) - 2xyz \leq \frac{7(x+y+z)^3}{27}.$$

Expanding the middle expression, we get $xyz + \sum_{sym} x^2y$, which is clearly nonnegative and the left inequality is proved. Expanding the rightmost expression and subtracting the middle expression, we get

$$\frac{7}{54}\sum_{sym}(x^3 - \frac{12}{7}x^2y + \frac{5}{7}xyz). \qquad (1)$$

By Schur's inequality, we have

$$\sum_{sym}(x^3 - 2x^2y + xyz) \geq 0. \qquad (2)$$

By the AM-GM inequality, we have

$$\sum_{sym} x^2y \geq 6(x^6y^6z^6)^{1/6} = \sum_{sym} xyz,$$

which is the same as

$$\sum_{sym}(x^2y - xyz) \geq 0. \qquad (3)$$

Multiplying (3) by 2/7 and adding it to (2), we see the symmetric sum in (1) is nonnegative. So the right inequality is proved.

Example 3. (*2004 APMO*) Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

for any positive real numbers a,b,c.

Solution. Expanding and expressing in symmetric sum notation, the desired inequality is

$$(abc)^2 + \sum_{sym}(a^2b^2+2a^2)+8 \geq \frac{9}{2}\sum_{sym} ab.$$

As $a^2+b^2 \geq 2ab$, we get $\sum_{sym} a^2 \geq \sum_{sym} ab$.

As $a^2b^2 + 1 \geq 2ab$, we get

$$\sum_{sym} a^2b^2 + 6 \geq 2\sum_{sym} ab.$$

Using these, the problem is reduced to showing

$$(abc)^2 + 2 \geq \sum_{sym}(ab - \frac{1}{2}a^2).$$

To prove this, we apply the AM-GM inequality twice and disguise c) of the $r = 1$ case of Schur's inequality as follow:

$$\begin{aligned}(abc)^2 + 2 &\geq 3(abc)^{2/3} \\ &\geq 9abc/(a+b+c) \\ &\geq 4(ab+bc+ca) - (a+b+c)^2 \\ &= 2(ab+bc+ca) - (a^2+b^2+c^2) \\ &= \sum_{sym}(ab - \frac{1}{2}a^2).\end{aligned}$$

Example 4. (*2000 USA Team Selection Test*) Prove that for any positive real numbers a, b, c, the following inequality holds

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***February 12, 2006.***

Problem 241. Determine the smallest possible value of

$$S = a_1 \cdot a_2 \cdot a_3 + b_1 \cdot b_2 \cdot b_3 + c_1 \cdot c_2 \cdot c_3,$$

if a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (*Source: 2002 Belarussian Math. Olympiad*)

Problem 242. Prove that for every positive integer n, 7 is a divisor of $3^n + n^3$ if and only if 7 is a divisor of $3^n n^3 + 1$. (*Source: 1995 Bulgarian Winter Math Competition*)

Problem 243. Let R^+ be the set of all positive real numbers. Prove that there is no function $f: R^+ \to R^+$ such that

$$(f(x))^2 \geq f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y. (*Source: 1998 Bulgarian Math Olympiad*)

Problem 244. An infinite set S of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1cm from each other. Does there exist any division of S into two disjoint infinite subsets R and B such that inside every triangle with vertices in R is at least one point of B and inside every triangle with vertices in B is at least one point of R? Give a proof to your answer. (*Source: 2002 Albanian Math Olympiad*)

Problem 245. $ABCD$ is a concave quadrilateral such that $\angle BAD = \angle ABC = \angle CDA = 45^\circ$. Prove that $AC = BD$.

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Solutions

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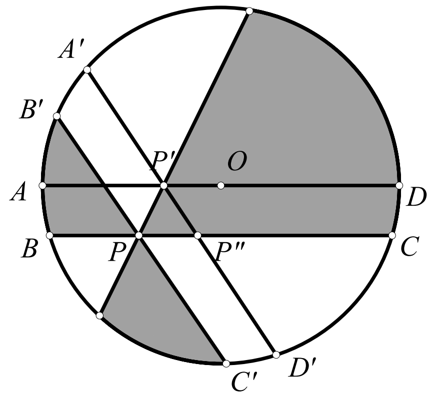
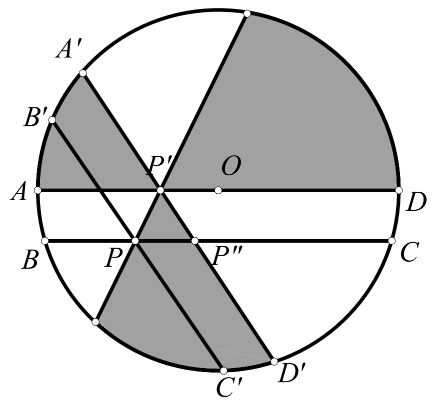
Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point P, different from the center of the pizza and they do three straight cuts through P, which pairwise intersect at 60˚ and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara? (*Source: 2002 Slovenian National Math Olympiad*)

Solution. (Official Solution)

Let Alice choose the piece that contains the center of the pizza first. We claim that the total area of the shaded regions below is greater than half of the area of the pizza.



Without loss of generality, we can assume the center of the pizza is at the origin O and one of the cuts is parallel to the x-axis (that is, BC is parallel to AD in the picture). Let P' be the intersection of the x-axis and the 60˚-cut. Let $A'D'$ be parallel to the 120˚-cut $B'C'$. Let P'' be the intersection of BC and $A'D'$. Then $\Delta PP'P''$ is equilateral. This implies the belts $ABCD$ and $A'B'C'D'$ have equal width. Since $AD > A'D'$, the area of the belt $ABCD$ is greater than the area of the belt $A'B'C'D'$. Now when the area of the belt $ABCD$ is subtracted from the total area of the shaded regions and the area of $A'B'C'D'$ is then added,



we get exactly half the area of the pizza. Therefore, the claim follows.

Problem 237. Determine (with proof) all polynomials p with real coefficients such that $p(x)\,p(x+1) = p(x^2)$ holds for every real number x. (*Source: 2000 Bulgarian Math Olympiad*)

Solution. **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Let $p(x)$ be such a polynomial. In case $p(x)$ is a constant polynomial, $p(x)$ must be 0 or 1. For the case $p(x)$ is nonconstant, let r be a root of $p(x)$. Then setting $x = r$ and $x + 1 = r$ in the equation, we see r^2 and $(r-1)^2$ are also roots of $p(x)$. Also, r^2 is a root implies $(r^2-1)^2$ is also a root. If $0 < |r| < 1$ or $|r| > 1$, then $p(x)$ will have infinitely many roots r, r^2, r^4, …, a contradiction. So $|r| = 0$ or 1 for every root r.

The case $|r| = 1$ and $|r-1| = 1$ lead to $r = (1 \pm i\sqrt{3})/2$, but then $|r^2 - 1| \neq 0$ or 1, a contradiction. Hence, either $|r| = 0$ or $|r-1| = 0$, that is, $r = 0$ or 1.

So $p(x) = x^m(x-1)^n$ for some nonnegative integers m, n. Putting this into the equation, we find $m = n$. Conversely, $p(x) = x^m(x-1)^m$ is easily checked to be a solution for every nonnegative integer m.

Problem 238. For which positive integers n, does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of the numbers 1, 2, …, n such that the number $x_1 + x_2 + \cdots + x_k$ is divisible by k for every $k \in \{1,2, \ldots, n\}$? (*Source: 1998 Nordic Mathematics Contest*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy), **LEE Kai Seng** (HKUST), **LO Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 7), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

For a solution n, since $x_1 + x_2 + \cdots + x_n = n(n+1)/2$ is divisible by n, n must be odd. The cases $n = 1$ and $n = 3$ (with permutation (1,3,2)) are solutions.

Assume $n \geq 5$. Then $x_1 + x_2 + \cdots + x_{n-1} = n(n+1)/2 - x_n \equiv 0 \pmod{n-1}$ implies $x_n \equiv (n+1)/2 \pmod{n-1}$. Since $1 \leq x_n \leq n$ and $3 \leq (n+1)/2 \leq n-2$, we get $x_n = (n+1)/2$. Similarly, $x_1 + x_2 + \cdots + x_{n-2} = n(n+1)/2 - x_n - x_{n-1} \equiv 0 \pmod{n-2}$ implies $x_{n-1} \equiv (n+1)/2 \pmod{n-2}$. Then also $x_{n-1} = (n+1)/2$, which leads to $x_n = x_{n-1}$, a contradiction. Therefore, $n = 1$ and 3 are the only solutions.

Problem 239. (*Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) In any acute triangle ABC, prove that

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)$$
$$\le\frac{\sqrt{2}}{2}\left(\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}\right).$$

Solution. (Proposer's Solution)

By cosine law and the *AM-GM* inequality,

$$1-2\sin^2\frac{A}{2}=\cos A=\frac{b^2+c^2-a^2}{2bc}$$
$$\ge\frac{b^2+c^2-a^2}{b^2+c^2}=1-\frac{a^2}{b^2+c^2}.$$

So $\sin\frac{A}{2}\le\frac{a}{\sqrt{2(b^2+c^2)}}$.

By sine law and $\cos(A/2) = \sin((B+C)/2)$, we get

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=$$

$$\frac{2\sin(A/2)\cos(A/2)}{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}=\frac{\sin(A/2)}{\cos(\frac{B-C}{2})}.$$

Then

$$\cos(\frac{B-C}{2})=\frac{b+c}{a}\sin\frac{A}{2}\le\frac{\sqrt{2}}{2}\frac{b+c}{\sqrt{b^2+c^2}}.$$

Adding two similar inequalities, we get the desired inequality.

Commended solvers: **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct. (*Source: 1968 All Soviet Union Math Competitions*)

Solution. **WONG Kwok Kit** (Carmel Divine Grace Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Suppose the 9 first places go to the same figure skater. Then 9 is the lowest sum.

Suppose the 9 first places are shared by two figure skaters. Then one of them gets at least 5 first places and that skater's other rankings are no worse than fourth places. So the lowest sum is at most $5 \times 1 + 4 \times 4 = 21$.

Suppose the 9 first places are shared by three figure skaters. Then the other 18 rankings of these figure skaters are no worse than 9 third and 9 fourth places. Then the lowest sum is at most $9(1 + 3 + 4)/3 = 24$.

Suppose the 9 first places are shared by four figure skaters. Then their rankings must be all the first, second, third and fourth places. So the lowest sum is at most $9(1 + 2 + 3 + 4)/4 < 24$.

Suppose the 9 first places are shared by $k > 4$ figure skaters. On one hand, these k skaters have a total of $9k > 36$ rankings. On the other hand, these k skaters can only be awarded first to fourth places, so they can have at most $4 \times 9 = 36$ rankings all together, a contradiction.

Now 24 is possible if skaters *A, B, C* all received 3 first, 3 third and 3 fourth places; skater *D* received 5 second and 4 fifth places; skater *E* received 4 second and 5 fifth places; and skater *F* received 9 sixth places, …, skater *T* received 9 twentieth places. Therefore, 24 is the answer.

Olympiad Corner

(continued from page 1)

Problem 4. (Cont.) it is possible to distribute the balls under the condition that A gets the same number of balls as the persons *B, C, D* and *E* together.

Problem 5. Let $ABCD$ be a given convex quadrilateral. Determine the locus of the point P lying inside the quadrilateral $ABCD$ and satisfying

$$[PAB]\cdot[PCD] = [PBC]\cdot[PDA],$$

where $[XYZ]$ denotes the area of triangle XYZ.

Problem 6. Determine all pairs of integers (x,y) satisfying the equation

$$y(x + y) = x^3 - 7x^2 + 11x - 3.$$

Schur's Inequality

(continued from page 2)

Solution. From the last part of the solution of example 3, we get

$$3(xyz)^{2/3} \ge 2(xy + yz + zx) - (x^2 + y^2 + z^2)$$

for any $x, y, z > 0$. (Note: this used Schur's inequality.) Setting

$x = \sqrt{a}$, $y = \sqrt{b}$ and $z = \sqrt{c}$

and arranging terms, we get

$$a+b+c-3\sqrt[3]{abc}$$
$$\le 2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca})$$
$$=(\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2$$
$$\le 3\max\{(\sqrt{a}-\sqrt{b})^2,(\sqrt{b}-\sqrt{c})^2,(\sqrt{c}-\sqrt{a})^2\}.$$

Dividing by 3, we get the desired inequality.

Example 5. (*2003 USA Team Selection Test*) Let a,b,c be real numbers in the interval $(0, \pi/2)$. Prove that

$$\frac{\sin a\sin(a-b)\sin(a-c)}{\sin(b+c)}+\frac{\sin b\sin(b-c)\sin(b-a)}{\sin(c+a)}$$

$$+\frac{\sin c\sin(c-a)\sin(c-b)}{\sin(a+b)}\ge 0.$$

Solution. Observe that

$$\sin(u - v)\sin(u + v) = (\cos 2v - \cos 2u)/2$$
$$= \sin^2 u - \sin^2 v.$$

Setting $x = \sin^2 a$, $y = \sin^2 b$, $z = \sin^2 c$, in adding up the terms, the left side of the inequality becomes

$$\frac{\sqrt{x}(x-y)(x-z)+\sqrt{y}(y-z)(y-x)+\sqrt{z}(z-x)(z-y)}{\sin(b+c)\sin(c+a)\sin(a+b)}.$$

This is nonnegative by the $r = 1/2$ case of Schur's inequality.

For many more examples on Schur's and other inequalities, we highly recommend the following book.

Titu Andreescu, Vasile Cîrtoaje, Gabriel Dospinescu and Mircea Lascu, Old and New Inequalities, GIL Publishing House, 2004.

Anyone interested may contact the publisher by post to GIL Publishing House, P. O. Box 44, Post Office 3, 450200, Zalau, Romania or by email to gil1993@zalau.astral.ro.

Mathematical Excalibur

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Olympiad Corner

Below was Slovenia's Selection Examinations for the IMO 2005.

First Selection Examination

Problem 1. Let M be the intersection of diagonals AC and BD of the convex quadrilateral $ABCD$. The bisector of angle ACD meets the ray BA at the point K. Prove that if $MA\cdot MC + MA\cdot CD = MB\cdot MD$, then $\angle BKC = \angle BDC$.

Problem 2. Let R_+ be the set of all positive real numbers. Find all functions $f\colon R_+\to R_+$ such that $x^2\,(f(x)+f(y)) = (x+y)\,f(f(x)\,y)$ holds for any positive real numbers x and y.

Problem 3. Find all pairs of positive integers (m, n) such that the numbers m^2-4n and n^2-4m are perfect squares.

Second Selection Examination

Problem 1. How many sequences of 2005 terms are there such that the following three conditions hold:

(a) no sequence has three consecutive terms equal to each other,
(b) every term of every sequence is equal to 1 or −1, and

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 16, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Muirhead's Inequality

Lau Chi Hin

Muirhead's inequality is an important generalization of the AM-GM inequality. It is a powerful tool for solving inequality problem. First we give a definition which is a generalization of arithmetic and geometric means.

Definition. Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$. The *p-mean* of $x_1, x_2, \ldots, x_n$ is defined by

$$[p] = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}^{p_1}x_{\sigma(2)}^{p_2}\cdots x_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of $\{1,2,\ldots,n\}$. (The summation sign means to sum $n!$ terms, one term for each permutation σ in S_n.)

For example, $[(1,0,\ldots,0)] = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the arithmetic mean of $x_1, x_2, \ldots, x_n$ and $[(1/n,1/n,\ldots,1/n)] = x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}$ is their geometric mean.

Next we introduce the concept of majorization in $\mathbb{R}^n$. Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ satisfy conditions

1. $p_1 \ge p_2 \ge \cdots \ge p_n$ and $q_1 \ge q_2 \ge \cdots \ge q_n$,
2. $p_1 \ge q_1$, $p_1+p_2 \ge q_1+q_2$, … , $p_1+p_2+\cdots+p_{n-1} \ge q_1+q_2+\cdots+q_{n-1}$ and
3. $p_1+p_2+\cdots+p_n = q_1+q_2+\cdots+q_n$.

Then we say $(p_1, p_2, \ldots, p_n)$ *majorizes* $(q_1, q_2, \ldots, q_n)$ and write

$$(p_1, p_2, \ldots, p_n) \succ (q_1, q_2, \ldots, q_n).$$

Theorem (Muirhead's Inequality). Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p, q \in \mathbb{R}^n$. If $p \succ q$, then $[p] \ge [q]$. Furthermore, for $p \ne q$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Since $(1,0,\ldots,0) \succ (1/n,1/n,\ldots,1/n)$, AM-GM inequality is a consequence.

Example 1. For any $a, b, c > 0$, prove that

$$(a+b)(b+c)(c+a) \ge 8abc.$$

Solution. Expanding both sides, the desired inequality is

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b \ge 6abc.$$

This is equivalent to $[(2,1,0)] \ge [(1,1,1)]$, which is true by Muirhead's inequality since $(2,1,0)\succ(1,1,1)$.

For the next example, we would like to point out a useful trick. When the product of $x_1, x_2, \ldots, x_n$ is 1, we have

$$[(p_1, p_2, \ldots, p_n)] = [(p_1-r, p_2-r, \ldots, p_n-r)]$$

for any real number r.

Example 2. (*IMO 1995*) For any $a, b, c > 0$ with $abc = 1$, prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is

$$\begin{aligned}&2(a^4b^4+b^4c^4+c^4a^4)\\&+2(a^4b^3c+a^4c^3b+b^4c^3a+b^4a^3c+c^4a^3b\\&+c^4b^3a)+2(a^3b^3c^2+b^3c^3a^2+c^3a^3b^2)\\&\ge 3(a^5b^4c^3+a^5c^4b^3+b^5c^4a^3+b^5a^4c^3\\&+c^5a^4b^3+c^5b^4a^3)+6a^4b^4c^4.\end{aligned}$$

This is equivalent to $[(4,4,0)]+2[(4,3,1)] + [(3,3,2)] \ge 3[(5,4,3)] + [(4,4,4)]$. Note $4+4+0 = 4+3+1 = 3+3+2 = 8$, but $5+4+3 = 4+4+4 = 12$. So we can set $r = 4/3$ and use the trick above to get $[(5,4,3)] = [(11/3,8/3,5/3)]$ and also $[(4,4,4)] = [(8/3,8/3,8/3)]$.

Observe that $(4,4,0) \succ (11/3,8/3,5/3)$, $(4,3,1) \succ (11/3,8/3,5/3)$ and $(3,3,2) \succ (8/3,8/3,8/3)$. So applying Muirhead's inequality to these three majorizations and adding the inequalities, we get the desired inequality.

Example 3. (*1998 IMO Shortlisted Problem*) For any $x, y, z > 0$ with $xyz = 1$, prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is

$$4(x^4+y^4+z^4+x^3+y^3+z^3)$$
$$\geq 3(1+x+y+z+xy+yz+zx+xyz).$$

This is equivalent to $4[(4,0,0)] + 4[(3,0,0)] \geq [(0,0,0)] + 3[(1,0,0)] + 3[(1,1,0)] + [(1,1,1)]$.

For this, we apply Muirhead's inequality and the trick as follow:

$[(4,0,0)] \geq [(4/3,4/3,4/3)] = [(0,0,0)]$,
$3[(4,0,0)] \geq 3[(2,1,1)] = 3[(1,0,0)]$,
$3[(3,0,0)] \geq 3[(4/3,4/3,1/3)] = 3[(1,1,0)]$
and $[(3,0,0)] \geq [(1,1,1)]$.

Adding these, we get the desired inequality.

Remark. For the following example, we will modify the trick above. In case $xyz \geq 1$, we have

$$[(p_1, p_2, p_3)] \geq [(p_1-r, p_2-r, p_3-r)]$$

for every $r \geq 0$. Also, we will use the following

Fact. For $p, q \in \mathbb{R}^n$, we have

$$\frac{[p]+[q]}{2}\geq\left[\frac{p+q}{2}\right].$$

This is because by the AM-GM inequality,

$$\frac{x_{\sigma(1)}^{p_1}\cdots x_{\sigma(n)}^{p_n}+x_{\sigma(1)}^{q_1}\cdots x_{\sigma(n)}^{q_n}}{2}$$
$$\geq x_{\sigma(1)}^{(p_1+q_1)/2}\cdots x_{\sigma(n)}^{(p_n+q_n)/2}.$$

Summing over $\sigma\in S_n$ and dividing by $n!$, we get the inequality.

Example 4. (*2005 IMO*) For any $x, y, z > 0$ with $xyz \geq 1$, prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is equivalent to $[(9,0,0)]+4[(7,5,0)]+[(5,2,2)]+[(5,5,5)] \geq [(6,0,0)] + [(5,5,2)] + 2[(5,4,0)] + 2[(4,2,0)] + [(2,2,2)]$.

To prove this, we note that

(1) $[(9,0,0)] \geq [(7,1,1)] \geq [(6,0,0)]$

(2) $[(7,5,0)] \geq [(5,5,2)]$

(3) $2[(7,5,0)] \geq 2[(6,5,1)] \geq 2[(5,4,0)]$

(4) $[(7,5,0)] + [(5,2,2)] \geq 2[(6,7/2,1)]$
$\geq 2[(11/2,7/2,3/2)] \geq 2[(4,2,0)]$

(5) $[(5,5,5)] \geq [(2,2,2)]$,

where (1) and (3) are by Muirhead's inequality and the remark, (2) is by Muirhead's inequality, (4) is by the fact, Muirhead's inequality and the remark and (5) is by the remark.

Considering the sum of the leftmost parts of these inequalities is greater than or equal to the sum of the rightmost parts of these inequalities, we get the desired inequalities.

__*Alternate Solution.*__ Since

$$\frac{x^5-x^2}{x^5+y^2+z^2}-\frac{x^5-x^2}{x^3(x^2+y^2+z^2)}$$
$$=\frac{(x^3-1)^2(y^2+z^2)}{x(x^2+y^2+z^2)(x^5+y^2+z^2)}\geq 0,$$

we have

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}$$
$$\geq\frac{x^5-x^2}{x^3(x^2+y^2+z^2)}+\frac{y^5-y^2}{y^3(y^2+z^2+x^2)}+\frac{z^5-z^2}{z^3(z^2+x^2+y^2)}$$
$$\geq\frac{1}{x^2+y^2+z^2}(x^2-\frac{1}{x}+y^2-\frac{1}{y}+z^2-\frac{1}{z})$$
$$\geq\frac{1}{x^2+y^2+z^2}(x^2+y^2+z^2-yz-zx-xy)$$
$$=\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2(x^2+y^2+z^2)}\geq 0.$$

Proofs of Muirhead's Inequality

Kin Yin Li

Let $p \succ q$ and $p \neq q$. From $i = 1$ to n, the first nonzero $p_i - q_i$ is positive by condition 2 of majorization. Then there is a negative $p_i - q_i$ later by condition 3. It follows that there are $j < k$ such that $p_j > q_j$, $p_k < q_k$ and $p_i = q_i$ for any possible i between j, k.

Let $b = (p_j+p_k)/2$, $d = (p_j-p_k)/2$ so that $[b-d,b+d] = [p_k, p_j] \supset [q_k, q_j]$. Let c be the maximum of $|q_j-b|$ and $|q_k-b|$, then $0 \leq c < d$. Let $r = (r_1,\ldots,r_n)$ be defined by $r_i = p_i$ except $r_j = b + c$ and $r_k = b - c$. By the definition of c, either $r_j = q_j$ or $r_k=q_k$. Also, by the definitions of b, c, d, we get $p \succ r$, $p \neq r$ and $r \succ q$. Now

$$n!([p]-[r])=\sum_{\sigma\in S_n}x_\sigma(x_{\sigma(j)}^{p_j}x_{\sigma(k)}^{p_k}-x_{\sigma(j)}^{r_j}x_{\sigma(k)}^{r_k})$$
$$=\sum_{\sigma\in S_n}x_\sigma(u^{b+d}v^{b-d}-u^{b+c}v^{b-c}),$$

where x_σ is the product of $x_{\sigma(i)}^{p_i}$ for $i \neq j$, k and $u = x_{\sigma(j)}$, $v = x_{\sigma(k)}$. For each permutation σ, there is a permutation ρ such that $\sigma(i) = \rho(i)$ for $i \neq j$, k and $\sigma(j) = \rho(k)$, $\sigma(k) = \rho(j)$. In the above sum, if we pair the terms for σ and ρ, then $x_\sigma = x_\rho$ and combining the parenthetical factors for the σ and ρ terms, we have

$$(u^{b+d}v^{b-d}-u^{b+c}v^{b-c})+(v^{b+d}u^{b-d}-v^{b+c}u^{b-c})$$
$$=u^{b-d}v^{b-d}(u^{d+c}-v^{d+c})(u^{d-c}-v^{d-c})\geq 0.$$

So the above sum is nonnegative. Then $[p] \geq [r]$. Equality holds if and only if $u = v$ for all pairs of σ and ρ, which yields $x_1 = x_2 = \cdots = x_n$. Finally we recall r has at least one more coordinate in agreement with q than p. So repeating this process finitely many times, we will eventually get the case $r = q$. Then we are done.

Next, __*for the advanced readers*__, we will outline a longer proof, which tells more of the story. It is consisted of two steps. The first step is to observe that if $c_1, c_2, \ldots, c_k \geq 0$ with sum equals 1 and $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, then

$$\sum_{i=1}^{k}c_i[v_i]\geq\left[\sum_{i=1}^{k}c_iv_i\right].$$

This follows by using the weighted AM-GM inequality instead in the proof of the fact above. (For the statement of the weighted AM-GM inequality, see *Mathematical Excalibur*, *vol.* 5, *no.* 4, *p.* 2, remark in column 1).

The second step is the difficult step of showing $p \succ q$ implies there exist nonnegative numbers $c_1, c_2, \ldots, c_{n!}$ with sum equals 1 such that

$$q=\sum_{i=1}^{n!}c_iP_i,$$

where $P_1, P_2, \ldots, P_{n!} \in \mathbb{R}^n$ whose coordinates are the $n!$ permutations of the coordinates of p. Muirhead's inequality follows immediately by applying the first step and observing that $[P_i]=[p]$ for $i=1,2,\ldots, n!$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***April 16, 2006.***

Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center A and radius 10 kilometers. A rocket is fired from A at the same speed as the spy plane such that it is always on the radius from A to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane. (*Source: 1965 Soviet Union Math Olympiad*)

Problem 247. (*a*) Find all possible positive integers $k \ge 3$ such that there are k positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(*b*) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (*a*) above.
(*Source: 2003 Belarussian Math Olympiad*)

Problem 248. Let $ABCD$ be a convex quadrilateral such that line CD is tangent to the circle with side AB as diameter. Prove that line AB is tangent to the circle with side CD as diameter if and only if lines BC and AD are parallel.

Problem 249. For a positive integer n, if $a_1,\cdots, a_n, b_1, \cdots, b_n$ are in [1,2] and $a_1^2+\cdots+a_n^2=b_1^2+\cdots+b_n^2$, then prove that

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n}\le\frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

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Solutions

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Problem 241. Determine the smallest possible value of

$$S = a_1 \cdot a_2 \cdot a_3 + b_1 \cdot b_2 \cdot b_3 + c_1 \cdot c_2 \cdot c_3,$$

if a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (*Source: 2002 Belarussian Math. Olympiad*)

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College), **CHAN Tsz Lung** (HKU Math PG Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, OR, USA, teacher), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum** and **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School).

The idea is to get the 3 terms as close as possible. We have $214 = 70 + 72 + 72 = 2\cdot5\cdot7 + 1\cdot8\cdot9 + 3\cdot4\cdot6$. By the AM-GM inequality, $S \ge 3(9!)^{1/3}$. Now $9! = 70\cdot72\cdot72 > 70\cdot73\cdot71 > 71^3$. So $S > 3\cdot71 = 213$. Therefore, 214 is the answer.

Problem 242. Prove that for every positive integer n, 7 is a divisor of 3^n+n^3 if and only if 7 is a divisor of 3^nn^3+1. (*Source: 1995 Bulgarian Winter Math Competition*)

Solution. **CHAN Tsz Lung** (HKU Math PG Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, OR, USA, teacher), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum**, **Tak Wai Alan WONG** (Markham, ON, Canada) and **YUNG Fai**.

Note $3^n \not\equiv 0 \pmod 7$. If $n \not\equiv 0 \pmod 7$, then $n^3 \equiv 1$ or $-1 \pmod 7$. So 7 is a divisor of 3^n+n^3 if and only if $-3^n \equiv n^3 \equiv 1 \pmod 7$ or $-3^n \equiv n^3 \equiv -1 \pmod 7$ if and only if 7 is a divisor of 3^nn^3+1.

Commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College), **LAM Shek Kin** (TWGHs Lui Yun Choy Memorial College) and **WONG Kai Cheuk** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 243. Let R^+ be the set of all positive real numbers. Prove that there is no function $f: R^+ \to R^+$ such that

$$(f(x))^2 \ge f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y. (*Source: 1998 Bulgarian Math Olympiad*)

Solution. **José Luis DíAZ-BARRERO**, (Universitat Politècnica de Catalunya, Barcelona, Spain).

Assume there is such a function. We rewrite the inequality as

$$f(x)-f(x+y)\ge\frac{f(x)y}{f(x)+y}.$$

Note the right side is positive. This implies $f(x)$ is a strictly decreasing.

First we prove that $f(x) - f(x + 1) \ge 1/2$ for $x > 0$. Fix $x > 0$ and choose a natural number n such that $n \ge 1 / f(x + 1)$. When $k = 0, 1, \ldots, n - 1$, we obtain

$$f(x+\frac{k}{n})-f(x+\frac{k+1}{n})$$

$$\ge\frac{f(x+\frac{k}{n})\frac{1}{n}}{f(x+\frac{k}{n})+\frac{1}{n}}\ge\frac{1}{2n}.$$

Adding the above inequalities, we get $f(x) - f(x+1) \ge 1/2$.

Let m be a positive integer such that $m \ge 2f(x)$. Then

$$f(x)-f(x+m)=\sum_{i=1}^{m}(f(x+i-1)-f(x+i))$$

$$\ge m/2 \ge f(x).$$

So $f(x+m) \le 0$, a contradiction.

Commended solvers: **Problem Solving Group @ Miniforum**.

Problem 244. An infinite set S of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1cm from each other. Does there exist any division of S into two disjoint infinite subsets R and B such that inside every triangle with vertices in R is at least one point of B and inside every triangle with vertices in B is at least one point of R? Give a proof to your answer. (*Source: 2002 Albanian Math Olympiad*)

Solution.(Official Solution)

Assume that such a division exists and let M_1 be a point of R. Then take four points M_2, M_3, M_4, M_5 different from M_1, which are the nearest points to M_1 in R. Let r be the largest distance between M_1 and each of these four points. Let H be the convex hull of these five points. Then the interior of

H lies inside the circle of radius r centered at M_1, but all other points of R is outside or on the circle. Hence the interior of H does not contain any other point of R.

Below we will say two triangles are *disjoint* if their interiors do not intersect. There are 3 possible cases:

(*a*) H is a pentagon. Then H may be divided into three disjoint triangles with vertices in R, each of them containing a point of B inside. The triangle with these points of B as vertices would contain another point of R, which would be in H. This is impossible.

(*b*) H is a quadrilateral. Then one of the M_i is inside H and the other M_j, M_k, M_l, M_m are at its vertices, say clockwise. The four disjoint triangles $M_iM_jM_k$, $M_iM_kM_l$, $M_iM_lM_m$, $M_iM_mM_i$ induce four points of B, which can be used to form two disjoint triangles with vertices in B which would contain two points in R. So H would then contain another point of R inside, other than M_i, which is impossible.

(*c*) H is a triangle. Then it contains inside it two points M_i, M_j. One of the three disjoint triangles $M_iM_kM_l$, $M_iM_lM_m$, $M_iM_mM_k$ will contain M_j. Then we can break that triangle into three smaller triangles using M_j. This makes five disjoint triangles with vertices in R, each having one point of B inside. With these five points of B, three disjoint triangles with vertices in B can be made so that each one of them having one point of R. Then H contains another point of R, different from M_1, M_2, M_3, M_4, M_5, which is impossible.

Problem 245. $ABCD$ is a concave quadrilateral such that $\angle BAD = \angle ABC = \angle CDA = 45^\circ$. Prove that $AC = BD$.

Solution. **CHAN Tsz Lung** (HKU Math PG Year 1), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum**, **WONG Kai Cheuk** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Man Kit** (Carmel Divine Grace Foundation Secondary School, Form 6) and **WONG Tsun Yu** (St. Mark's School, Form 6).

Let line BC meet AD at E, then $\angle BEA = 180^\circ - \angle ABC - \angle BAD = 90^\circ$. Note ΔAEB and ΔCED are 45°-90°-45° triangles. So $AE = BE$ and $CE = DE$. Then $\Delta AEC \cong \Delta BED$. So $AC = BD$.

Commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College), **CHAN Pak Woon** (HKU Math UG Year 1), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YUEN Wah Kong** (St. Joan of Arc Secondary School).

Olympiad Corner

(continued from page 1)

Problem 1. (*Cont.*)

(c) the sum of all terms of every sequence is at least 666?

Problem 2. Let O be the center of the circumcircle of the acute-angled triangle ABC, for which $\angle CBA < \angle ACB$ holds. The line AO intersects the side BC at the point D. Denote by E and F the centers of the circumcircles of triangles ABD and ACD respectively. Let G and H be two points on the rays BA and CA such that $AG=AC$ and $AH=AB$, and the point A lies between B and G as well as between C and H. Prove the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Problem 3. Let a, b and c be positive numbers such that $ab + bc + ca = 1$. Prove the inequality

$$3\sqrt[3]{\frac{1}{abc} + 6(a+b+c)} \le \frac{\sqrt[3]{3}}{abc}.$$

Proofs of Muirhead's Inequality

(continued from page 2)

For the proof of the second step, we follow the approach in J. Michael Steele's book *The Cauchy-Schwarz Master Class*, MAA-Cambridge, 2004. For a $n\times n$ matrix M, we will denote its entry in the j-th row, k-th column by M_{jk}. Let us introduce the term *permutation matrix* for $\sigma \in S_n$ to mean the $n\times n$ matrix $M(\sigma)$ with $M(\sigma)_{jk} = 1$ if $\sigma(j)=k$ and $M(\sigma)_{jk} = 0$ otherwise. Also, introduce the term *doubly stochastic matrix* to mean a square matrix whose entries are nonnegative real numbers and the sum of the entries in every row and every column is equal to one. The proof of the second step follows from two results:

Hardy-Littlewood-Polya's Theorem. If $p \succ q$, then there is a $n\times n$ doubly stochastic matrix D such that $q = Dp$, where we write p and q as column matrices.

Birkhoff's Theorem. For every doubly stochastic matrix D, there exist nonnegative numbers $c(\sigma)$ with sum equals 1 such that

$$D = \sum_{\sigma \in S_n} c(\sigma) M(\sigma).$$

Granting these results, for P_i's in the second step, we can just let $P_i = M(\sigma_i)p$.

Hardy-Littlewood-Polya's theorem can be proved by introducing r as in the first proof. Following the idea of Hardy-Littlewood-Polya, we take T to be the matrix with

$$T_{jj} = \frac{d+c}{2d} = T_{kk},\ \ T_{jk} = \frac{d-c}{2d} = T_{kj},$$

all other entries on the main diagonal equal 1 and all other entries of the matrix equal 0. We can check T is doubly stochastic and $r = Tp$. Then we repeat until $r = q$.

Birkhoff's theorem can be proved by induction on the number N of positive entries of D using Hall's theorem (see *Mathematical Excalibur*, *vol.* 1, *no.* 5, *p.* 2). Note $N \ge n$. If $N = n$, then the positive entries are all 1's and D is a permutation matrix already. Next for $N > n$, suppose the result is true for all doubly stochastic matrices with less than N positive entries. Let D have exactly N positive entries. For $j = 1,\ldots, n$, let W_j be the set of k such that $D_{jk} > 0$. We need a *system of distinct representatives* (*SDR*) for $W_1,\ldots,W_n$. To get this, we check the condition in Hall's theorem. For every collection $W_{j_1},\ldots,W_{j_m}$, note m is the sum of all positive entries in column $j_1,\ldots,j_m$ of D. This is less than or equal to the sum of all positive entries in those rows that have at least one positive entry among column $j_1,\ldots,j_m$. This latter sum is the number of such rows and is also the number of elements in the union of $W_{j_1},\ldots,W_{j_m}$.

So the condition in Hall's theorem is satisfied and there is a *SDR* for $W_1,\ldots, W_n$. Let $\sigma(i)$ be the representative in W_i, then $\sigma \in S_n$. Let $c(\sigma)$ be the minimum of $D_{1\sigma(1)},\ldots,D_{n\sigma(n)}$. If $c(\sigma) = 1$, then D is a permutation matrix. Otherwise, let

$$D' = (1 - c(\sigma))^{-1}(D - c(\sigma)\, M(\sigma)).$$

Then $D = c(\sigma)\, M(\sigma) + (1 - c(\sigma))\, D'$ and D' is a double stochastic matrix with at least one less positive entries than D. So we may apply the cases less than N to D' and thus, D has the required sum.

Mathematical Excalibur

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Olympiad Corner

Below was the Find Round of the 36th Austrian Math Olympiad 2005.

Part 1 (May 30, 2005)

Problem 1. Show that an infinite number of multiples of 2005 exist, in which each of the 10 digits 0,1,2,…,9 occurs the same number of times, not counting leading zeros.

Problem 2. For how many integer values of a with $|a| \leq 2005$ does the system of equations $x^2 = y + a$, $y^2 = x + a$ have integer solutions?

Problem 3. We are given real numbers a, b and c and define s_n as the sum $s_n = a^n + b^n + c^n$ of their n-th powers for non-negative integers n. It is known that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$ hold. Show that

$$| s_n^2 - s_{n-1} \cdot s_{n+1} | = 8$$

holds for all integers $n > 1$.

Problem 4. We are given two equilateral triangles ABC and PQR with parallel sides, "one pointing up" and "one pointing down." The common area of the triangles' interior is a hexagon. Show that the lines joining opposite corners of this hexagon are concurrent.

(continued on page 4)

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http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 16, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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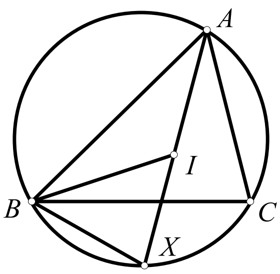
Angle Bisectors Bisect Arcs

Kin Y. Li

In general, angle bisectors of a triangle do not bisect the sides opposite the angles. However, ***angle bisectors always bisect the arcs opposite the angles on the circumcircle of the triangle!*** In math competitions, this fact is very useful for problems concerning angle bisectors or incenters of a triangle involving the circumcircle. Recall that the incenter of a triangle is the point where the three angle bisectors concur.

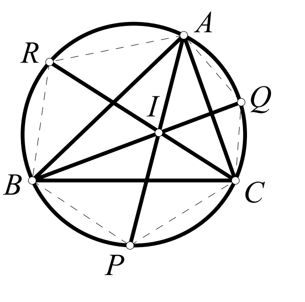
Theorem. Suppose the angle bisector of $\angle BAC$ intersect the circumcircle of ΔABC at $X \neq A$. Let I be a point on the line segment AX. Then I is the incenter of ΔABC if and only if $XI = XB = XC$.



Proof. Note $\angle BAX = \angle CAX = \angle CBX$. So $XB = XC$. Then

$$\begin{aligned} & I \text{ is the incenter of } \Delta ABC \\ \Leftrightarrow\ & \angle CBI = \angle ABI \\ \Leftrightarrow\ & \angle IBX - \angle CBX = \angle BIX - \angle BAX \\ \Leftrightarrow\ & \angle IBX = \angle BIX \\ \Leftrightarrow\ & XI = XB = XC. \end{aligned}$$

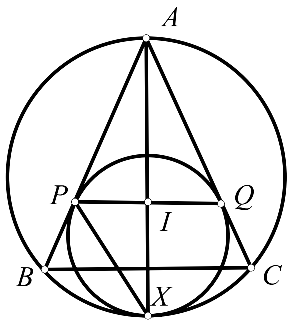
Example 1. (*1982 Australian Math Olympiad*) Let ABC be a triangle, and let the internal bisector of the angle A meet the circumcircle again at P. Define Q and R similarly. Prove that $AP + BQ + CR > AB + BC + CA$.



Solution. Let I be the incenter of ΔABC. By the theorem, we have $2IR = AR + BR > AB$ and similarly $2IP > BC$, $2IQ > CA$. Also $AI + BI > AB$, $BI + CI > BC$ and $CI + AI > CA$. Adding all these inequalities together, we get

$$2(AP + BQ + CR) > 2(AB + BC + CA).$$

Example 2. (*1978 IMO*) In ABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.



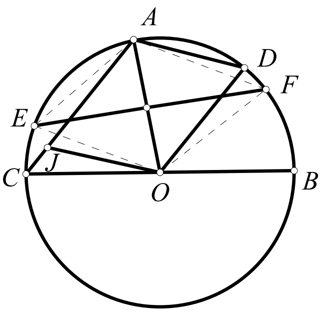
Solution. Let I be the midpoint of line segment PQ and X be the intersection of the angle bisector of $\angle BAC$ with the arc BC not containing A.

By symmetry, AX is a diameter of the circumcircle of ΔABC and X is the midpoint of the arc PXQ on the inside circle, which implies PX bisects $\angle QPB$. Now $\angle ABX = 90^\circ = \angle PIX$ so that X, I, P, B are concyclic. Then

$$\angle IBX = \angle IPX = \angle BPX = \angle BIX.$$

So $XI = XB$. By the theorem, I is the incenter of ΔABC.

Example 3. (*2002 IMO*) Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incenter of the triangle CEF.

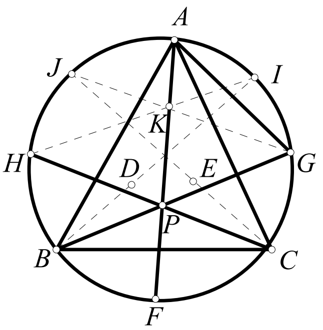


Solution. The condition $\angle AOB$ < 120° ensures I is inside ΔCEF (when $\angle AOB$ increases to 120°, I will coincide with C). Now radius OA and chord EF are perpendicular and bisect each other. So $EOFA$ is a rhombus. Hence A is the midpoint of arc EAF. Then CA bisects $\angle ECF$. Since $OA = OC$, $\angle AOD = 1/2\angle AOB = \angle OAC$. Then DO is parallel to AJ. Hence $ODAJ$ is a parallelogram. Then $AJ = DO = EO = AE$. By the theorem, J is the incenter of ΔCEF.

Example 4. (*1996 IMO*) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.



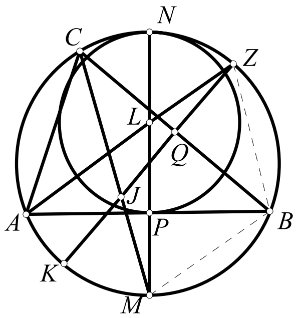
Solution. Let lines AP, BP, CP intersect the circumcircle of ΔABC again at F, G, H respectively. Now

$$\angle APB - \angle ACB = \angle FPG - \angle AGB = \angle FAG.$$

Similarly, $\angle APC - \angle ABC = \angle FAH$. So AF bisects $\angle HAG$. Let K be the incenter of ΔHAG. Then K is on AF and lines HK, GK pass through the midpoints I, J of minor arcs AG, AH respectively. Note lines BD, CE also pass through I, J as they bisect $\angle ABP$, $\angle ACP$ respectively.

Applying Pascal's theorem (see *vol.*10, *no*. 3 of <u>*Math Excalibur*</u>) to B, G, J, C, H, I on the circumcircle, we see that $P=BG\cap CH$, $K=GJ\cap HI$ and $BI\cap CJ= BD\cap CE$ are collinear. Hence, $BD\cap CE$ is on line PK, which is the same as line AP.

Example 5. (*2006 APMO*) Let A, B be two distinct points on a given circle O and let P be the midpoint of line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.



Solution. Let the perpendicular to AB through P intersect circle O at N and M with N and C on the same side of line AB. By symmetry, segment NP is a diameter of the circle of O_1 and its midpoint L is the center of O_1. Let line AL intersect circle O again at Z. Let line ZQ intersect line CM at J and circle O again at K.

Since AB and AC are tangent to circle O_1, AL bisects $\angle CAB$ so that Z is the midpoint of arc BC. Since Q is the midpoint of segment BC, $\angle ZQB = 90^\circ = \angle LPA$ and $\angle JQC = 90^\circ = \angle MPB$. Next

$$\angle ZBQ = \angle ZBC = \angle ZAC = \angle LAP.$$

So ΔZQB, ΔLPA are similar. Since M is the midpoint of arc AMB,

$$\angle JCQ = \angle MCB = \angle MCA = \angle MBP.$$

So ΔJQC, ΔMPB are similar.

By the intersecting chord theorem, $AP\cdot BP = NP\cdot MP = 2LP\cdot MP$. Using the similar triangles above, we have

$$\frac{1}{2} = \frac{LP\cdot MP}{AP\cdot BP} = \frac{ZQ\cdot JQ}{BQ\cdot CQ}.$$

By the intersecting chord theorem, $KQ\cdot ZQ = BQ\cdot CQ$ so that

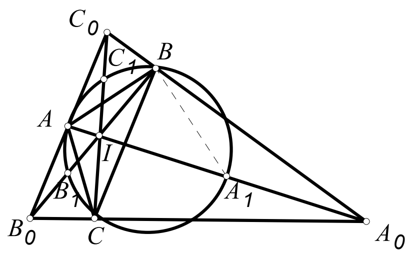
$$KQ = (BQ\cdot CQ)/ZQ = 2JQ.$$

This implies J is the midpoint of KQ. Hence the circle with center J and diameter KQ is tangent to circle O at K and tangent to BC at Q. Since J is on the bisector of $\angle BCA$, this circle is also tangent to AC. So this circle is O_2.

Example 6. (*1989 IMO*) In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$,

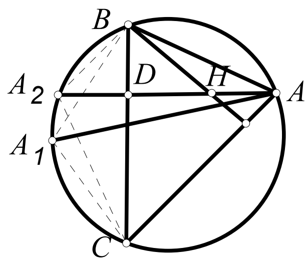
(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.



Solution. (i) Let I be the incenter of ΔABC. Since internal angle bisector and external angle bisector are perpendicular, we have $\angle B_0BA_0 = 90^\circ$. By the theorem, $A_1I = A_1B$. So A_1 must be the midpoint of the hypotenuse A_0I of right triangle IBA_0. So the area of ΔBIA_0 is twice the area of ΔBIA_1.

Cutting the hexagon $AC_1BA_1CB_1$ into six triangles with common vertex I and applying a similar area fact like the last statement to each of the six triangles, we get the conclusion of (i).

(ii) Using (i), we only need to show the area of hexagon $AC_1BA_1CB_1$ is at least twice the area of ΔABC.



(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 16, 2006.***

Problem 251. Determine with proof the largest number x such that a cubical gift of side x can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Problem 252. Find all polynomials $f(x)$ with integer coefficients such that for every positive integer n, $2^n - 1$ is divisible by $f(n)$.

Problem 253. Suppose the bisector of $\angle BAC$ intersect the arc opposite the angle on the circumcircle of ΔABC at A_1. Let B_1 and C_1 be defined similarly. Prove that the area of $\Delta A_1B_1C_1$ is at least the area of ΔABC.

Problem 254. Prove that if $a, b, c > 0$, then

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \geq 4\sqrt{3abc\,(a+b+c)}.$$

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group's performance at least once in one of its day-offs.

Find with proof the minimum total number of performances by these groups.

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Solutions

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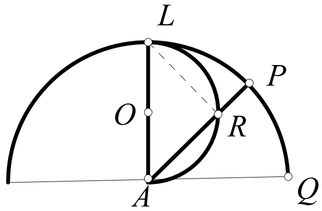
Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center A and radius 10 kilometers. A rocket is fired from A at the same speed as the spy plane such that it is always on the radius from A to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane.

(*Source: 1965 Soviet Union Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School).



Let the spy plane be at Q when the rocket was fired. Let L be the point on the circle obtained by rotating Q by 90˚ in the forward direction of motion with respect to the center A. Consider the semicircle with diameter AL on the same side of line AL as Q. We will show the path from A to L along the semicircle satisfies the conditions.

For any point P on the arc QL, let the radius AP intersect the semicircle at R. Let O be the midpoint of AL. Since

$$\angle QAP = \angle RLA = 1/2\angle ROA$$

and $AL = 2AO$, the length of arc AR is the same as the length of arc QP. So the conditions are satisfied.
Finally, the rocket will hit the spy plane at L after $5\pi/1000$ hour it was fired.

<u>Comments:</u> One solver guessed the path should be a curve and decided to try a circular arc to start the problem. The other solvers derived the equation of the path by a differential equation as follows: using <u>polar coordinates</u>, since the spy plane has a constant angular velocity of 1000/10 = 100 rad/sec, so at time t, the spy plane is at $(10, 100t)$ and the rocket is at $(r(t), \theta(t))$. Since the rocket and the spy plane are on the same radius, so $\theta(t) = 100t$. Now they have the same speed, so

$$(r'(t))^2+(r(t)\theta'(t))^2=10^6.$$

Then

$$\frac{r'(t)}{\sqrt{100-r(t)^2}}=100.$$

Integrating both sides from 0 to t, we get the equation $r = 10\sin(100t) = 10\sin\theta$, which describes the path above.

Problem 247. (a) Find all possible positive integers $k \geq 3$ such that there are k positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(b) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (a) above.

(*Source: 2003 Belarussian Math Olympiad*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy) and **YUNG Fai**.

(a) We shall prove by induction that the conditions are true for every positive integer $k \geq 3$.

For $k = 3$, the numbers 6, 10, 15 satisfy the conditions. Assume it is true for some $k \geq 3$ with the numbers being $a_1, a_2, \ldots, a_k$. Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers such that each p_i is greater than $a_1a_2\ldots a_k$. For $I = 1$ to k, let $b_i= a_ip_i$ and let $b_{k+1}= p_1p_2\ldots p_k$. Then

$gcd(b_i, b_j)=gcd(a_i, a_j) >1$ for $1\leq i < j \leq k$,

$gcd(b_i, b_{k+1}) = p_i > 1$ for $1 \leq i \leq k$,

$gcd(b_h, b_i, b_j) = gcd(a_h, a_i, a_j) = 1$
for $1\leq h \leq i < j \leq k$ and

$gcd(b_i, b_j, b_{k+1}) = 1$ for $1 \leq i < j \leq k$,

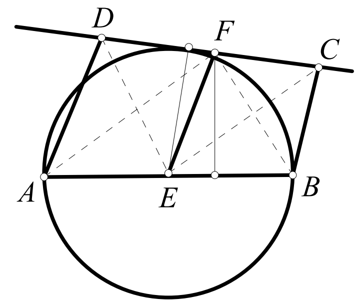
completing the induction.

(b) Assume there are infinitely many positive integers $a_1, a_2, a_3, \ldots$ satisfying the conditions in (a). Let a_1 have exactly m prime divisors. For $i = 2$ to $m + 2$, since each of the $m + 1$ numbers $gcd(a_1, a_i)$ is divisible by one of these m primes, by the pigeonhole principle, there are i, j with $2 \leq i < j \leq m + 2$ such that $gcd(a_1, a_i)$ and $gcd(a_1, a_j)$ are divisible by the same prime. Then $gcd(a_1, a_i, a_j) > 1$, a contradiction.

Commended solvers: **CHAN Nga Yi** (Carmel Divine Grace Foundation Secondary School, Form 6) and **CHAN Yat Sing** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 248. Let $ABCD$ be a convex quadrilateral such that line CD is tangent to the circle with side AB as diameter. Prove that line AB is tangent to the circle with side CD as diameter if and only if lines BC and AD are parallel.

Solution. **Jeff CHEN** (Virginia, USA) and **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).



Let E be the midpoints of AB. Since CD is tangent to the circle, the distance from E to line CD is $h_1 = AB/2$. Let F be the midpoint of CD and let h_2 be the distance from F to line AB. Observe that the areas of ΔCEF and $\Delta DEF = CD{\cdot}AB/8$. Now

line AB is tangent to the circle with side CD as diameter
$\Leftrightarrow$ $h_2=CD/2$
$\Leftrightarrow$ areas of ΔAEF, ΔBEF, ΔCEF and ΔDEF are equal to $AB{\cdot}CD/8$
$\Leftrightarrow$ $AD \| EF$, $BC \| EF$
$\Leftrightarrow$ $AD \| BC$.

Problem 249. For a positive integer n, if $a_1,\cdots, a_n$, b_1, $\cdots$, b_n are in [1,2] and $a_1^2+\cdots+a_n^2=b_1^2+\cdots+b_n^2$, then prove that

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n}\le\frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Solution. **Jeff CHEN** (Virginia, USA).

For x, y in [1,2], we have

$$\begin{aligned} & 1/2 \le x/y \le 2 \\ \Leftrightarrow\ & y/2 \le x \le 2y \\ \Leftrightarrow\ & (y/2-x)(2y-x)\le 0 \\ \Leftrightarrow\ & x^2+y^2 \le 5xy/2. \end{aligned}$$

Let $x = a_i$ and $y = b_i$, then $a_i^2 + b_i^2 \le 5a_ib_i/2$. Summing and manipulating, we get

$$-\sum_{i=1}^n a_ib_i \le -\frac{2}{5}\sum_{i=1}^n (a_i^2+b_i^2) = -\frac{4}{5}\sum_{i=1}^n a_i^2.$$

Let $x = (a_i^3/b_i)^{1/2}$ and $y = (a_ib_i)^{1/2}$. Then $x/y = a_i/b_i$ in [1,2]. So $a_i^3/b_i + a_ib_i \le 5a_i^2/2$.
Summing, we get

$$\sum_{i=1}^n \frac{a_i^3}{b_i}+\sum_{i=1}^n a_ib_i \le \frac{5}{2}\sum_{i=1}^n a_i^2.$$

Adding the two displayed inequalities, we get

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n}\le\frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

Solution. **YUNG Fai.**

Assume the contrary that there is a dissection of the region into nonconvex quadrilateral R_1, R_2, …, R_n. For a nonconvex quadrilateral R_i, there is a vertex where the angle is $\theta_i > 180^\circ$, which we refer to as the *large* vertex of the quadrilateral. The three other vertices, where the angles are less than 180˚ will be referred to as *small* vertices.

Since the boundary of the region is a convex polygon, all the large vertices are in the interior of the region. At a large vertex, one angle is $\theta_i > 180^\circ$, while the remaining angles are angles of small vertices of some of the quadrilaterals and add up to $360^\circ - \theta_i$. Now

$$\sum_{i=1}^n (360^\circ-\theta_i)$$

accounts for all the angles associated with all the small vertices. This is a contradiction since this will leave no more angles from the quadrilaterals to form the angles of the region.

Olympiad Corner

(continued from page 1)

Part 2, Day 1 (June 8, 2005)

Problem 1. Determine all triples of positive integers (a,b,c), such that $a + b + c$ is the least common multiple of a, b and c.

Problem 2. Let a, b, c, d be positive real numbers. Prove

$$\frac{a+b+c+d}{abcd}\le\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}.$$

Problem 3. In an acute-angled triangle ABC, circle k_1 with diameter AC and k_2 with diameter BC are drawn. Let E be the foot of B on AC and F be the foot of A on BC. Furthermore, let L and N be the points in which the line BE intersects with k_1 (with L lying on the segment BE) and K and M be the points in which the line AF intersects with k_2 (with K on the segment AF). Prove that $KLMN$ is a cyclic quadrilateral.

Part 2, Day 2 (June 9, 2005)

Problem 4. The function f is defined for all integers {0, 1, 2, …, 2005}, assuming non-negative integer values in each case. Furthermore, the following conditions are fulfilled for all values of x for which the function is defined:

$$f(2x+1)=f(2x),\quad f(3x+1)=f(3x) \quad \text{and} \quad f(5x+1)=f(5x).$$

How many different values can the function assume at most?

Problem 5. Determine all sextuples (a,b,c,d,e,f) of real numbers, such that the following system of equations is fulfilled:

$$4a=(b+c+d+e)^4,\ 4b=(c+d+e+f)^4,$$
$$4c=(d+e+f+a)^4,\ 4d=(e+f+a+b)^4,$$
$$4e=(f+a+b+c)^4,\ 4f=(a+b+c+d)^4.$$

Problem 6. Let Q be a point in the interior of a cube. Prove that an infinite number of lines passing through Q exists, such that Q is the mid-point of the line-segment joining the two points P and R in which the line and the cube intersect.

Angle Bisectors Bisect Arcs

(continued from page 2)

Let H be the orthocenter of ΔABC. Let line AH intersect BC at D and the circumcircle of ΔABC again at A_2. Note

$$\begin{aligned} \angle A_2BC &= \angle A_2AC \\ &= \angle DAC \\ &= 90^\circ - \angle ACD \\ &= \angle HBC. \end{aligned}$$

Similarly, we have $\angle A_2CB = \angle HCB$. Then $\Delta BA_2C \cong \Delta BHC$. Since A_1 is the midpoint of arc BA_1C, it is at least as far from chord BC as A_2. So the area of $\Delta\ BA_1C$ is at least the area of $\Delta\ BA_2C$. Then the area of quadrilateral BA_1CH is at least twice the area of ΔBHC.

Cutting hexagon $AC_1BA_1CB_1$ into three quadrilaterals with common vertex H and comparing with cutting ΔABC into three triangles with common vertex H in terms of areas, we get the conclusion of (ii).

Remarks. In the solution of (ii), we saw the orthocenter H of ΔABC has the property that $\Delta BA_2C \cong \Delta BHC$ (hence, also $HD = A_2D$). These are useful facts for problems related to the orthocenters involving the circumcircles.

Mathematical Excalibur

Volume 11, Number 3 *June 2006 – October 2006*

Olympiad Corner

The following were the problems of the IMO 2006.

Day 1 (July 12, 2006)

Problem 1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 3. Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 25, 2006***.

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Summation by Parts

Kin Y. Li

In calculus, we have a formula called <u>integration by parts</u>

$$\int_s^t f(x)g(x)dx = F(t)g(t) - F(s)g(s) - \int_s^t F(x)g'(x)dx,$$

where $F(x)$ is an anti-derivative of $f(x)$. There is a discrete version of this formula for series. It is called <u>summation by parts</u>, which asserts

$$\sum_{k=1}^n a_kb_k = A_nb_n - \sum_{k=1}^{n-1} A_k(b_{k+1}-b_k),$$

where $A_k = a_1+a_2+\cdots+a_k$. This formula follows easily by observing that $a_1 = A_1$ and for $k > 1$, $a_k = A_k - A_{k-1}$ so that

$$\begin{aligned}\sum_{k=1}^n a_kb_k &= A_1b_1+(A_2-A_1)b_2+\cdots+(A_n-A_{n-1})b_n\\ &= A_nb_n - A_1(b_2-b_1)-\cdots-A_{n-1}(b_n-b_{n-1})\\ &= A_nb_n - \sum_{k=1}^{n-1}A_k(b_{k+1}-b_k).\end{aligned}$$

From this identity, we can easily obtain some famous inequalities.

<u>Abel's Inequality.</u> Let $m \leq \sum_{i=1}^k a_i \leq M$ for $k = 1,2,\ldots,n$ and $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Then

$$b_1m \leq \sum_{k=1}^n a_kb_k \leq b_1M.$$

<u>Proof.</u> Let $A_k = a_1+a_2+\cdots+a_k$. Applying summation by parts, we have

$$\sum_{k=1}^n a_kb_k = A_nb_n + \sum_{k=1}^{n-1}A_k(b_k-b_{k+1}).$$

The right side is at least

$$mb_n + \sum_{k=1}^{n-1} m(b_k-b_{k+1}) = mb_1$$

and at most

$$Mb_n + \sum_{k=1}^{n-1} M(b_k-b_{k+1}) = Mb_1.$$

<u>K. L. Chung's Inequality.</u> Suppose $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1,2,\ldots,n$. Then

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n b_i^2.$$

<u>Proof.</u> Applying summation by parts and Cauchy-Schwarz' inequality, we have

$$\begin{aligned}\sum_{i=1}^n a_i^2 &= \left(\sum_{i=1}^n a_i\right)a_n + \sum_{k=1}^{n-1}\left(\sum_{i=1}^k a_i\right)(a_k-a_{k+1})\\ &\leq \left(\sum_{i=1}^n b_i\right)a_n + \sum_{k=1}^{n-1}\left(\sum_{i=1}^k b_i\right)(a_k-a_{k+1})\\ &= \sum_{i=1}^n a_ib_i\\ &\leq \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}.\end{aligned}$$

Squaring and simplifying, we get

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n b_i^2.$$

Below we will do some more examples to illustrate the usefulness of the summation by parts formula.

<u>Example 1.</u> (*1978 IMO*) Let n be a positive integer and $a_1, a_2, \cdots, a_n$ be a sequence of distinct positive integers. Prove that

$$\sum_{k=1}^n \frac{a_k}{k^2} \geq \sum_{k=1}^n \frac{1}{k}.$$

<u>Solution.</u> Since the a_i's are distinct positive integers, $A_k = a_1 + a_2 + \cdots + a_k$ is at least $1 + 2 + \cdots + k = k(k+1)/2$.

Applying summation by parts, we have

$$\begin{aligned}\sum_{k=1}^n \frac{a_k}{k^2} &= \frac{A_n}{n^2} + \sum_{k=1}^{n-1} A_k\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)\\ &\geq \frac{n(n+1)/2}{n^2} + \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\frac{(2k+1)}{k^2(k+1)^2}\\ &= \frac{1}{2}\left(1+\frac{1}{n}+\sum_{k=1}^{n-1}\frac{2k+1}{k(k+1)}\right)\\ &= \frac{1}{2}\left(1+\frac{1}{n}+\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{k+1}\right)\right)\\ &= \frac{1}{2}\left(\sum_{k=1}^n\frac{1}{k}+\sum_{k=0}^{n-1}\frac{1}{k+1}\right)\\ &= \sum_{k=1}^n\frac{1}{k}.\end{aligned}$$

Example 2. (*1982 USAMO*) If x is a positive real number and n is a positive integer, then prove that

$$[nx] \ge \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \cdots + \frac{[nx]}{n},$$

where $[t]$ denotes the greatest integer less than or equal to t.

Solution. Let $a_k = [kx]/k$. Then

$$A_k = \sum_{i=1}^{k} \frac{[ix]}{i}.$$

In terms of A_k, we are to prove $[nx] \ge A_n$. The case $n = 1$ is easy. Suppose the cases 1 to $n-1$ are true. Applying summation by parts, we have

$$\sum_{k=1}^{n} [kx] = \sum_{k=1}^{n} a_k k = A_n n - \sum_{k=1}^{n-1} A_k.$$

Using this and the inductive hypothesis,

$$\begin{aligned} A_n n &= \sum_{k=1}^{n} [kx] + \sum_{k=1}^{n-1} A_k \\ &\le \sum_{k=1}^{n} [kx] + \sum_{k=1}^{n-1} [kx] \\ &= [nx] + \sum_{k=1}^{n-1} \big([kx] + [(n-k)x]\big) \\ &\le [nx] + \sum_{k=1}^{n-1} [kx + (n-k)x] \\ &= n[nx], \end{aligned}$$

which yields case n.

Example 3. Consider a polygonal line $P_0P_1P_2\ldots P_n$ such that $\angle P_0P_1P_2 = \angle P_1P_2P_3 = \cdots = \angle P_{n-2}P_{n-1}P_n$, all measure in counterclockwise direction. If $P_0P_1 > P_1P_2 > \cdots > P_{n-1}P_n$, show that P_0 and P_n cannot coincide.

Solution. Let a_k be the length of $P_{k-1}P_k$. Consider the complex plane. Each P_k corresponds to a complex number. We may set $P_0 = 0$ and $P_1 = a_1$. Let $\theta = \angle P_0P_1P_2$ and $z = -\cos\theta + i\sin\theta$, then $P_n = a_1 + a_2 z + \cdots + a_n z^{n-1}$. Applying summation by parts, we get

$$\begin{aligned} P_n &= (a_1 - a_2) + (a_2 - a_3)(1+z) + \cdots \\ &\quad + a_n(1 + z + \cdots + z^{n-1}). \end{aligned}$$

If $\theta = 0$, then $z = 1$ and $P_n > 0$. If $\theta \ne 0$, then assume $P_n = 0$. We get $P_n(1-z) = 0$, which implies

$$\begin{aligned} &(a_1 - a_2)(1-z) + (a_2 - a_3)(1-z^2) + \cdots \\ &\quad + a_n(1-z^n) = 0. \end{aligned}$$

Then

$$\begin{aligned} &(a_1 - a_2) + (a_2 - a_3) + \cdots + a_n = \\ &(a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots + a_n z^n. \end{aligned}$$

However, since $|z| = 1$ and $z \ne 1$, by the triangle inequality,

$$\begin{aligned} &|(a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots + a_n z^n| \\ &< |(a_1 - a_2)z| + |(a_2 - a_3)z^2| + \cdots + |a_n z^n| \\ &= (a_1 - a_2) + (a_2 - a_3) + \cdots + a_n, \end{aligned}$$

which is a contradiction to the last displayed equation. So $P_n \ne 0 = P_0$.

Example 4. Show that the series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Solution. Let $a_k = \sin k$ and $b_k = 1/k$. Using the identity

$$\sin m \sin\frac{1}{2} = \frac{\cos(m - \frac12) - \cos(m + \frac12)}{2},$$

we get

$$A_k = \sin 1 + \cdots + \sin k = \frac{\cos\frac12 - \cos(k + \frac12)}{2\sin\frac12}.$$

Then $|A_k| \le 1/(\sin \frac12)$ and hence

$$\lim_{n\to\infty} A_n b_n = 0.$$

Applying summation by parts, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\sin k}{k} &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k \\ &= \lim_{n\to\infty} \Big(A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)\Big) \\ &= \sum_{k=1}^{\infty} A_k \left(\frac{1}{k} - \frac{1}{k+1}\right). \end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \left| A_k \left(\frac{1}{k} - \frac{1}{k+1}\right) \right| \le \frac{1}{\sin\frac12} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{\sin\frac12},$$

so $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Example 5. Let $a_1 \ge a_2 \ge \cdots \ge a_n$ with $a_1 \ne a_n$, $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} |x_i| = 1$. Find the least number m such that

$$\left| \sum_{i=1}^{n} a_i x_i \right| \le m(a_1 - a_n)$$

always holds.

Solution. Let $S_i = x_1 + x_2 + \cdots + x_i$. Let

$$p = \sum_{x_i > 0} x_i, \quad q = -\sum_{x_i < 0} x_i.$$

Then $p - q = 0$ and $p + q = 1$. So $p = q = \frac{1}{2}$. Thus, $-\frac{1}{2} \le S_k \le \frac{1}{2}$ for $k = 1, 2, \cdots, n$.

Applying summation by parts, we get

$$\begin{aligned} \left| \sum_{i=1}^{n} a_i x_i \right| &= \left| S_n a_n - \sum_{k=1}^{n-1} S_k (a_{k+1} - a_k) \right| \\ &\le \sum_{k=1}^{n-1} |S_k| (a_k - a_{k+1}) \\ &\le \sum_{k=1}^{n-1} \frac{1}{2} (a_k - a_{k+1}) \\ &= \frac{1}{2}(a_1 - a_n). \end{aligned}$$

When $x_1 = 1/2$, $x_n = -1/2$ and all other $x_i = 0$, we have equality. So the least such m is 1/2.

Example 6. Prove that for all real numbers $a_1, a_2, \ldots, a_n$, there is an integer m among $1, 2, \ldots, n$ such that if

$$0 \le \theta_n \le \theta_{n-1} \le \cdots \le \theta_1 \le \frac{\pi}{2},$$

then $\left| \sum_{i=1}^{n} a_i \sin\theta_i \right| \le \left| \sum_{i=1}^{m} a_i \right|$.

Solution. Let $A_i = a_1 + a_2 + \cdots + a_i$ and $b_i = \sin\theta_i$, then $1 \ge b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Next let $|A_m|$ be the maximum among $|A_1|, |A_2|, \ldots, |A_n|$. With $a_{n+1} = b_{n+1} = 0$, we apply summation by parts to get

$$\begin{aligned} \left| \sum_{i=1}^{n} a_i \sin\theta_i \right| &= \left| \sum_{i=1}^{n+1} a_i b_i \right| \\ &= \left| \sum_{i=1}^{n} A_i (b_{i+1} - b_i) \right| \\ &\le \sum_{i=1}^{n} |A_m| (b_i - b_{i+1}) \\ &= |A_m| b_1 \\ &\le |A_m|. \end{aligned}$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***November 25, 2006.***

Problem 256. Show that there is a rational number q such that

$$\sin 1^\circ \sin 2^\circ \cdots \sin 89^\circ \sin 90^\circ = q\sqrt{10}.$$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by n, where $[x]$ denotes the greatest integer less than or equal to x.
(*Source: 1993 Jiangsu Math Contest*)

Problem 258. (*Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romaina*) Show that if A, B, C are in the interval $(0, \pi/2)$, then

$$f(A,B,C) + f(B,C,A) + f(C,A,B) \ge 3,$$

where $f(x,y,z) = \dfrac{4\sin x + 3\sin y + 2\sin z}{2\sin x + 3\sin y + 4\sin z}$.

Problem 259. Let AD, BE, CF be the altitudes of acute triangle ABC. Through D, draw a line parallel to line EF intersecting line AB at R and line AC at Q. Let P be the intersection of lines EF and CB. Prove that the circumcircle of $\triangle PQR$ passes through the midpoint M of side BC.
(*Source: 1994 Hubei Math Contest*)

Problem 260. In a class of 30 students, number the students 1, 2, …, 30 from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student *good* if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.
(*Source: 1998 Hubei Math Contest*)

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Solutions

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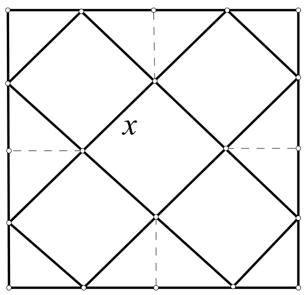
Problem 251. Determine with proof the largest number x such that a cubical gift of side x can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Solution. **CHAN Tsz Lung** (Math, HKU) and **Jeff CHEN** (Virginia, USA).

Let A and B be two points inside or on the unit square such that the line segment AB has length d. After folding, the distance between A and B along the surface of the cube will be at most d because the line segment AB on the unit square after folding will provide one path between the two points along the surface of the cube, which may or may not be the shortest possible.

In the case A is the center of the unit square and B is the point opposite to A on the surface of the cube with respect to the center of the cube, then the distance along the surface of the cube between them is at least $2x$. Hence, $2x \le d \le \sqrt{2}/2$. Therefore, $x \le \sqrt{2}/4$.

The maximum $x = \sqrt{2}/4$ is attainable can be seen by considering the following picture of the unit square.



Commended solvers: **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).

Problem 252. Find all polynomials $f(x)$ with integer coefficients such that for every positive integer n, $2^n - 1$ is divisible by $f(n)$.

Solution. **Jeff CHEN** (Virginia, USA) and **G.R.A. 20 Math Problem Group** (Roma, Italy).

We will prove that the only such polynomials $f(x)$ are the constant polynomials 1 and −1.

Assume $f(x)$ is such a polynomial and $|f(n)| \ne 1$ for some $n > 1$. Let p be a prime which divides $f(n)$, then p also divides $f(n+kp)$ for every integer k. Therefore, p divides $2^{n+kp}-1$ for all integers $k \ge 0$.

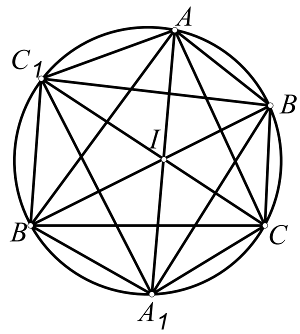
When $k = 0$, p divides $2^n - 1$, which implies $2^n \equiv 1 \pmod p$. By Fermat's little theorem, $2^p \equiv 2 \pmod p$. Finally, when $k = 1$, we get

$$1 \equiv 2^{n+p} = 2^n\, 2^p \equiv 1\cdot 2 = 2 \pmod p$$

implying p divides $2 - 1 = 1$, which is a contradiction.

Problem 253. Suppose the bisector of $\angle BAC$ intersect the arc opposite the angle on the circumcircle of ΔABC at A_1. Let B_1 and C_1 be defined similarly. Prove that the area of $\Delta A_1B_1C_1$ is at least the area of ΔABC.

Solution. **CHAN Tsz Lung** (Math, HKU), **Jeff CHEN** (Virginia, USA) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).



By a well-known property of the incenter I (see page 1 of Mathematical Excalibur, vol. 11, no. 2), we have $AC_1 = C_1I$ and $AB_1 = B_1I$. Hence, $\Delta AC_1B_1 \cong \Delta IC_1B_1$. Similarly, $\Delta BA_1C_1 \cong \Delta IA_1C_1$ and $\Delta CB_1A_1 \cong \Delta IB_1A_1$. Letting $[\cdots]$ denote area, we have

$$[AB_1CA_1BC_1] = 2[A_1B_1C_1].$$

If ΔABC is not acute, say $\angle BAC$ is not acute, then

$$[ABC] \le \frac{1}{2}[ABA_1C]$$

$$\le \frac{1}{2}[AB_1CA_1BC_1] = [A_1B_1C_1].$$

Otherwise, ΔABC is acute and we can apply the fact that

$$[ABC] \le \frac{1}{2}[AB_1CA_1BC_1] = [A_1B_1C_1]$$

(see example 6 on page 2 of Mathematical Excalibur, vol. 11, no. 2).

Commended solvers: **Samuel Liló Abdalla** (Brazil) and **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Problem 254. Prove that if a, b, $c > 0$, then

$$\sqrt{abc}\,(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \geq 4\sqrt{3abc\,(a+b+c)}.$$

Solution 1. **José Luis Díaz-Barrero** (Universitat Politècnica de Catalunya, Barcelona, Spain) and **G.R.A. 20 Math Problem Group** (Roma, Italy).

Dividing both sides by $\sqrt{abc(a+b+c)}$, the inequality is equivalent to

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a+b+c}}+\frac{(\sqrt{a+b+c})^3}{\sqrt{abc}} \geq 4\sqrt{3}.$$

By the AM-GM inequality,

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq 3(\sqrt{abc})^{1/3}.$$

Therefore, it suffices to show

$$\frac{3(\sqrt{abc})^{1/3}}{\sqrt{a+b+c}}+\frac{(\sqrt{a+b+c})^3}{\sqrt{abc}}=\frac{3}{t}+t^3 \geq 4\sqrt{3},$$

where again by the AM-GM inequality,

$$t=\frac{\sqrt{a+b+c}}{(\sqrt{abc})^{1/3}}=\sqrt{\frac{a+b+c}{(abc)^{1/3}}} \geq \sqrt{3}.$$

By the AM-GM inequality a third time,

$$\frac{3}{t}+t^3=\frac{3}{t}+\frac{t^3}{3}+\frac{t^3}{3}+\frac{t^3}{3} \geq \frac{4t^2}{\sqrt{3}} \geq 4\sqrt{3}.$$

Solution 2. **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School).

By the AM-GM inequality, we have

$$a+b+c \geq 3(abc)^{1/3} \qquad (1)$$

and

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq 3(abc)^{1/6}. \qquad (2)$$

Applying (2), (1), the AM-GM inequality and (1) in that order below, we have

$$\begin{aligned}
&\sqrt{abc}\,(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2\\
&\geq 3(abc)^{2/3}+3(abc)^{1/3}(a+b+c)\\
&\geq 4\left(3(abc)^{2/3}(abc)(a+b+c)^3\right)^{1/4}\\
&\geq 4\left(3(abc)^{2/3}(abc)3(abc)^{1/3}(a+b+c)^2\right)^{1/4}\\
&= 4\sqrt{3abc(a+b+c)}.
\end{aligned}$$

Commended solvers: **Samuel Liló Abdalla** (Brazil), **CHAN Tsz Lung** (Math, HKU), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group's performance at least once in one of its day-offs.

Find with proof the minimum total number of performances by these groups.

Solution. **CHAN Tsz Lung** (Math, HKU).

Here are three important observations:

(1) Each group perform at least once.

(2) If more than one groups perform on the same day, then each of these groups will have to perform on another day so the other groups can see its performance in their day-offs.

(3) If a group performs exactly once, on the day it performs, it is the only group performing.

We will show the minimum number of performances is 22. The following performance schedule shows the case 22 is possible.

Day 1: Group 1
Day 2: Group 2
Day 3: Groups 3, 4, 5, 6
Day 4: Groups 7, 8, 9, 3
Day 5: Groups 10, 11, 4, 7
Day 6: Groups 12, 5, 8, 10
Day 7: Groups 6, 9, 11, 12.

Assume it is possible to do at most 21 performances. Let k groups perform exactly once, then $k + 2(12 - k) \leq 21$ will imply $k \geq 3$.

Case 1: Exactly 3 groups perform exactly once, say group 1 on day 1, group 2 on day 2 and group 3 on day 3.

(a) If at least 4 groups perform on one of the remaining 4 days, say groups 4, 5, 6, 7 on day 4, then by (2), each of them has to perform on one of the remaining 3 days. By the pigeonhole principle, two of groups 4, 5, 6, 7 will perform on the same day again later, say groups 4 and 5 perform on day 5. Then they will have to perform separately on the last 2 days for the other to see. Then groups 1, 2, 3 once each, groups 4, 5 thrice each and groups 6, 7, …, 12 twice each at least , resulting in at least

$$3 + 2 \times 3 + 7 \times 2 = 23$$

performances, contradiction.

(b) If at most 3 groups perform on each of the remaining 4 days, then there are at most $3 \times 4 = 12$ slots for performances. However, each of groups 4 to 12 has to perform at least twice, yielding at least $9 \times 2 = 18$ (> 12) performances, contradiction.

Case 2: More than 3 groups perform exactly once, say k groups with $k > 3$. By argument similar to case 1(a), we see at most 3 groups can perform on each of the remaining $7 - k$ days (meaning at most $3(7-k)$ performance slots). Again, the remaining $12 - k$ groups have to perform at least twice, yielding $2(12-k) \leq 3(7 - k)$, which implies $k \leq -3$, contradiction.

Commended solvers: **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **Raúl A. SIMON** (Santiago, Chile).

Comments: This was a problem in the 1994 Chinese IMO team training tests. In the Chinese literature, there is a solution using the famous Sperner's theorem which asserts that for a set with n elements, the number of subsets so that no two with one contains the other is at most $\binom{n}{[n/2]}$. We hope to present this solution in a future article.

Olympiad Corner

(continued from page 1)

Day 2 (July 13, 2006)

Problem 4. Determine all pairs (x,y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\cdots P(P(x))\cdots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Mathematical Excalibur

Volume 11, Number 4 — *November 2006 – December 2006*

Olympiad Corner

The 9th China Hong Kong Math Olympiad was held on Dec. 2, 2006. The following were the problems.

Problem 1. Let M be a subset of $\{1, 2, \ldots, 2006\}$ with the following property: For any three elements x, y and z ($x < y < z$) of M, $x + y$ does not divide z. Determine the largest possible size of M. Justify your claim.

Problem 2. For a positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. (For example $f_1(123) = (1+2+3)^2 = 36$.) Let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{2007}(2^{2006})$. Justify your claim.

Problem 3. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O. Let E be the intersection of diagonals AC and BD. If P is a point inside $ABCD$ such that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ,$$

prove that O, P and E are collinear.

Problem 4. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive numbers. If there exists a positive number M such that for every $n = 1, 2, 3, \ldots,$

$$a_1^2 + a_2^2 + \ldots + a_n^2 < M a_{n+1}^2,$$

then prove that there exists a positive number M' such that for every $n = 1, 2, 3, \ldots,$

$$a_1 + a_2 + \ldots + a_n < M' a_{n+1}.$$

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 25, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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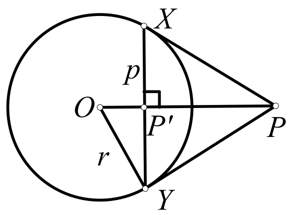
Pole and Polar

Kin Y. Li

Let C be a circle with center O and radius r. Recall the inversion with respect to C (see <u>*Mathematical Excalibur*</u>, *vol.* 9, *no.* 2, *p.*1) sends every point $P \neq O$ in the same plane as C to the image point P' on the ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. The <u>*polar*</u> of P is the line p perpendicular to the line OP at P'. Conversely, for any line p not passing through O, the <u>*pole*</u> of p is the point P whose polar is p. The function sending P to p is called the <u>*pole-polar transformation*</u> (or <u>*reciprocation*</u>) with respect to O and r (or with respect to C).



Following are some useful facts:

(1) If P is outside C, then recall P' is found by drawing tangents from P to C, say tangent at X and Y. Then $P' = OP \cap XY$, where $\cap$ denotes intersection. By symmetry, $OP \perp XY$. So the polar p of P is the line XY.

Conversely, for distinct points X, Y on C, the pole of the line XY is the intersection of the tangents at X and Y. Also, it is the point P on the perpendicular bisector of XY such that O, X, P, Y are concyclic since $\angle OXP = 90^\circ = \angle OYP$.

(2) (***<u>La Hire's Theorem</u>***) Let x and y be the polars of X and Y, respectively. Then X is on line $y \Leftrightarrow Y$ is on line x.

<u>*Proof.*</u> Let X', Y' be the images of X, Y for the inversion with respect to C. Then $OX \cdot OX' = r^2 = OY \cdot OY'$ implies X, X', Y, Y' are concyclic. Now

$$\begin{aligned} X \text{ is on } y &\Leftrightarrow \angle XY'Y = 90^\circ \\ &\Leftrightarrow \angle XX'Y = 90^\circ \\ &\Leftrightarrow Y \text{ is on } x. \end{aligned}$$

(3) Let x,y,z be the polars of distinct points X,Y,Z respectively. Then $Z = x \cap y \Leftrightarrow z = XY$.

<u>*Proof.*</u> By La Hire's theorem, Z on $x \cap y \Leftrightarrow X$ on z and Y on $z \Leftrightarrow z = XY$.

(4) Let W, X, Y, Z be on C. The polar p of $P = XY \cap WZ$ is the line through $Q = WX \cap ZY$ and $R = XZ \cap YW$.

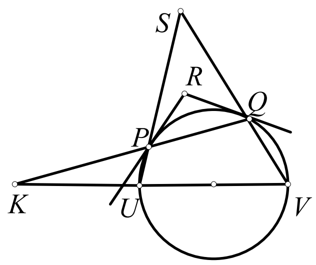
<u>*Proof.*</u> Let S, T be the poles of $s = XY$, $t = WZ$ respectively. Then $P = s \cap t$. By fact (3), $S = x \cap y$, $T = w \cap z$ and $p = ST$. For hexagon $WXXZYY$, we have

$$Q = WX \cap ZY, \quad S = XX \cap YY, \quad R = XZ \cap YW,$$

where XX denotes the tangent line at X. By Pascal's theorem (see <u>*Mathematical Excalibur*</u>, *vol.* 10, *no.* 3, *p.*1), Q,S,R are collinear. Similarly, considering the hexagon $XWWYZZ$, we see Q,T,R are collinear. Therefore, $p = ST = QR$.

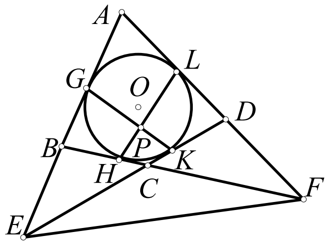
Next we will present some examples using the pole-polar transformation.

<u>Example 1.</u> Let UV be a diameter of a semicircle. P,Q are two points on the semicircle with $UP < UQ$. The tangents to the semicircle at P and Q meet at R. If $S = UP \cap VQ$, then prove that $RS \perp UV$.



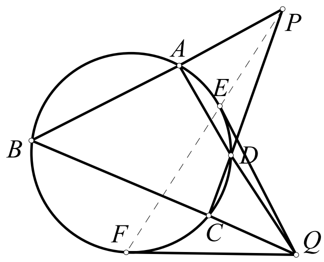
<u>Solution (due to CHENG Kei Tsi).</u> Let $K = PQ \cap UV$. With respect to the circle, by fact (4), the polar of K passes through $UP \cap VQ = S$. Since the tangents to the semicircle at P and Q meet at R, by fact (1), the polar of R is PQ. Since K is on line PQ, which is the polar of R, by La Hire's theorem, R is on the polar of K. So the polar of K is the line RS. As K is on the diameter UV extended, by the definition of polar, we get $RS \perp UV$.

Example 2. Quadrilateral $ABCD$ has an inscribed circle Γ with sides AB, BC, CD, DA tangent to Γ at G, H, K, L respectively. Let $AB\cap CD = E$, $AD\cap BC = F$ and $GK\cap HL = P$. If O is the center of Γ, then prove that $OP\perp EF$.



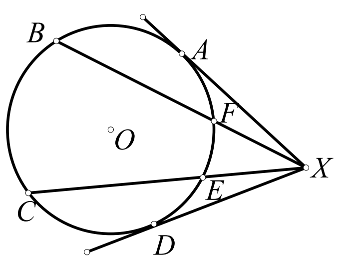
Solution. Consider the pole-polar transformation with respect to the inscribed circle. By fact (1), the polars of E, F are lines GK, HL respectively. Since $GK\cap HL = P$, by fact (3), the polar of P is line EF. By the definition of polar, we get $OP\perp EF$.

Example 3. (*1997 Chinese Math Olympiad*) Let $ABCD$ be a cyclic quadrilateral. Let $AB\cap CD = P$ and $AD\cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that P, E, F are collinear.



Solution. Consider the pole-polar transformation with respect to the circumcircle of $ABCD$. Since $P = AB\cap CD$, by fact (4), the polar of P passes through $AD\cap BC = Q$. By La Hire's theorem, P is on the polar of Q, which by fact (1), is the line EF.

Example 4. (*1998 Austrian-Polish Math Olympiad*) Distinct points A, B, C, D, E, F lie on a circle in that order. The tangents to the circle at the points A and D, the lines BF and CE are concurrent. Prove that the lines AD, BC, EF are either parallel or concurrent.

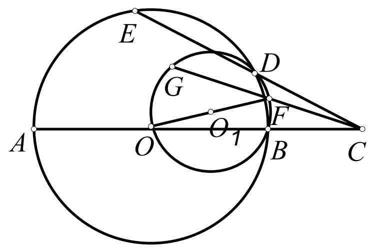


Solution. Let O be the center of the circle and $X = AA\cap DD\cap BF\cap CE$.

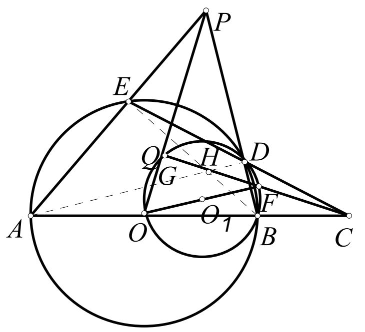
If $BC||EF$, then by symmetry, lines BC and EF are perpendicular to line OX. Since $AD\perp OX$, we get $BC||EF||AD$.

If lines BC, EF intersect, then by fact (4), the polar of $X = CE\cap BF$ passes through $BC\cap EF$. Since the tangents at A and D intersect at X, by fact (1), the polar of X is line AD. Therefore, AD, BC and EF are concurrent in this case.

Example 5. (*2006 China Western Math Olympiad*) As in the figure below, AB is a diameter of a circle with center O. C is a point on AB extended. A line through C cuts the circle with center O at D, E. OF is a diameter of the circumcircle of ΔBOD with center O_1. Line CF intersect the circumcircle again at G. Prove that O,A,E,G are concyclic.



Solution (due to WONG Chiu Wai). Let $AE\cap BD=P$. By fact (4), the polar of P with respect to the circle having center O is the line through $BA\cap DE = C$ and $AD\cap EB = H$. Then $OP\perp CH$. Let $Q = OP\cap CH$.



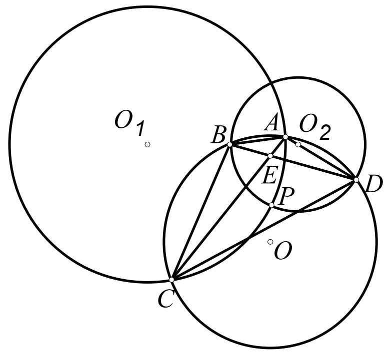
We claim $Q = G$. Once this shown, we will have $P = BD \cap OG$. Then $PE\cdot PA = PD\cdot PB = PG\cdot PO$, which implies O, A, E, G are concyclic.

To show $Q = G$, note that $\angle PQH$, $\angle PDH$ and $\angle PEH$ are 90˚, which implies P, E, Q, H, D are concyclic. Then $\angle PQD = \angle PED = \angle DBO$, which implies Q, D, B, O are concyclic. Therefore, $Q = G$ since they are both the point of intersection (other than O) of the circumcircle of ΔBOD and the circle with diameter OC.

Example 6. (*2006 China Hong Kong Math Olympiad*) A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O. Let E be the intersection of diagonals AC and BD. If P is a point inside $ABCD$ such that

$$\angle PAB+\angle PCB=\angle PBC+\angle PDC=90^\circ,$$

prove that O, P and E are collinear.



Solution (due to WONG Chiu Wai). Let Γ, Γ_1, Γ_2 be the circumcircles of quadrilateral $ABCD$, ΔPAC, ΔPBD with centers O, O_1, O_2 respectively. We first show that the polar of O_1 with respect to Γ is line AC. Since OO_1 is the perpendicular bisector of AC, by fact (1), all we need to show is that

$$\angle AOC + \angle AO_1C = 180^\circ.$$

For this, note

$$\begin{aligned}&\angle APC\\ &= 360^\circ - (\angle PAB + \angle PCB + \angle ABC)\\ &= 270^\circ - \angle ABC\\ &= 90^\circ + \angle ADC\end{aligned}$$

and so

$$\begin{aligned}\angle AO_1C &= 2(180^\circ - \angle APC)\\ &= 2(90^\circ - \angle ADC)\\ &= 180^\circ - 2\angle ADC\\ &= 180^\circ - \angle AOC.\end{aligned}$$

Similarly, the polar of O_2 with respect to Γ is line BD. By fact (3), since $E = AC\cap BD$, the polar of E with respect to Γ is line O_1O_2. So $OE\perp O_1O_2$.

(Next we will consider radical axis and radical center, see Mathematical Excalibur, *vol*. 4, *no*. 3, *p*. 2.) Among Γ, Γ_1, Γ_2, two of the pairwise radical axes are lines AC and BD. This implies E is the radical center. Since Γ_1, Γ_2 intersect at P, so PE is the radical axis of Γ_1, Γ_2, which implies $PE\perp O_1O_2$. Combining with $OE\perp O_1O_2$ proved above, we see O, P and E are collinear.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***January 25, 2007.***

Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Problem 262. Let O be the center of the circumcircle of ΔABC and let AD be a diameter. Let the tangent at D to the circumcircle intersect line BC at P. Let line PO intersect lines AC, AB at M, N respectively. Prove that $OM = ON$.

Problem 263. For positive integers m, n, consider a $(2m+1)\times(2n+1)$ table, where in each cell, there is exactly one ant. At a certain moment, every ant moves to a horizontal or vertical neighboring cell. Prove that after that moment, there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers a, b, prove that $ab^p - ba^p$ is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n}(x_j^4 - x_j^5),$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.

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Solutions

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Problem 256. Show that there is a rational number q such that

$$\sin 1^\circ \sin 2^\circ \cdots \sin 89^\circ \sin 90^\circ = q\sqrt{10}.$$

Solution 1. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Let $\omega = e^{2\pi i/180}$. Then

$$P(z) = \sum_{n=0}^{179} z^n = \prod_{k=1}^{179}(z-\omega^k).$$

Using $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i} = \dfrac{e^{2ix}-1}{2ie^{ix}}$, we have

$$\prod_{k=1}^{90}\sin k^\circ = \prod_{k=1}^{90}\frac{\omega^k-1}{2i\omega^{k/2}}.$$

Also, $\displaystyle\prod_{k=1}^{90}\sin k^\circ = \prod_{k=91}^{179}\sin k^\circ = \prod_{k=91}^{179}\frac{\omega^k-1}{2i\omega^{k/2}}.$

Then

$$\left|\prod_{k=1}^{90}\sin k^\circ\right|^2 = \prod_{k=1}^{179}\frac{|\omega^k-1|}{2} = \frac{|P(1)|}{2^{179}} = \frac{90}{2^{178}}.$$

Therefore, $\displaystyle\prod_{k=1}^{90}\sin k^\circ = \frac{3}{2^{89}}\sqrt{10}.$

Solution 2. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Let S be the left-handed side. Note

$$\begin{aligned}\sin 3\theta &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta\\ &= \sin\theta(\cos^2\theta - \sin^2\theta + 2\cos^2\theta)\\ &= 4\sin\theta(\tfrac{3}{4}\cos^2\theta - \tfrac{1}{4}\sin^2\theta)\\ &= 4\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta).\end{aligned}$$

So, $\displaystyle\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta) = \frac{\sin 3\theta}{4}.$

Using this, we have

$$\begin{aligned}S &= \sin 30^\circ \sin 60^\circ \prod_{n=1}^{29}\sin n^\circ \sin(60^\circ - n^\circ)\sin(60^\circ + n^\circ)\\ &= \frac{\sqrt{3}}{4^{30}}\sin 3^\circ \sin 6^\circ \sin 9^\circ \cdots \sin 87^\circ\\ &= \frac{\sqrt{3}}{4^{30}}\sin 30^\circ \sin 60^\circ \prod_{m=1}^{9}\sin 3m^\circ \sin(60^\circ - 3m^\circ)\sin(60^\circ + 3m^\circ)\\ &= \frac{3}{4^{40}}\sin 9^\circ \sin 18^\circ \sin 27^\circ \cdots \sin 81^\circ\\ &= \frac{3}{4^{40}}\sin 9^\circ \cos 9^\circ \sin 18^\circ \cos 18^\circ \sin 27^\circ \cos 27^\circ \sin 36^\circ \cos 36^\circ \sin 45^\circ\\ &= \frac{3\sqrt{2}}{2^{85}}\sin 18^\circ \sin 36^\circ \sin 54^\circ \sin 72^\circ\\ &= \frac{3\sqrt{2}}{2^{85}}\sin 18^\circ \cos 18^\circ \sin 36^\circ \cos 36^\circ\\ &= \frac{3\sqrt{2}}{2^{87}}\sin 36^\circ \sin 72^\circ\\ &= \frac{3\sqrt{2}}{2^{87}}\frac{\sqrt{10-2\sqrt{5}}}{4}\frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{3}{2^{89}}\sqrt{10}.\end{aligned}$$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by n, where $[x]$ denotes the greatest integer less than or equal to x. (*Source: 1993 Jiangsu Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **Fai YUNG**.

We claim the unique number is $A = n+1$.

If $n = 2$, then $1 \le A < n^2 = 4$ and only $A = 3$ works. If $n > 2$, then $[n^2/A]+1$ divisible by n implies

$$\frac{n^2}{A} + 1 \ge \left[\frac{n^2}{A}\right] + 1 \ge n.$$ This leads to

$$A \le \frac{n^2}{n-1} = n + 1 + \frac{1}{n-1}.$$ So $A \le n + 1$.

The case $A = n + 1$ works because

$$\left[\frac{n^2}{n+1}\right] + 1 = (n-1) + 1 = n.$$

The case $A = n$ does not work because $[n^2/n] + 1 = n + 1$ is not divisible by n when $n > 1$.

For $0 < A < n$, assume $[n^2/A]+1 = kn$ for some positive integer k. This leads to

$$kn - 1 = \left[\frac{n^2}{A}\right] \le \frac{n^2}{A} < \left[\frac{n^2}{A}\right] + 1 = kn,$$

which implies $n < kA \le (n^2+A)/n < n+1$. This is a contradiction as kA is an integer and cannot be strictly between n and $n + 1$.

Problem 258. (*Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romaina*) Show that if A, B, C are in the interval $(0, \pi/2)$, then

$$f(A,B,C) + f(B,C,A) + f(C,A,B) \ge 3,$$

where

$$f(x,y,z) = \frac{4\sin x + 3\sin y + 2\sin z}{2\sin x + 3\sin y + 4\sin z}.$$

Solution. **Samuel Liló Abdalla** (Brazil), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Fai YUNG**.

Note

$$f(x,y,z) + 1 = \frac{6\sin x + 6\sin y + 6\sin z}{2\sin x + 3\sin y + 4\sin z}.$$

For a, b, $c > 0$, by the *AM-HM* inequality, we have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9.$$

Multiplying by $\frac{2}{3}$ on both sides, we get

$$(a+b+c)\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 6. \quad (*)$$

Let $r = \sin A$, $s = \sin B$, $t = \sin C$, $a = 1/(2r+3s+4t)$, $b = 1/(2s+3t+4r)$ and

$c = 1/(2t + 3r + 4s)$. Then

$$\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 6r + 6s + 6t.$$

Using (*), we get

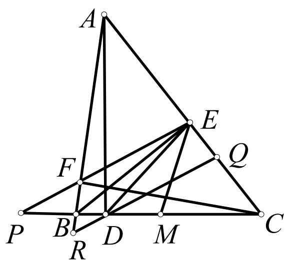
$$f(A,B,C)+f(B,C,A)+f(C,A,B)+3$$
$$=\frac{6r+6s+6t}{2r+3s+4t}+\frac{6r+6s+6t}{2s+3t+4r}+\frac{6r+6s+6t}{2t+3r+4s}$$
$$\geq 6.$$

The result follows.

Problem 259. Let AD, BE, CF be the altitudes of acute triangle ABC. Through D, draw a line parallel to line EF intersecting line AB at R and line AC at Q. Let P be the intersection of lines EF and CB. Prove that the circumcircle of ΔPQR passes through the midpoint M of side BC.

(*Source: 1994 Hubei Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA).



Observe that

(1) $\angle BFC = 90^\circ = \angle BEC$ implies B, F, E, C concyclic;

(2) $\angle AEB = 90^\circ = \angle ADB$ implies A,B,D,E concyclic.

By (1), we have $\angle ACB = \angle AFE$. From $EF||QR$, we get $\angle AFE = \angle ARQ$. So $\angle ACB = \angle ARQ$. Then B, Q, R, C are concyclic. By the intersecting chord theorem,

$$RD\cdot QD=BD\cdot CD \qquad (*)$$

Since $\angle BEC = 90^\circ$ and M is the midpoint of BC, we get $MB = ME$ and $\angle EBM = \angle BEM$. Now

$$\angle EBM = \angle EPM + \angle BEP$$
$$\angle BEM = \angle DEM + \angle BED.$$

By (1) and (2), $\angle BEP = \angle BCF = 90^\circ - \angle ABC = \angle BAD = \angle BED$. So $\angle EPM = \angle DEM$. Then right triangles EPM and DEM are similar. We have $ME/MP = MD/ME$ and so

$$MB^2=ME^2=MD\cdot MP=MD(MD+PD)$$
$$=MD^2+MD\cdot PD.$$

Then $MD\cdot PD = MB^2 - MD^2$
$= (MB-MD)(MB+MD)$
$= BD(MC+MD)$
$= BD\cdot CD.$

Using (*), we get $RD\cdot QD=MD\cdot PD$. By the converse of the intersecting chord theorem, P, Q, R, M are concyclic.

Commended solvers: **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Problem 260. In a class of 30 students, number the students 1, 2, …, 30 from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student <u>*good*</u> if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.

(*Source: 1998 Hubei Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA) and **Fai YUNG**.

Suppose each student has m friends and n is the maximum number of good students. There are $15m$ pairs of friendship.

For m odd, $m = 2k - 1$ for some positive integer k. For $j = 1, 2, \ldots, k$, student j has at least $(2k-j) \geq k > m/2$ worse friends, hence student j is good. For the other $n - k$ good students, every one of them has at least k worse friends. Then

$$\sum_{j=1}^{k}(2k-j)+(n-k)k \leq 15(2k-1).$$

Solving for n, we get

$$n \leq 30.5-\left(\frac{15}{k}+\frac{k}{2}\right) \leq 30.5-\sqrt{30} < 26.$$

For m even, $m = 2k$ for some positive integer k. For $j = 1, 2, \ldots, k$, student j has at least $(2k + 1 - j) > k = m/2$ worse friends, hence student j is good. For the other $n - k$ good students, every one of them has at least $k + 1$ worse friends. Then

$$\sum_{j=1}^{k}(2k+1-j)+(n-k)(k+1) \leq 15\cdot 2k.$$

Solving for n, we get

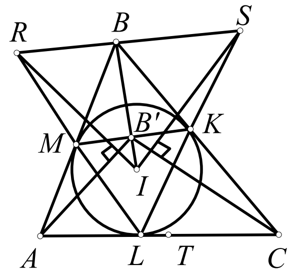
$$n \leq 31.5-\left(\frac{31}{k+1}+\frac{k+1}{2}\right) \leq 31.5-\sqrt{62} < 24.$$

Therefore, $n \leq 25$. For an example of $n = 25$, in the odd case, we need to take $k = 5$ (so $m = 9$). Consider the 6×5 matrix M with $M_{ij}= 5(i - 1) + j$. For M_{1j}, let his friends be M_{6j}, M_{1k} and M_{2k} for all $k \neq j$. For M_{ij} with $1 < i < 6$, let his friends be M_{6j}, $M_{(i-1)k}$ and $M_{(i+1)k}$ for all $k \neq j$. For M_{6j}, let his friends be M_{ij} and M_{5k} for all $i < 6$ and $k \neq j$. It is easy to check 1 to 25 are good.

Pole and Polar

(continued from page 2)

<u>***Example 7.***</u> (*1998 IMO*) Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that angle RIS is acute.



<u>***Solution.***</u> Consider the pole-polar transformation with respect to the incircle. Due to tangency, the polars of B, K, L, M are lines MK, BC, CA, AB respectively. Observe that B is sent to $B' = IB\cap MK$ under the inversion with respect to the incircle. Since B' is on line MK, which is the polar of B, by La Hire's theorem, B is on the polar of B'. Since $MK||RS$, so the polar of B' is line RS. Since R,B,S are collinear, their polars concur at B'.

Next, since the polars of K, L intersect at C and since L, K, S are collinear, their polars concur at C. Then the polar of S is $B'C$. By the definition of polar, we get $IS\perp B'C$. By a similar reasoning, we also get $IR\perp B'A$. Then $\angle RIS = 180^\circ - \angle AB'C$.

To finish, we will show B' is inside the circle with diameter AC, which implies $\angle AB'C > 90^\circ$ and hence $\angle RIS < 90^\circ$. Let T be the midpoint of AC. Then

$$2\overrightarrow{B'T} = \overrightarrow{B'C} + \overrightarrow{B'A}$$
$$= (\overrightarrow{B'K} + \overrightarrow{KC}) + (\overrightarrow{B'M} + \overrightarrow{MA})$$
$$= \overrightarrow{KC} + \overrightarrow{MA}.$$

Since $\overrightarrow{KC}$ and $\overrightarrow{MA}$ are nonparallel,

$$B'T < \frac{KC+MA}{2} = \frac{CL+AL}{2} = \frac{AC}{2}.$$

Therefore, B' is inside the circle with diameter AC.

Mathematical Excalibur

Volume 11, Number 5 *January 2007 – February 2007*

Olympiad Corner

Below are the 2006 British Math Olympiad (Round 2) problems.

Problem 1. Find the minimum possible value of x^2+y^2 given that x and y are real numbers satisfying $xy(x^2-y^2) = x^2+y^2$ and $x \neq 0$.

Problem 2. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy $x^2 - y^2 = 2^k$ for some non-negative integer k.

Problem 3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that $\angle BPC + \angle BAC = 180°$.

Problem 4. An exam consisting of six questions is sat by 2006 children. Each question is marked right or wrong. Any three children have right answers to at least five of the six questions between them. Let N be the total number of right answers achieved by all the children (i.e. the total number of questions solved by child 1 + the total solved by child 2 + ··· + the total solved by child 2006). Find the least possible value of N.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 25, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Difference Operator

Kin Y. Li

Let h be a nonzero real number and $f(x)$ be a function. When $f(x+h)$ and $f(x)$ are real numbers, we call

$$\Delta_h f(x) = f(x+h) - f(x)$$

the <u>*first difference of f at x with step h*</u>. For functions f, g and real number c, we have

$$\Delta_h(f+g)(x) = \Delta_h f(x) + \Delta_h g(x) \text{ and}$$
$$\Delta_h(cf)(x) = c\Delta_h f(x).$$

Also, $\Delta_h^0 f(x)$ or $I f(x)$ stands for $f(x)$.

For any integer $n \geq 1$, we define the <u>*n-th difference*</u> by $\Delta_h^n f(x) = \Delta_h(\Delta_h^{n-1} f)(x)$. For example,

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$
$$\Delta_h^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x).$$

By induction, we can check that

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} C_n^k f(x+kh), \qquad (\alpha)$$

where $C_n^0 = 1$ and for $k > 0$,

$$C_n^k = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

(<u>*Note*</u>: for these formulas, we may even let n be a real number!!!)

If h=1, we simply write Δ and omit the subscript h. For example, in case of a sequence $\{x_n\}$, we have $\Delta x_n = x_{n+1} - x_n$.

<u>*Facts.*</u>(1) For function $f(x)$, n=0,1,2,…,

$$f(x+n) = \sum_{k=0}^{n} C_n^k \Delta^k f(x);$$

in particular, if $\Delta^m f(n)$ is a nonzero constant for every positive integer n, then $f(n) = \sum_{k=0}^{m} C_n^k \Delta^k f(0);$

(2) if $P(x) = ax^n + \cdots$ is a polynomial of degree n, then for all x, $\Delta_h^n P(x) = an!h^n$ and $\Delta_h^m P(x) = 0$ for $m>n$.

Let k be a positive integer. As a function of x, C_x^k has the properties:

(a) $C_x^{k-1} + C_x^k = C_{x+1}^k$ (so $\Delta C_x^k = C_x^{k-1}$);

(b) for $0 \leq r \leq k$, $\Delta^r C_x^k = C_x^{k-r}$;
for $r > k$, $\Delta^r C_x^k = 0$;

(c) $C_1^k + C_2^k + \cdots + C_n^k = C_{n+1}^{k+1}$ (just add C_1^{k+1}=0 to the left and apply (a) repeatedly).

Similar to fact (1), if $f(x)$ is a degree m polynomial, then

$$f(x) = \sum_{k=0}^{m} C_x^k \Delta^k f(0). \qquad (\beta)$$

(This is because both sides are degree m polynomials and from property (b), the k-th differences at 0 are the same for $k = 0$ to m, which implies the values of both sides at 0, 1, 2, …, m are the same.)

<u>***Example 1.***</u> Sum $S_n = 1^4 + 2^4 + \cdots + n^4$ in terms of n.

<u>***Solution.***</u> Let $f(x) = x^4$. By (β) and (c),

$$S_n = \sum_{j=1}^{n} f(j) = \sum_{j=1}^{n}\sum_{k=0}^{4} C_j^k \Delta^k f(0)$$
$$= \sum_{k=0}^{4}\Big(\sum_{j=1}^{n} C_j^k\Big)\Delta^k f(0) = \sum_{k=0}^{4} C_{n+1}^{k+1}\Delta^k f(0).$$

x :	0	1	2	3	4
$f(x)$:	0	1	16	81	256
$\Delta f(x)$:	1	15	65	175	
$\Delta^2 f(x)$:	14	50	110		
$\Delta^3 f(x)$:	36	60			
$\Delta^4 f(x)$:	24				

Therefore,

$$S_n = \binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}$$
$$= n(n+1)(2n+1)(3n^2+3n-1)/30.$$

<u>***Example 2.***</u> (*2000 Chinese IMO Team Selection Test*) Given positive integers k, m, n satisfying $1 \leq k \leq m \leq n$. Find

$$\sum_{i=0}^{n} (-1)^i \frac{1}{n+k+i}\,\frac{(m+n+i)!}{i!(n-i)!(m+i)!}.$$

<u>***Solution.***</u> Define

$$g(x) = \frac{(x+m+1)(x+m+2)\cdots(x+m+n)}{x+n+k}.$$

From $1 \leq k \leq m \leq n$, we see $m+1 \leq n+k \leq m+n$. So $g(x)$ is a polynomial of degree $n-1$. By fact (2) and formula (α),

$$0 = (-1)^n \Delta^n g(0) = \sum_{i=0}^{n} (-1)^i C_n^i g(i)$$
$$= \sum_{i=0}^{n} (-1)^i \frac{n!}{i!(n-i)!}\,\frac{(m+n+i)!}{(m+i)!}\,\frac{1}{n+k+i}.$$

The required sum is $(-1)^n\ \Delta^n g(0)/n! = 0$.

Example 3. (*1949 Putnam Exam*) The sequence $x_0, x_1, x_2, \ldots$ is defined by the conditions $x_0 = a$, $x_1 = b$ and for $n \ge 1$,

$$x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n},$$

where a and b are given numbers. Express $\lim_{n\to\infty} x_n$ in terms of a and b.

Solution. The recurrence relation can be written as

$$\Delta x_n = -\frac{\Delta x_{n-1}}{2n}.$$

Repeating this $n-2$ times, we get

$$\Delta x_n = \left(-\frac{1}{2}\right)^n \frac{1}{n!}\Delta x_1 = \left(-\frac{1}{2}\right)^n \frac{1}{n!}(b-a).$$

Then

$$x_n = x_0 + \sum_{i=0}^{n-1}\Delta x_i = a + (b-a)\sum_{i=0}^{n-1}\left(-\frac{1}{2}\right)^i \frac{1}{i!}.$$

Using the fact

$$e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!},$$

we get $\lim_{n\to\infty} x_n = a + (b-a)e^{-1/2}$.

Example 4. (*2004 Chinese Math Olympiad*) Given a positive integer c, let $x_1, x_2, x_3, \ldots$ satisfy $x_1 = c$ and

$$x_n = x_{n-1} + \left[\frac{2x_{n-1} - (n+2)}{n}\right] + 1$$

for $n = 2,3,\ldots$, where $[x]$ is the greatest integer less than or equal to x. Find a general formula of x_n in terms of n.

Solution. First tabulate some values.

	x_1	x_2	x_3	x_4	x_5	x_6
c=1	1	1	1	1	1	1
c=2	2	3	4	5	6	7
c=3	3	5	7	10	13	17
c=4	4	7	11	16	22	29
c=5	5	9	14	20	27	35
c=6	6	11	17	25	34	45
c=7	7	13	21	31	43	57

Next tabulate first differences in each column.

column 1: 1,1,1,1,1,1,…
column 2: 2,2,2,2,2,2,…
column 3: 3,3,4,3,3,4,…
column 4: 4,5,6,4,5,6,…
column 5: 5,7,9,5,7,9,…
column 6: 6,10,12,6,10,12,…

We suspect they are periodic with period 3. Let $x(c,n)$ be the value of x_n for the sequence with $x_1 = c$. For rows 1 and 2, the first differences seem to be constant and for row 4, the second differences seem to be constant. Using fact (1) and induction, we get

$$x(1,n) = 1,\quad x(2,n) = n + 1 \qquad \text{(i)}$$

and $x(4,n) = (n^2+3n+4)/2$ for all n. Now

$$x(4,n) - x(1,n) = \frac{(n+1)(n+2)}{2}.$$

To check the column difference periodicity, we claim that for a fixed c,

$$x(c+3, n) = x(c, n) + (n+1)(n+2)/2.$$

If $n = 1$, then $x(c+3,1) = c + 3 = x(c,1) + 3$ and so case $n = 1$ is true. Suppose the case $n-1$ is true. By the recurrence relation, $x(c+3,n)$ equals

$$x(c+3,n-1) + \left[\frac{2x(c+3,n-1)-(n+2)}{n}\right] + 1.$$

From the case $n-1$, we get $x(c+3, n-1) = x(c, n-1) + n(n+1)/2$. Using this, the displayed expression simplifies to

$$x(c,n-1) + \left[\frac{2x(c,n-1)-(n+2)}{n}\right] + \frac{n^2+3n+4}{2},$$

which is $x(c, n) + (n+1)(n+2)/2$ by the recurrence relation. This completes the induction for the claim.

Now the claim implies

$$x(c,n) = x(d,n) + \left(\frac{c-d}{3}\right)\frac{(n+1)(n+2)}{2}, \qquad \text{(ii)}$$

where $d = 1,2$ or 3 subject to $c \equiv d \pmod 3$. Since $x(1,n)$ and $x(2,n)$ are known, all we need to find is $x(3,n)$.

For the case $c = 3$, studying $x_1,x_3,x_5,\ldots$ and $x_2,x_4,x_6,\ldots$ separately, we can see that the second differences of these sequences seem to be constant. Using fact (1) and induction, we get

$x(3,n) = (n^2+4n+7)/4$ if n is odd and
$x(3,n) = (n^2+4n+8)/4$ if n is even. (iii)

Formula (ii) along with formulas (i) and (iii) provided the required answer for the problem.

Example 5. Let $g(x)$ be a polynomial of degree n with real coefficients. If $a \ge 3$, then prove that one of the numbers $|1 - g(0)|, |a - g(1)|, |a^2 - g(2)|, \ldots, |a^{n+1} - g(n+1)|$ is at least 1.

Solution. Let $f(x) = a^x - g(x)$. We have

$$\Delta a^x = a^{x+1} - a^x = (a-1)a^x,$$
$$\Delta^2 a^x = (a-1)\,\Delta a^x = (a-1)^2 a^x,$$
$$\ldots,$$
$$\Delta^{n+1} a^x = (a-1)^{n+1} a^x.$$

In particular, $\Delta^{n+1} a^0 = (a-1)^{n+1}$. Now

$$\Delta^{n+1} f(0) = \Delta^{n+1} a^0 - \Delta^{n+1} g(0) = (a-1)^{n+1}.$$

Since $a \ge 3$, we get $2^{n+1} \ge \Delta^{n+1} f(0)$. Assume $|a^k - g(k)| < 1$ for $k = 0,1,\ldots, n+1$. Then

$$\Delta^{n+1} f(0) = \sum_{k=0}^{n+1}(-1)^{n+1-k} C_{n+1}^k\left(a^k - g(k)\right)$$

$$< \sum_{k=0}^{n+1} C_{n+1}^k = 2^{n+1},$$

which is a contradiction.

Example 6. (*1984 USAMO*) Let $P(x)$ be a polynomial of degree $3n$ such that

$$P(0) = P(3) = \cdots = P(3n) = 2,$$
$$P(1) = P(4) = \cdots = P(3n-2) = 1,$$
$$P(2) = P(5) = \cdots = P(3n-1) = 0.$$

If $P(3n+1)$, then find n.

Solution. By fact (2) and (α),

$$0 = (-1)^{n+1}\Delta^{3n+1}P(0) = \sum_{k=0}^{3n+1}(-1)^k C_{3n+1}^k P(3n+1-k)$$

$$= 730 + 2\sum_{j=0}^{n}(-1)^{3j+1}C_{3n+1}^{3j+1} + \sum_{j=1}^{n}(-1)^{3j}C_{3n+1}^{3j}.$$

We can write this as $2a+b = -3^6$, where

$$a = \sum_{j=0}^{n}(-1)^{3j+1}C_{3n+1}^{3j+1},\quad b = \sum_{j=0}^{n}(-1)^{3j}C_{3n+1}^{3j}.$$

To find a and b, we consider the cube root of unity $\omega = e^{2\pi i/3}$, the binomial expansion of $f(x) = (1-x)^{3n+1}$ and let

$$c = \sum_{j=1}^{n}(-1)^{3j-1}C_{3n+1}^{3j-1}.$$

Now $0 = f(1) = b-a-c$, $f(\omega) = b - a\omega - c\omega^2$ and $f(\omega^2) = b - a\omega^2 - c\omega$. Solving, we see

$$a = -\left(f(1) + \omega^2 f(\omega) + \omega f(\omega^2)\right)/3$$
$$= 2\left(\sqrt{3}\right)^{3n-1}\cos\frac{3n-1}{6}\pi$$

and

$$b = \left(f(1) + f(\omega) + f(\omega^2)\right)/3$$
$$= 2\left(\sqrt{3}\right)^{3n-1}\cos\frac{3n+1}{6}\pi.$$

Studying the equation $2a + b = -3^6$, we find that it has no solution when n is odd and one solution when n is even, namely when $n = 4$.

Example 7. (*1980 Putnam Exam*) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \ge 0$?

Solution. Among all sequences satisfying $u_{n+1} = 2u_n - n^2$ for all $n \ge 0$, the difference v_n of any two such sequences will satisfy $v_{n+1} = 2v_n$ for all $n \ge 0$. Then $v_n = 2^n v_0$ for all $n \ge 0$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***March 25, 2007.***

Problem 266. Let

$$N = 1+10+10^2+\cdots+10^{1997}.$$

Determine the 1000^{th} digit after the decimal point of the square root of N in base 10.

Problem 267. For any integer a, set

$$n_a = 101a - 100\cdot 2^a.$$

Show that for $0 \le a, b, c, d \le 99$, if

$$n_a + n_b \equiv n_c + n_d \pmod{10100},$$

then $\{a,b\} = \{c,d\}$.

Problem 268. In triangle ABC, $\angle ABC = \angle ACB = 40^\circ$. Points P and Q are inside the triangle such that $\angle PAB = \angle QAC = 20^\circ$ and $\angle PCB = \angle QCA = 10^\circ$. Must B, P, Q be collinear? Give a proof.

Problem 269. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$, $a_{n+1} = f(a_n)$ for all $n \ge 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

Problem 270. The distance between any two of the points A, B, C, D on a plane is at most 1. Find the minimum of the radius of a circle that can cover these four points.

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Solutions

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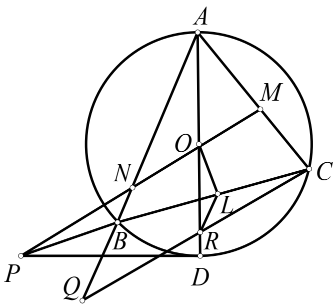
Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **Naoki S. D. LING**, **Anna Ying PUN** (HKU, Math, Year 1), **Simon YAU Chi Keung**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Consider the tens digits of the 13 consecutive positive integers. By the pigeonhole principle, there are at least $[13/2] + 1 = 7$ of them with the same tens digit. The sums of digits for these 7 numbers are consecutive. Hence, one of the sums of digits is divisible by 7.

Problem 262. Let O be the center of the circumcircle of ΔABC and let AD be a diameter. Let the tangent at D to the circumcircle intersect line BC at P. Let line PO intersect lines AC, AB at M, N respectively. Prove that $OM=ON$.

Solution 1. **Jeff CHEN** (Virginia, USA).



We may assume B is between P and C (otherwise interchange B and C, then N and M). Through C, draw a line parallel to line MN and intersect line AN at Q. Let line AO intersect line CQ at R. Since $MN \| CQ$, triangles AMN and ACQ are similar. To show $OM = ON$, it suffices to show $RC = RQ$.

Let L be the midpoint of BC. We will show $LR \parallel BQ$ (which implies $RC = RQ$).

Now $\angle OLP = \angle OLB = 90^\circ = \angle ODP$, which implies O, P, D, L are concyclic. Then $\angle ODL = \angle OPL$. From $OP \parallel RC$, we get $\angle RDL = \angle RCL$, which implies L, R, D, C are concyclic. Then

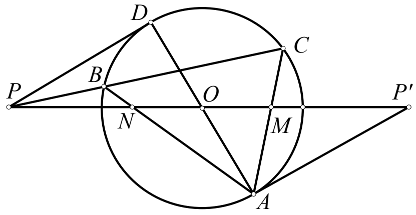
$$\begin{aligned}\angle RLC &= 180^\circ - \angle RDC = 180^\circ - \angle ADC \\ &= 180^\circ - \angle ABC = \angle QBC.\end{aligned}$$

Therefore, $LR \parallel BQ$ as claimed.

Solution 2. **CHEUNG Wang Chi** (Raffles Junior College, Singapore).

Set O as the origin and line MN as the x-axis.

Let P' be the reflection of P with respect to O. Then the coordinates of P and P' are of the form $(p,0)$ and $(-p,0)$.



The equation of the circumcircle as a conic section is of the form $x^2+y^2-r^2=0$. The equation of the pair of lines AP' and BC as a (degenerate) conic section is

$$\big(y - m(x+p)\big)\big(y - n(x-p)\big) = 0,$$

where m is the slope of line AP' and n is the slope of line BC. Since these two conic sections intersect at A, B, C, so the equation of the pair of lines AB and AC as a (degenerate) conic section is of the form

$$x^2 + y^2 - r^2 = \lambda\big(y - m(x+p)\big)\big(y - n(x-p)\big),$$

for some real number λ. When we set $y = 0$, we see the x-coordinates of M and N satisfies $x^2 - r^2 = \lambda mn(x^2 - p^2)$, whose roots are some positive number and its negative. Therefore, $OM = ON$.

Commended solvers: **Courtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Anna Ying PUN** (HKU, Math, Year 1).

Problem 263. For positive integers m, n, consider a $(2m+1)\times(2n+1)$ table, where in each cell, there is exactly one ant. At a certain moment, every ant moves to a horizontal or vertical neighboring cell. Prove that after that moment, there exists a cell with no ant.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **Naoki S. D. LING**, **Anna Ying PUN** (HKU, Math, Year 1), **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Assign the value $(-1)^{i+j}$ to the cell in the i-th row, j-th column of the table. Then two horizontal or vertical neighboring cells will have values of opposite sign. Since $2m+1$ and $2n+1$ are odd, there is exactly one more cell with negative values than cells with positive values. Before the moment, there is one more ant in cells with negative values than ants in cells with positive values. After the moment, two of the ants from cells with negative values will occupy a common cell with a positive value. Then there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers a, b, prove that $ab^p - ba^p$ is divisible by $6p$.

Solution. **Samuel Liló ABDALLA** (São Paulo, Brazil), **Claudio ARCONCHER** (Jundiaí, Brazil), **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **HO Ka Fai** (Carmel Divine Grace Foundation Secondary School, Form 6), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Anna Ying PUN** (HKU, Math, Year 1), **Simon YAU Chi Keung**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Observe that

$ab^p - ba^p = ab[(b^{p-1}-1)-(a^{p-1}-1)].$

For $q = 2$, 3 or p, if a or b is divisible by q, then the right side is divisible by q.

Otherwise, a and b are relatively prime to q. Now $p - 1$ is divisible by $q - 1$, which is 1, 2 or $p - 1$. By Fermat's little theorem, both $a^{q-1}, b^{q-1} \equiv 1 \pmod q$. So $a^{p-1}, b^{p-1} \equiv 1 \pmod q$. Hence, the bracket factor above is divisible by q. Thus $ab^p - ba^p$ is divisible by 2, 3 and p. Therefore, it is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n}(x_j^4 - x_j^5),$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.
(*Source: 1999 Chinese IMO Team Selection Test*)

Solution. **Jeff CHEN** (Virginia, USA), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Anna Ying PUN** (HKU, Math, Year 1) and **YIM Wing Yin** (HKU, Year 1).

Let $f(x) = x^4 - x^5 = x^4(1-x)$. Since $f''(x) = 4x^2(3-5x)$, we see that $f(x)$ is strictly convex on [0, 3/5]. Suppose $n \ge 3$. Without loss of generality, we may assume $x_1 \ge x_2 \ge \cdots \ge x_n$. If $x_1 \le 3/5$, then since

$$\left(\frac{3}{5},\frac{2}{5},0,\cdots,0\right) \succ (x_1, x_2, \cdots, x_n),$$

by the majorization inequality (see <u>Math Excalibur</u>, *vol.* 5, *no.* 5, *pp.* 2,4),

$$\sum_{i=1}^{n} f(x_i) \le f\left(\frac{3}{5}\right) + f\left(\frac{2}{5}\right).$$

If $x_1 > 3/5$, then $1 - x_1, x_2, \ldots, x_n$ are in [0, 2/5]. Since

$$(1-x_1, 0, \ldots, 0) \succ (x_2, \ldots, x_n),$$

by the majorization inequality,

$$\sum_{i=1}^{n} f(x_i) \le f(x_1) + f(1-x_1).$$

Thus the problem is reduced to the case $n = 2$. So now consider nonnegative a, b with $a + b = 1$. We have

$$\begin{aligned} f(a) + f(b) &= a^4(1-a) + b^4(1-b) \\ &= a^4b + b^4a = ab(a^3+b^3) \\ &= ab[(a+b)^3 - 3ab(a+b)] \\ &= 3ab(1-3ab)/3 \\ &\le 1/12 \end{aligned}$$

by the *AM-GM* inequality. Equality case holds when $ab = 1/6$ in addition to $a + b = 1$, for example when

$$(a,b) = \left(\frac{3+\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}\right).$$

Therefore, the maximum is 1/12.

Difference Operator

(continued from page 2)

Next we look for a particular solution of $u_{n+1} = 2u_n - n^2$ for all $n \ge 0$. Observe that $n^2 = u_n - (u_{n+1} - u_n) = (I - \Delta)u_n$. From the sum of geometric series, we guess

$$u_n = (I-\Delta)^{-1}n^2 = (I + \Delta + \Delta^2 + \cdots)n^2$$

$$= n^2 + (2n+1) + 2 = n^2 + 2n + 3$$

should work. Indeed, this is true since $(n+1)^2 + 2(n+1) + 3 = 2(n^2 + 2n + 3) - n^2$.

Combining, we see that the general solution to $u_{n+1} = 2u_n - n^2$ for all $n \ge 0$ is

$$u_n = n^2 + 2n + 3 + 2^n v_0 \text{ for any real } v_0.$$

Finally, to have $u_0 = a$, we must choose $v_0 = a - 3$. Hence, the sequence we seek is

$$u_n = n^2 + 2n + 3 + 2^n(a-3) \quad \text{for all } n \ge 0.$$

Since $$\lim_{n\to\infty} \frac{2^n}{n^2+2n+3} = +\infty,$$

u_n will be negative for large n if $a - 3 < 0$. Conversely, if $a - 3 \ge 0$, then all $u_n > 0$. Therefore, the answer is $a \ge 3$.

<u>Example 8.</u> (*1971 Putnam Exam*) Let c be a real number such that n^c is an integer for every positive integer n. Show that c is a non-negative integer.

<u>Solution.</u> 2^c is an integer implies $c \ge 0$.

Next we will do the case c is between 0 and 1 using the mean value theorem. This motivates and clarifies the general argument for the case $c \ge 1$. Assume $0 \le c < 1$. Then choose a positive integer $n > c^{1/(1-c)}$. Applying the mean value theorem to $f(x) = x^c$ on $[n, n+1]$, we know there exists a number w between n and $n + 1$ such that $\Delta n^c = (n+1)^c - n^c = cw^{c-1}$. On the left side, we have an integer, but on the right side, since $w > n > c^{1/(1-c)}$, we have $0 \le cw^{c-1} < 1$. Hence, $c = 0$.

For $c \ge 1$, let us mention there is an extension of the mean value theorem, which asserts that if f is continuous on $[a,b]$, k-times differentiable on (a,b), $0 < h \le (b-a)/k$ and $x + kh \le b$, then there exists a number v such that $a < v < b$ and

$$\frac{\Delta_h^k f(x)}{h^k} = f^{(k)}(v).$$

Taking this for the moment, we will finish the argument as follows. Let k be the integer such that $k-1 \le c < k$. Choose an integer n so large that

$$c(c-1)(c-2)\cdots(c-k+1)n^{c-k} < 1.$$

Applying the extension of the mean value theorem mentioned above to $f(x) = x^c$ on $[n, n+k]$, there is a number v between n and $n + k$ such that

$$\Delta^k n^c = c(c-1)(c-2)\cdots(c-k+1)v^{c-k}.$$

Again, the left side is an integer, but the right side is in the interval [0, 1). Therefore, both sides are 0 and $c = k-1$.

Now the extension of the mean value theorem can be proved by doing math induction on k. The case $k = 1$ is the mean value theorem. Next, suppose the extension is true for the case $k-1$. Let $0 < h \le (b-a)/k$. On $[a, b-h]$, define

$$g(x) = \frac{\Delta_h f(x)}{h} = \frac{f(x+h) - f(x)}{h}.$$

Applying the case $k-1$ to $g(x)$, we know there exists a number v_0 such that $a < v_0 < b-h$ and

$$\frac{\Delta_h^k f(x)}{h^k} = \frac{\Delta_h^{k-1} g(x)}{h^{k-1}} = g^{(k-1)}(v_0).$$

By the mean value theorem, there exists h_0 such that $0 < h_0 < h$ and

$$g^{(k-1)}(v_0) = \frac{f^{(k-1)}(v_0+h) - f^{(k-1)}(v_0)}{h}$$

$$= f^{(k)}(v_0 + h_0).$$

Finally, $v = v_0 + h_0$ is between a and b.

Mathematical Excalibur

Volume 12, Number 1 *March 2007 – April 2007*

Olympiad Corner

Below are the 2007 Asia Pacific Math Olympiad problems.

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H be the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

Problem 3. Consider n disks $C_1, C_2, \cdots, C_n$ in a plane such that for each $1 \le i < n$, the center C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the score of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z} = 1$. Prove that

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}} \ge 1.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 31, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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From *How to Solve It* to Problem Solving in Geometry

K. K. Kwok
Munsang College (HK Island)

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Geometry is the science of
correct reasoning on
incorrect figures.

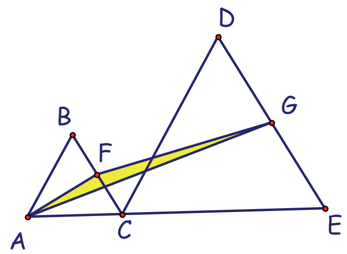
If you can't solve a problem,
then there is an easier
problem you can solve, find
it.
```

George Pölya

幾何是：在靜止中看出動態，從變幻中覓得永恆

數學愛好者，強

Example 1. In the figure below, C is a point on AE. ΔABC and ΔCDE are equilateral triangles. F and G are the midpoints of BC and DE respectively. If the area of ΔABC is 24 cm^2, the area of ΔCDE is 60 cm^2, find the area of ΔAFG.



Idea and solution outline:
This question is easy enough and can be solved by many different approaches. One of them is to recognize that the extensions of AF and CG are parallel. (Why? At what angles do they intersect line AE?) Thus $[AFC] = [AFG]$.

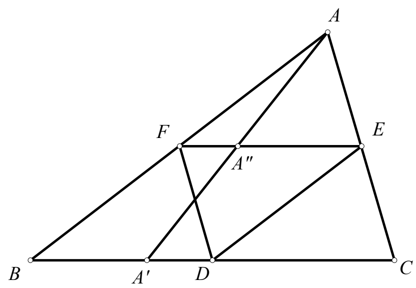
Example 2. In ΔABC, $AB = AC$. A point P on the plane satisfies $\angle ABP = \angle ACP$. Show that P is either on BC or on the perpendicular bisector of BC.

Solution:
Apply the sine law to ΔABP and ΔACP, we have

$$\sin\angle APB = \frac{AB\sin\angle ABP}{AP} = \frac{AC\sin\angle ACP}{AP} = \sin\angle APC.$$

Thus, either $\angle APB = \angle APC$ or $\angle APB + \angle APC = 180^\circ$. The first case implies $\Delta ABP \cong \Delta ACP$, so $BP = CP$ and P lies on the perpendicular bisector of BC. The second case implies P lies on BC.

Example 3. [Tournament of Towns 1993] Vertices A, B and C of a triangle are connected to points A', B' and C' lying in their respective opposite sides of the triangle (not at vertices). Can the midpoints of the segments AA', BB' and CC' lie in a straight line?

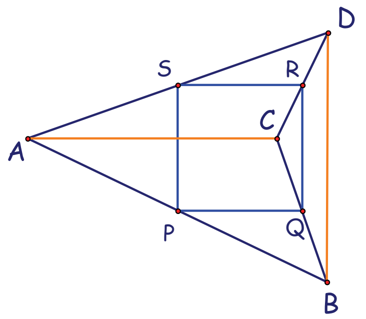


Solution outline:
Let D, E and F be midpoints of BC, AC, and AB respectively. Given any point A' on BC, let AA' intersect EF at A''. Then it is easy to see that A'' is indeed the midpoint of AA'.

Therefore, the midpoints of the segments AA', BB' and CC' lie respectively on EF, DF and DE, and cannot be collinear.

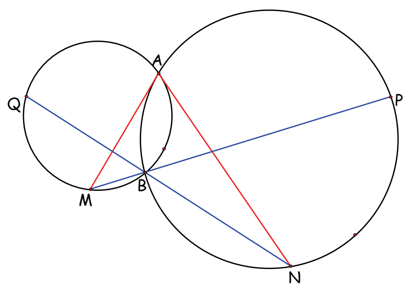
Example 4. [Tournament of Towns 1993] Three angles of a non-convex, non-self-intersecting quadrilateral are equal to 45 degrees (i.e. the last equals 225 degrees). Prove that the midpoints of its sides are vertices of a square.

Idea:
Do you know a similar, but easier problem? For example, the famous *Varignon Theorem*: By joining the midpoints of the sides of an arbitrary quadrilateral, a parallelogram is formed.



Solution outline:
Extend BC to cut AD at O. Then ΔOAB and ΔOCD are both isosceles right-angled triangle. It follows that a 90° rotation about O will map A into B and C into D, so that $AC = BD$ and they are perpendicular to each other.

Example 5. [Tournament of Towns 1994] Two circles intersect at the points A and B. Tangent lines drawn to both of the circles at the point A intersect the circles at the points M and N. The lines BM and BN intersect the circles once more at the points P and Q respectively. Prove that the segments MP and NQ are equal.



Idea:
MP and NQ are sides of the triangles ΔAQN and ΔAMP respectively, so it is natural for us to prove that the two triangles are congruent. It is easy to observe that the two triangles are similar, so what remains to prove is either $AQ = AM$ or $AP = AN$. Note that we can transmit the information between the two circles by using the theorem on alternate segment at A.

Solution outline:

(1) Observe that $\Delta AQN \sim \Delta AMP$.

(2) $AP = AN$ follows from computing

$\angle APN = \angle APB + \angle BPN$

$= \angle ANB + \angle BAN$ [$\angle$s in same segment]

$= \angle ANB + \angle AQN$ [$\angle$ in alt. segment]

$= 180^\circ - \angle QAN$

$= 180^\circ - \angle MAP$ [by step (1)]

$= \angle AMB + \angle APB$

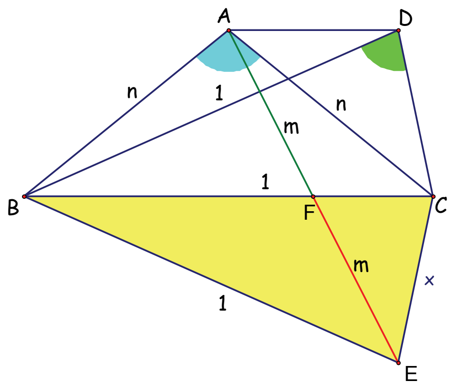
$= \angle AMB + \angle MAB$ [$\angle$ in alt. segment]

$= \angle ABP$ [ext. $\angle$ of Δ]

$= \angle ANP$ [$\angle$s in the same segment].

Example 6. $ABCD$ is a trapezium with $AD \parallel BC$. It is known that $BC = BD = 1$, $AB = AC$, $CD < 1$ and $\angle BAC + \angle BDC = 180^\circ$, find CD.

Idea:
The condition $\angle BAC + \angle BDC = 180^\circ$ leads us to consider a cyclic quadrilateral. If we reflect ΔBDC across BC, a cyclic quadrilateral is formed.



Solution outline:
(1) Let E be the reflection of D across BC.

(2) $\angle BAC + \angle BDC = 180^\circ$
$\Rightarrow \angle BAC + \angle BEC = 180^\circ$
$\Rightarrow ABEC$ is cyclic,

$AD \parallel BC \Rightarrow AF = FE,$

$AB = AC \Rightarrow \angle BEF = \angle FEC$

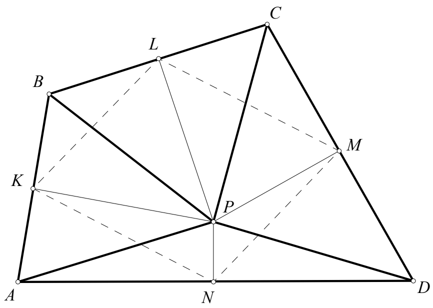
$$\Rightarrow \frac{FC}{BF} = \frac{EC}{BE} = EC.$$

(3) Let $AF = FE = m$, $AB = AC = n$ and $DC = EC = x$. It follows from Ptolemy's theorem that $AE \times BC = AC \times BE + AB \times EC$, i.e. $2m = n(1 + x)$. Now

$$\frac{2}{1+x} = \frac{n}{m} = \frac{AC}{AF} = \frac{BE}{BF} = \frac{BC}{BF} = \frac{BF+FC}{BF}$$
$$= 1 + \frac{FC}{BF} = 1 + \frac{EC}{BE} = 1 + x,$$

i.e. $(1 + x)^2 = 2$. Therefore, $x = \sqrt{2} - 1$.

Example 7. [Tournament of Towns 1995] Let P be a point inside a convex quadrilateral $ABCD$. Let the angle bisector of $\angle APB$, $\angle BPC$, $\angle CPD$ and $\angle DPA$ meet AB, BC, CD and DA at K, L, M and N respectively. Find a point P such that $KLMN$ is a parallelogram.



Idea:
The *angle bisector theorem* enables us to replace the ratios that K, L, N and M divided the sides of the quadrilateral by the ratios of the distance from P to A, B, C and D. For instance, we have

$$\frac{AK}{KB} = \frac{AP}{BP} \text{ and } \frac{AN}{ND} = \frac{AP}{DP}$$

If $BP = DP$, we have $\frac{AK}{KB} = \frac{AN}{ND}$ and hence $KN \parallel BD$. Similarly, we have $LM \parallel BD$ and so $KN \parallel LM$.

Therefore, we shall look for a point P such that $BP = DP$ and $AP = CP$.

Solution outline:
(1) Let P be the point of intersection of the perpendicular bisectors of the diagonals AC and BD. Then $AP = CP$ and $BP = DP$.

(2) By the angle bisector theorem, we have

$$\frac{AK}{KB} = \frac{AP}{BP} = \frac{AP}{DP} = \frac{AN}{ND}$$

and so $KN \parallel BD$. Similarly, $LM \parallel BD$, $KL \parallel AC$ and $MN \parallel AC$.

Hence $KN \parallel LM$ and $KL \parallel NM$, which means that $KLMN$ is a parallelogram.

Remark: Indeed, point P in the solution above is the only point that satisfies the condition given in the problem.

Example 8. [IMO 2001] Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Remark: This is a difficult problem in number theory. However, we would like to present a solution aided by geometrical insights!

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 31, 2007.***

Problem 271. There are 6 coins that look the same. Five of them have the same weight, each of these is called a good coin. The remaining one has a different weight from the 5 good coins and it is called a bad coin. Devise a scheme to weigh groups of the coins using a scale (not a balance) three times only to determine the bad coin and its weight.

Problem 272. ΔABC is equilateral. Find the locus of all point Q inside the triangle such that

$$\angle QAB + \angle QBC + \angle QCA = 90^\circ.$$

Problem 273. Let R and r be the circumradius and the inradius of triangle ABC. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\ge\frac{R}{r}.$$

(*Source: 2000 Beijing Math Contest*)

Problem 274. Let $n < 11$ be a positive integer. Let p_1, p_2, p_3, p be prime numbers such that $p_1 + p_3^n$ is prime. If $p_1+p_2=3p$, $p_2 + p_3 = p_1^n(p_1 + p_3)$ and $p_2>9$, then determine $p_1p_2p_3^n$.

(*Source: 1997 Hubei Math Contest*)

Problem 275. There is a group of children coming from 11 countries (at least one child from each of the 11 countries). Their ages are from 7 to 13. Prove that there are 5 children in the group, for each of them, the number of children in the group with the same age is greater than the number of children in the group from the same country.

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Solutions

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Problem 266. Let

$$N = 1+10+10^2+\cdots+10^{1997}.$$

Determine the 1000th digit after the decimal point of the square root of N in base 10. (*Source: 1998 Putnam Exam*)

Solution. **Jeff CHEN** (Virginia, USA), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina), **Anna Ying PUN** (HKU, Math, Year 1) and **Fai YUNG**.

The answer is the same as the unit digit of $10^{1000}\sqrt{N}$. We have

$$10^{1000}\sqrt{N}=10^{1000}\sqrt{\frac{10^{1998}-1}{9}}=\frac{\sqrt{10^{3998}-10^{2000}}}{3}.$$

Since

$$(10^{1999}-7)^2<10^{3998}-10^{2000}<(10^{1999}-4)^2,$$

so it follows that $10^{1000}\sqrt{N}$ is between $(10^{1999}-7)/3=33\cdots31$ and $(10^{1999}-4)/3=33\cdots32$. Therefore, the answer is 1.

Commended solvers: **Simon YAU** and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

Problem 267. For any integer a, set

$$n_a = 101a - 100\cdot 2^a.$$

Show that for $0 \le a, b, c, d \le 99$, if

$$n_a + n_b \equiv n_c + n_d \pmod{10100},$$

then $\{a,b\}=\{c,d\}$. (*Source: 1994 Putnam Exam*)

Solution. **Jeff CHEN** (Virginia, USA), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina), **Anna Ying PUN** (HKU, Math, Year 1) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

If $n_a+n_b \equiv n_c+n_d \pmod{10100}$, then $a+b \equiv n_a+n_b \equiv n_c+n_d \equiv c+d \pmod{100}$ and $2^a+2^b \equiv n_a+n_b \equiv n_c+n_d \equiv 2^c+2^d \pmod{101}$.

By Fermat's little theorem, $2^{100}\equiv 1 \pmod{101}$ and so $2^{a+b}\equiv 2^{c+d} \pmod{101}$. Next

$$\begin{aligned}(2^a-2^c)(2^a-2^d) &= 2^a(2^a-2^c-2^d)+2^{c+d}\\ &\equiv 2^a(-2^b)+2^{a+b}\\ &= 0 \pmod{101}.\end{aligned}$$

So $2^a \equiv 2^c \pmod{101}$ or $2^a \equiv 2^d \pmod{101}$.

Now we claim that if $0 \le s \le t \le 99$ and $2^s \equiv 2^t \pmod{101}$, then $s=t$. To see this, let k be the least positive integer such that $2^k\equiv 1 \pmod{101}$. Dividing 100 by k, we get $100 = kq+r$ with $0 \le r < k$. Since $2^r = 2^{100-kq} \equiv 1 \pmod{101}$ too, so $r = 0$, then k is a divisor of 100.

Clearly, $1 < 2^1, 2^2, 2^4, 2^5 < 101$ and $2^{10} = 1024 \equiv 14 \pmod{101}$, $2^{20}\equiv 14^2 \equiv -6 \pmod{101}$, $2^{25}\equiv (-6)32 \equiv 10 \pmod{101}$, $2^{50} \equiv 10^2 \equiv -1 \pmod{101}$. Hence $k=100$. Finally $2^{t-s}\equiv 1 \pmod{101}$ and $0 \le t-s < 100$ imply $t-s=0$, proving the claim.

By the claim, we get $a=c$ or $a=d$. From $a+b \equiv c+d \pmod{100}$ and $0 \le a, b, c, d \le 99$, we get $a = c$ implies $b = d$ and similarly $a = d$ implies $b = c$. The conclusion follows.

Problem 268. In triangle ABC, $\angle ABC = \angle ACB = 40^\circ$. Points P and Q are inside the triangle such that $\angle PAB = \angle QAC = 20^\circ$ and $\angle PCB = \angle QCA = 10^\circ$. Must B, P, Q be collinear? Give a proof. (*Source: 1994 Shanghai Math Competition*)

Solution. **Jeff CHEN** (Virginia, USA), **Courtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Kelvin LEE** (Winchester College, England) **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina) and **NG Ngai Fung** (STFA Leung Kau Kui College).

Since lines AP, BP, CP concur, by the trigonometric form of Ceva's theorem,

$$\frac{\sin\angle CBP\sin\angle BAP\sin\angle PCA}{\sin\angle PBA\sin\angle PAC\sin\angle PCB}=1,$$

which implies

$$\frac{\sin\angle CBP}{\sin\angle PBA}=\frac{\sin 80^\circ\sin 10^\circ}{\sin 20^\circ\sin 30^\circ}=\frac{\cos 10^\circ\sin 10^\circ}{\sin 20^\circ/2}=1.$$

So $\angle CBP = \angle PBA = 20^\circ$. Replacing P by Q above, we similarly have

$$\frac{\sin\angle CBQ}{\sin\angle QBA}=\frac{\sin 20^\circ\sin 30^\circ}{\sin 80^\circ\sin 10^\circ}=1.$$

So $\angle QBA = \angle CBQ = 20^\circ$. Then B, P, Q are on the bisector of $\angle ABC$.

Commended solvers: **CHIU Kwok Sing** (Belilios Public School), **FOK Pak Hei** (Pui Ching Middle School), **Anna Ying PUN** (HKU, Math, Year 1) and **Simon YAU**.

Problem 269. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$, $a_{n+1} = f(a_n)$ for all $n \ge 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or

$a_2 = 0$. (*Source: 2000 Putnam Exam*)

Solution. **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina) and **Anna Ying PUN** (HKU, Math, Year 1).

Observe that for any integers m and n, $m-n$ divides $f(m)-f(n)$ since for all nonnegative integer k, m^k-n^k has $m-n$ as a factor. For nonnegative integer n, let $b_n=a_{n+1}-a_n$, then by the last sentence, b_n divides b_{n+1} for all n.

Since $a_0 = a_m = 0$, $a_1 = a_{m+1}$ and so $b_0=b_m$. If $b_0 = 0$, then $a_1 = a_{m+1} = b_m+a_m = 0$.

If $b_0 \neq 0$, then using b_n divides b_{n+1} for all n and $b_0=b_m$, we get $b_n = \pm\, b_0$ for $n=1,2,\cdots,m$. Since $b_0+b_1+\cdots+b_m = a_m-a_0 = 0$, half of the integers $b_0, \cdots, b_m$ are positive and half are negative. Then there is $k < m$ such that $b_{k-1} = -b_k$, which implies $a_{k-1}=a_{k+1}$. Then $a_m=a_{m+2}$ and so $0=a_m=a_{m+2}=f(f(a_m))=f(f(a_0))=a_2$.

Problem 270. The distance between any two of the points *A, B, C, D* on a plane is at most 1. Find the minimum of the radius of a circle that can cover these four points. (*Source 1998 Tianjin Math Competition*)

Solution. **Jeff CHEN** (Virginia, USA).

<u>Case 1</u>: (one of the point, say D, is inside or on a side of ΔABC) If ΔABC is acute, then one of the angle, say $\angle BAC \geq 60^\circ$. By the extended sine law, the circumcircle of ΔABC covers the four points with diameter

$$2R = \frac{BC}{\sin \angle BAC} \leq \frac{2}{\sqrt{3}}.$$

(Note equality occurs in case ΔABC is equilateral.) If ΔABC is right or obtuse, then the circle using the longest side as diameter covers the four points with $R \leq 1/2$.

<u>Case 2</u>: ($ABCD$ is a convex quadrilateral) If there is a pair of opposite angles, say angles A and C, are at least 90°, then the circle with BD as diameter will cover the four points with $R \leq 1/2$. Otherwise, there is a pair of neighboring angles, say angles A and B, both of which are less than 90°.

If $\angle ADB \geq \angle ACB \geq 90^\circ$, then the circle with AB as diameter covers the four points and radius $R \leq 1/2$.

If $\angle ADB \geq \angle ACB$ and $\angle ACB < 90^\circ$, then D is in or on the circumcircle of ΔABC with radius $R \leq 1/\sqrt{3}$ as in case 1.

So summarizing all cases, we see the minimum radius that works for all possible arrangements of *A,B,C and D* is $R = 1/\sqrt{3}$.

Commended solvers: **NG Ngai Fung** (STFA Leung Kau Kui College) and **Anna Ying PUN** (HKU, Math, Year 1).

Olympiad Corner

(continued from page 1)

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all possible positions of this light.

From *How to Solve It* to Problem Solving in Geometry

(continued from page 2)

Idea:
Observe that

$$ac + bd = (b + d + a - c)(b + d - a + c)$$
$$\Leftrightarrow a^2 + c^2 - ac = b^2 + d^2 + bd$$

The last equality suggests one to think about using the cosine law as follow:

$$\begin{aligned} & a^2 + c^2 - 2ac\cos 60^\circ \\ &= a^2 + c^2 - ac \\ &= b^2 + d^2 + bd \\ &= b^2 + d^2 - 2bd\cos 120^\circ. \end{aligned}$$

Solution:

(1) **Lemma**: Let x, y, and z be positive integers with $z < x$ and $z < y$. If xy/z is an integer, then xy/z is composite.
[Can you prove this lemma? Is there any trivial case you can see immediately? How about proving the lemma by mathematical induction in z?]

The case $z = 1$ is trivial. In case $z > 1$, inductively suppose the lemma is true for all positive integers z' less than z. Then z has a prime divisor p, say $z = pz'$. Since xy/z is an integer, either p divides x or p divides y, say p divides x. Then $x = px'$. So $xy/z=x'y/z'$ with $z' < x'$ and $z' < z < y$. By the induction hypothesis, $xy/z=x'y/z'$ is composite.

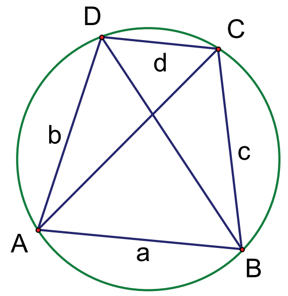
(2) The equality

$$ac + bd = (b + d + a - c)(b + d - a + c)$$

is equivalent to

$$a^2 + c^2 - ac = b^2 + d^2 + bd$$

In view of this, we can construct cyclic quadrilateral $ABCD$ with $AB = a$, $BC = c$, $CD = d$, $DA = b$, $\angle ABC = 60^\circ$ and $\angle ADC = 120^\circ$.



(3) Considering the ratios of areas and using Ptolemy's theorem, we have

$$\frac{AC}{BD} = \frac{ab+cd}{ac+bd} \quad \text{and} \quad AC \times BD = ad + bc.$$

(4) Therefore,

$$\frac{ab+cd}{ac+bd} = \frac{AC}{BD} = \frac{AC^2}{AC \times BD} = \frac{a^2+c^2-ac}{ad+bc},$$

which implies

$$ab+cd = \frac{(ac+bd)(a^2+c^2-ac)}{ad+bc} \qquad (*).$$

(5) To get the conclusion from the lemma, it remains to show

$$ad + bc < ac + bd$$

and $$ad + bc < a^2 + c^2 - ac.$$

Now

$$\begin{aligned} &(ac + bd) - (ad + bc) \\ &= (a - b)(c - d) > 0 \\ \Rightarrow\ & ad + bc < ac + bd. \end{aligned}$$

Also,

$$\begin{aligned} &(ab + cd) - (ac + bd) \\ &= (a - d)(b - c) > 0 \\ \Rightarrow\ & ab + cd > ac + bd \\ \Rightarrow\ & ad + bc < a^2 + c^2 - ac \quad \text{(by (*))}. \end{aligned}$$

Now the result follows from the lemma.

Mathematical Excalibur

Volume 12, Number 2 *May 2007 – August 2007*

Olympiad Corner

Below are the problems of the 2006 Belarussian Math Olympiad, Final Round, Category C.

Problem 1. Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets,

a) there is a number c in the third subset such that $a + b = 2c$?
b) there are two numbers c_1 and c_2 in the third subset such that $a + b = c_1 + c_2$?

Problem 2. Points X, Y, Z are marked on the sides AB, BC, CD of the rhombus $ABCD$, respectively, so that $XY \| AZ$. Prove that XZ, AY and BD are concurrent.

Problem 3. Let a, b, c be real positive numbers such that $abc = 1$. Prove that

$$2(a^2+b^2+c^2)+a+b+c \geq 6+ab+bc+ca.$$

Problem 4. Given triangle ABC with $\angle A = 60^\circ$, $AB = 2005$, $AC = 2006$. Bob and Bill in turn (Bob is the first) cut the triangle along any straight line so that two new triangles with area more than or equal to 1 appear.

(continued on page 4)

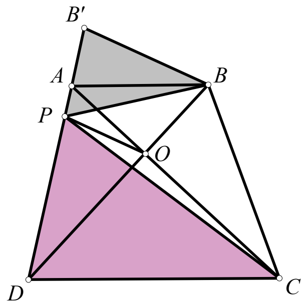
From *How to Solve It* to Problem Solving in Geometry (II)

K. K. Kwok
Munsang College (HK Island)

We will continue with more examples.

Example 9. In the trapezium $ABCD$, $AB \| CD$ and the diagonals intersect at O. P, Q are points on AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that $OP = OQ$.

Idea:
We shall try to find OP in terms of "more basic" lengths, e.g. AB, CD, OA, OC, …. To achieve that, we can construct a triangle that is similar to ΔDPC.



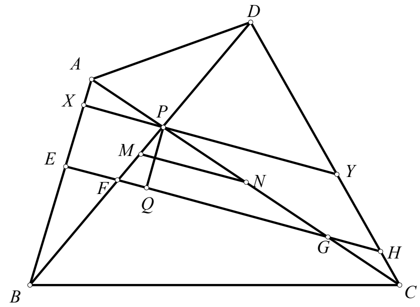
Solution Outline:
(1) Extend DA to B' such that $BB' = BA$. Then $\angle PB'B = \angle B'AB = \angle PDC$. So $\Delta DPC \sim \Delta B'PB$.

(2) It follows that
$$\frac{DP}{PB'} = \frac{CD}{BB'} = \frac{CD}{BA} = \frac{DO}{BO}$$
and so $PO \| BB'$.

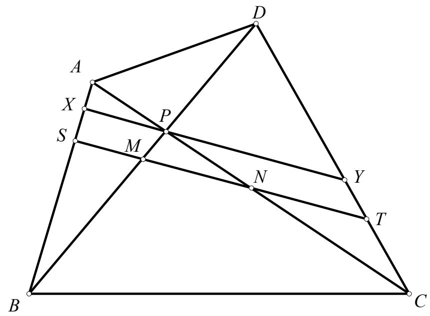
(3) Since $\Delta DPO \sim \Delta DB'B$, we have
$$OP = BB' \times \frac{DO}{DB} = AB \times \frac{DO}{DB}.$$

(4) Similarly, we have $OQ = AB \times \dfrac{CO}{CA}$ and the result follows.

Example 10. In quadrilateral $ABCD$, the diagonals intersect at P. M and N are midpoint of BD and AC respectively. Q is the reflected image of P about MN. The line through P and parallel to MN cuts AB and CD at X and Y respectively. The line through Q parallel to MN cuts AB, BD, AC and CD at E, F, G and H respectively. Prove that $EF = GH$.



Idea:
The diagram is not simple. We shall try to express the lengths involved in terms of "more basic" lengths, e.g. PA, PB, PC and PD.



Solution Outline:
(1) First observe that $PM = MF$ and $PN = NG$, hence $BF = PD$ and $CG = PA$.

(2) $\dfrac{EF}{XP} = \dfrac{BF}{BP} = \dfrac{PD}{BP}$, $EF = \dfrac{PD \times XP}{BP}$.

Similarly, we have $GH = \dfrac{PA \times YP}{CP}$.

(3) Let the line MN cuts AB and CD at S and T respectively. Then

$$\frac{SM}{XP} = \frac{BM}{BP} = \frac{BD}{2BP}, \frac{SN}{XP} = \frac{AN}{AP} = \frac{AC}{2AP}.$$

Subtracting the equalities get
$$\frac{MN}{XP} = \frac{1}{2}\left(\frac{AC}{AP} - \frac{BD}{BP}\right).$$

Similarly, we have
$$\frac{MN}{YP} = \frac{1}{2}\left(\frac{BD}{PD} - \frac{AC}{PC}\right).$$

(4) $EF = GH \Leftrightarrow \dfrac{PD\times XP}{BP} = \dfrac{PA\times YP}{CP}$

$\Leftrightarrow \dfrac{PD\times MN}{BP\times YP} = \dfrac{PA\times MN}{CP\times XP}$. By (3),

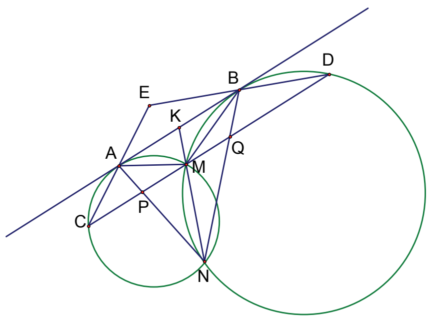
$$\frac{PD}{BP}\left(\frac{BD}{PD}-\frac{AC}{PC}\right)=\frac{PA}{CP}\left(\frac{AC}{AP}-\frac{BD}{BP}\right)$$

$$\Leftrightarrow \frac{BD}{BP}-\frac{PD\times AC}{BP\times PC}=\frac{AC}{CP}-\frac{PA\times BD}{CP\times BP}$$

$$\Leftrightarrow \frac{BD}{BP}+\frac{PA\times BD}{CP\times BP}=\frac{AC}{CP}+\frac{PD\times AC}{BP\times PC}.$$

By addition, both sides of the last equation equal $\dfrac{AC\times BD}{BP\times CP}$.

Example 11. [IMO 2000] Two circles Γ_1 and Γ_2 intersect at M and N. Let L be the common tangent to Γ_1 and Γ_2 so that M is closer to L than N is. Let L touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to L meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.



Idea:
First, note that if $EP = EQ$, then E lies on the perpendicular bisector of PQ.

Observe that $AB \parallel CD$ implies A and B are the midpoints of arc CAM and arc DBM respectively, from which we see ΔACM and ΔBDM are isosceles.

Second, we have $\angle EAB = \angle ECM = \angle AMC = \angle BAM$ and similarly, $\angle EBA = \angle ABM$. That means E is the reflected image of M about AB. In particular, $EM \perp AB$ and hence $EM \perp PQ$.

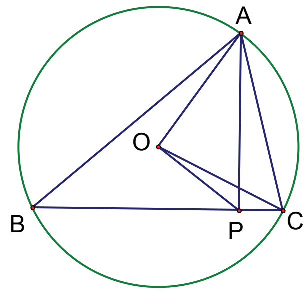
Therefore, the result follows if we can show that M is the midpoint of PQ.

Solution outline:
(1) Extend NM to meet AB at K.

(2) $AK^2 = KN\times KM = BK^2 \Rightarrow K$ is the midpoint of $AB \Rightarrow M$ is the midpoint of PQ.

(3) Following the steps discussed above, we get $EM \perp PQ$ and hence $EP = EQ$.

Example 12. [IMO 2001] Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \ge \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.



Idea:
(1) Examine the conclusion $\angle CAB + \angle COP < 90^\circ$, which is equivalent to $2\angle CAB + 2\angle COP < 180^\circ$. That is,

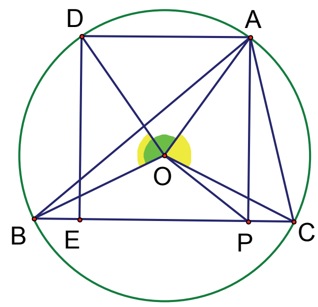
$$\angle COB + 2\angle COP < 180^\circ.$$

On the other hand, we have $\angle COB + 2\angle OCP = 180^\circ$. Therefore, we shall show $\angle COP < \angle OCP$ or $PC < OP$.

(2) Examine the condition $\angle BCA \ge \angle ABC + 30^\circ$, which is equivalent to $2\angle BCA - 2\angle ABC \ge 60^\circ$. That is,

$$\angle BOA - \angle AOC \ge 60^\circ.$$

What is the meaning of $\angle BOA - \angle AOC$?



Solution outline:
(1) Let D and E be the reflected image of A and P about the perpendicular bisector of BC respectively. Let R be the circumradius.

(2) $\angle BCA \ge \angle ABC + 30^\circ$
$\Rightarrow \angle BOA - \angle AOC \ge 60^\circ$
$\Rightarrow \angle DOA \ge 60^\circ$
$\Rightarrow EP = DA \ge R.$

(3) $OP + R = OP + OC = OE + OC$
$> EC = EP + PC \ge R + PC$
$\Rightarrow OP > PC \quad \Rightarrow \angle COP < \angle OCP.$

(4) $2\angle CAB + 2\angle COP$
$= \angle COB + 2\angle COP$
$< \angle COB + 2\angle OCP < 180^\circ$
and the result follows.

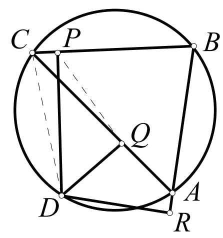
Example 13. [Simson's Theorem] The feet of the perpendiculars drawn from any point on the circumcircle of a triangle to the sides of the triangle are collinear.

Solution:
In the figure below, D is a point on the circumcircle of ΔABC, P, Q, and R are feet of perpendiculars from D to BC, AC, and BA respectively.

Note that $DQAR$, $DCPQ$, and $DPBR$ are cyclic quadrilaterals. So

$$\angle DQR = \angle DAR = \angle BCD = 180^\circ - \angle PQD,$$

i.e. $\angle DQR + \angle PQD = 180^\circ$. Thus, P, Q, and R are collinear.



Example 14. [IMO 2003] Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisector of $\angle ABC$ and $\angle ADC$ meet on AC.

Solution :
From Simson's theorem, P, Q, and R are collinear. Now

$\angle DPC = \angle DQC = 90^\circ$
$\Rightarrow D, P, C$ and Q are concyclic
$\Rightarrow \angle DCA = \angle DPQ = \angle DPR.$

Similarly, since D, Q, R and A are concyclic, we get $\angle DAC = \angle DRP$. It follows that $\Delta DCA \sim \Delta DPR$.

Similarly, $\Delta DAB \sim \Delta DQP$ and $\Delta DBC \sim \Delta DRQ$. So,

$$\frac{DA}{DC}=\frac{DR}{DP}=\frac{DB\cdot\frac{QR}{BC}}{DB\cdot\frac{PQ}{BA}}=\frac{QR}{PQ}\cdot\frac{BA}{BC}.$$

Therefore, $PQ = QR \Leftrightarrow \dfrac{DA}{DC} = \dfrac{BA}{BC}$.

Example 15. [IMO 2001] In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 20, 2007.***

Problem 276. Let n be a positive integer. Given a $(2n-1)\times(2n-1)$ square board with exactly one of the following arrows $\uparrow$, $\downarrow$, $\rightarrow$, $\leftarrow$ at each of its cells. A beetle sits in one of the cells. Per year the beetle creeps from one cell to another in accordance with the arrow's direction. When the beetle leaves the cell, the arrow at that cell makes a counterclockwise 90-degree turn. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)!-3$ years.
(*Source: 2001 Belarussian Math Olympiad*)

Problem 277. (*Due to Koopa Koo, Univ. of Washington, Seattle, WA, USA*) Prove that the equation

$$x^2+y^2+z^2+2xyz=1$$

has infinitely many integer solutions (then try to get all solutions – Editiors).

Problem 278. Line segment SA is perpendicular to the plane of the square $ABCD$. Let E be the foot of the perpendicular from A to line segment SB. Let P, Q, R be the midpoints of SD, BD, CD respectively. Let M, N be on line segments PQ, PR respectively. Prove that AE is perpendicular to MN.

Problem 279. Let R be the set of all real numbers. Determine (with proof) all functions $f: R\to R$ such that for all real x and y,

$$f\big(f(x)+y\big)=2x+f\big(f(f(y))-x\big).$$

Problem 280. Let n and k be fixed positive integers. A basket of peanuts is distributed into n piles. We gather the piles and rearrange them into $n+k$ new piles. Prove that at least $k+1$ peanuts are transferred to smaller piles than the respective original piles that contained them. Also, give an example to show the constant $k+1$ cannot be improved.

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Solutions

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Problem 271. There are 6 coins that look the same. Five of them have the same weight, each of these is called a <u>*good*</u> coin. The remaining one has a different weight from the 5 good coins and it is called a <u>*bad*</u> coin. Devise a scheme to weigh groups of the coins using a scale (not a balance) three times only to determine the bad coin and its weight.
(*Source: 1998 Zhejiang Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **St. Paul's College Math Team**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Number the coins 1 to 6. For the first weighting, let us weigh coins 1, 2, 3 and let the weight be $3a$. For the second weighting, let us weigh coins 1, 2, 4, 5 and let the weight be $4b$.

If $a=b$, then coin 6 is bad and we can use the third weighting to find the weight of this coin.

If $a\neq b$, then the bad coin is among coins 1 to 5. For the third weighting, let us weigh coins 2, 4 and let the weight be $2c$.

If coin 1 is bad, then c and $4b-3a$ are both the weight of a good coin. So $3a-4b+c=0$. Similarly, if coin 2 or 3 or 4 or 5 is bad, we get respective equations $3a-2b-c=0$, $b-c=0$, $a-2b+c=0$ and $a-c=0$.

We can check that if any two of these equations are satisfied simultaneously, then we will arrive at $a=b$, a contradiction. Therefore, exactly one of these five equations will hold.

If the first equation $3a-4b+c=0$ holds, then coin 1 is bad and its weight can be found by the first and third weightings to be $3a-2c$. Similarly, for $k=2$ to 5, if the k-th equation holds, then coin k is bad and its weight can be found to be $3c-2b$, $3a-2c$, $4b-3a$ and $4b-3a$ respectively.

Problem 272. $\Delta\,ABC$ is equilateral. Find the locus of all point Q inside the triangle such that

$$\angle QAB+\angle QBC+\angle QCA=90^\circ.$$

(*Source: 2000 Chinese IMO Team Training Test*)

Solution. **Alex Kin-Chit O** (STFA Cheng Yu Tung Secondary School) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

We take the origin at the center O of $\Delta\,ABC$. Let $\omega\neq 1$ be a cube root of unity and A,B,C,Q correspond to the complex numbers 1, ω, $\omega^2=\overline{\omega}$, z respectively. Then

$$\angle QAB+\angle QBC+\angle QCA=90^\circ$$

if and only if

$$\frac{\omega-1}{z-1}\cdot\frac{\overline{\omega}-\omega}{z-\omega}\cdot\frac{1-\overline{\omega}}{z-\overline{\omega}}=\frac{(\omega-\overline{\omega})\,|\,\omega-1|^2}{z^3-1}$$

is purely imaginary, which is equivalent to z^3 is real. These are the complex numbers whose arguments are multiples of $\pi/3$. Therefore, the required locus is the set of points on the three altitudes.

Commended solvers: **Jeff CHEN** (Virginia, USA), **St. Paul's College Math Team, Simon YAU** and **YIM Wing Yin** (HKU, Year 1).

Problem 273. Let R and r be the circumradius and the inradius of triangle ABC. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq\frac{R}{r}.$$

(*Source: 2000 Beijing Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **Kelvin LEE** (Winchester College, England), **NG Eric Ngai Fung** (STFA Leung Kau Kui College), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6) and **YIM Wing Yin** (HKU, Year 1).

Without loss of generality, let a, b, c be the sides and $a\geq b\geq c$. By the extended sine law, $R=a/(2\sin A)=b/(2\sin B)=c/(2\sin C)$. Now the area of the triangle is $(bc\sin A)/2=abc/(4R)$ and is also rs, where $s=(a+b+c)/2$ is the semi- perimeter. So $abc=4Rrs$.

Next, observe that for any positive x and y, we have $(x^2-y^2)(1/x-1/y)\leq 0$, which after expansion yields

$$\frac{x^2}{y}+\frac{y^2}{x}\geq x+y. \qquad (*)$$

By the cosine law and the extended sine law, we get

$$\frac{\cos A}{\sin^2 A}=\frac{(b^2+c^2-a^2)/2bc}{(a/2R)^2}$$

$$=\frac{2R^2}{abc}\left(\frac{b^2+c^2-a^2}{a}\right)=\frac{R}{2rs}\left(\frac{b^2}{a}+\frac{c^2}{a}-a\right).$$

Adding this to the similar terms for B and C, we get

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}$$

$$=\frac{R}{2rs}\left(\frac{b^2}{a}+\frac{a^2}{b}+\frac{c^2}{b}+\frac{b^2}{c}+\frac{a^2}{c}+\frac{c^2}{a}-a-b-c\right)$$

$$\geq \frac{R}{2rs}(a+b+c)=\frac{R}{r} \text{ by (*).}$$

Commended solvers: **CHEUNG Wang Chi** (Singapore).

Problem 274. Let $n < 11$ be a positive integer. Let p_1, p_2, p_3, p be prime numbers such that $p_1 + p_3^n$ is prime. If $p_1 + p_2 = 3p$, $p_2 + p_3 = p_1^n(p_1 + p_3)$ and $p_2 > 9$, then determine $p_1p_2p_3^n$. (*Source: 1997 Hubei Math Contest*)

Solution. **CHEUNG Wang Chi** (Singapore), **NG Eric Ngai Fung** (STFA Leung Kau Kui College), **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Assume $p_1 \geq 3$. Then $p_1+p_2 > 12$ and $3p$ is even, which would imply p is even and at least 5, contradicting p is prime. So $p_1=2$ and $p_2=3p-2$.

Modulo 3, the given equation $p_2 + p_3 = p_1^n(p_1+p_3)$ leads to

$$\begin{aligned} 0 &\equiv 3p \\ &= p_2+2 = 2^n(2+p_3)+2 \\ &= 2^{n+1} + 2 + (2^n-1)p_3 \\ &\equiv (-1)^{n+1}+2+((-1)^n-1)p_3 \pmod 3. \end{aligned}$$

The case n is even results in the contradiction $0 \equiv 1 \pmod 3$. So n is odd and we get $0 \equiv p_3 \pmod 3$. So $p_3 = 3$.

Finally, the cases $n = 1, 3, 5, 7, 9$ lead to $p_1 + p_3^n = 5, 29, 245, 2189, 19685$ respectively. Since 245, 19685 are divisible by 5 and 2189 is divisible by 11, n can only be 1 or 3 for $p_1+p_3^n$ to be prime. Now $p_2 = p_1^n(p_1+p_3) - p_3 = 2^n5-3 > 9$ implies $n = 3$. Then the answer is

$$p_1p_2p_3^n = 2\cdot 37\cdot 3^3 = 1998.$$

Problem 275. There is a group of children coming from 11 countries (at least one child from each of the 11 countries). Their ages are from 7 to 13. Prove that there are 5 children in the group, for each of them, the number of children in the group with the same age is greater than the number of children in the group from the same country.

Solution. **Jeff CHEN** (Virginia, USA).

For i =7 to 13 and j = 1 to 11, let a_{ij} be the number of children of age i from country j in the group. Then

$$b_i = \sum_{j=1}^{11} a_{ij} \geq 0 \text{ and } c_j = \sum_{i=7}^{13} a_{ij} \geq 1$$

are the number of children of age i in the group and the number of children from country j respectively. Note that

$$c_j = \sum_{i=7}^{13} a_{ij} = \sum_{b_i \neq 0} a_{ij}\text{ , where } \sum_{b_i \neq 0} \text{ is}$$

used to denote summing i from 7 to 13 skipping those i for which b_i=0. Now

$$\sum_{b_i\neq 0}\sum_{j=1}^{11} a_{ij}\left(\frac{1}{c_j}-\frac{1}{b_i}\right)$$

$$=\sum_{j=1}^{11}\frac{\sum_{b_i\neq 0} a_{ij}}{c_j}-\sum_{b_i\neq 0}\frac{\sum_{j=1}^{11} a_{ij}}{b_i}$$

$$\geq \sum_{j=1}^{11} 1-\sum_{i=7}^{13} 1 = 4.$$

Since $a_{ij}(1/c_j - 1/b_i) < a_{ij}/c_j \leq 1$, there are at least five terms $a_{ij}(1/c_j - 1/b_i) > 0$. So there are at least five ordered pairs (i,j) such that $a_{ij} > 0$ (so we can take a child of age i from country j) and we have $b_i > c_j$.

Olympiad Corner

(continued from page 1)

Problem 4. (Cont.) After that an obtused-angled triangle (or any of two right-angled triangles) is deleted and the procedure is repeated with the remained triangle. The player loses if he cannot do the next cutting. Determine, which player wins if both play in the best way.

Problem 5. AA_1, BB_1 and CC_1 are the altitudes of an acute triangle ABC. Prove that the feet of the perpendiculars from C_1 onto the segments AC, BC, BB_1 and AA_1 lie on the same straight line.

Problem 6. Given real numbers a, b, k (k>0). The circle with the center (a,b) has at least three common points with the parabola $y = kx^2$; one of them is the origin (0,0) and two of the others lie on the line $y=kx+b$. Prove that $b \geq 2$.

Problem 7. Let x, y, z be real numbers greater than 1 such that

$$xy^2 - y^2 + 4xy + 4x - 4y = 4004,$$

$$\text{and } xz^2 - z^2 + 6xz + 9x - 6z = 1009.$$

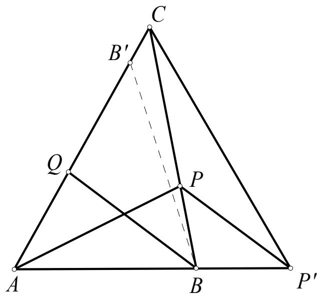
Determine all possible values of the expression $xyz+3xy+2xz-yz+6x-3y-2z$.

Problem 8. A $2n\times 2n$ square is divided into $4n^2$ unit squares. What is the greatest possible number of diagonals of these unit squares one can draw so that no two of them have a common point (including the endpoints of the diagonals)?

From *How to Solve It* to Problem Solving in Geometry (II)

(continued from page 2)

Idea:
By examining the conditions given, we may see that the point C is not too important.



We will focus on how to represent the condition $AB + BP = AQ + QB$ in the diagram. For that, we construct points P' and B' on AB and AQ extended respectively so that $PB = P'B$ and $QB' = QB$. Then

$$\begin{aligned} &AB + BP = AQ + QB \\ &\Rightarrow AB + BP' = AQ + QB' \Rightarrow AP' = AB' \\ &\Rightarrow AP'B' \text{ is equilateral (as } \angle B'AP' = 60^\circ). \end{aligned}$$

Solution outline:

(1) Let $\angle ABQ = \angle QBP = \theta$. Since $PB = P'B$, we have $\angle PP'B = \theta$.

(2) Since AP bisects $\angle QAB$ and $\Delta AB'P'$ is equilateral, it follows that B' is the reflected image of P' about AP. So, $PP' = PB'$ and $\angle QB'P = \angle AP'P = \theta$.

(3) Since $QB = QB'$ and $\angle QBP = \theta = \angle QB'P$, by Example 2, P lies on either BB' or the perpendicular bisector of BB'. If P does not lie on BB', we will have $PB = PB' = PP'$. This will imply $\Delta BPP'$ is equilateral, $\theta = 60^\circ$ and $\angle QAB + \angle ABP = 60^\circ + 2\theta = 180^\circ$, which is absurd. So, P must lie on BB'. Therefore, $B' = C$.

(4) Since $QB=QB'=QC$, $\angle QCB = \angle QBC = \theta$. So $\angle QAB + 2\theta + \theta = 180^\circ \Rightarrow 60^\circ + 3\theta = 180^\circ \Rightarrow \theta = 40^\circ$. Therefore, $\angle ABC = 80^\circ$, $\angle ACB = 40^\circ$.

Mathematical Excalibur

Volume 12, Number 3 — *September 2007 – October 2007*

Olympiad Corner

Below were the problems of the 2007 International Math Olympiad, which was held in Hanoi, Vietnam.

Day 1 (*July 25, 2007*)

Problem 1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : 1 \le j \le n\}$$

and let $d = \max\{d_i : 1 \le i \le n\}$.

(a) Prove that, for any real numbers $x_1 \le x_2 \le \ldots \le x_n$,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \ldots \le x_n$ such that equality holds in (*).

Problem 2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF=EG=EC$. Prove that ℓ is the bisector of angle DAB.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 25, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Convex Hull

Kin Yin Li

A set S in a plane or in space is convex if and only if whenever points X and Y are in S, the line segment XY must be contained in S. The intersection of any collection of convex sets is convex. For an arbitrary set W, the convex hull of W is the intersection of all convex sets containing W. This is the smallest convex set containing W. For a finite set W, the boundary of the convex hull of W is a polygon, whose vertices are all in W.

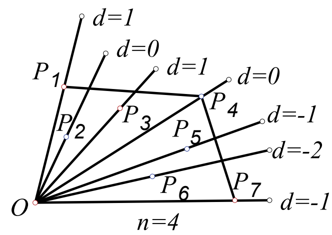
In a previous article (see *pp*. 1-2, *vol*. 5, *no*. 1 of Math. Excalibur), we solved problem 1 of the 2000 IMO using convex hull. Below we will discuss more geometric combinatorial problems that can be solved by studying convex hulls of sets.

Example 1. There are $n > 3$ coplanar points, no three of which are collinear and every four of them are the vertices of a convex quadrilateral. Prove that the n points are the vertices of a convex n-sided polygon.

Solution. Assume one of these points, say P, is inside the convex hull of the n points. Let Q be a vertex of the convex hull. The diagonals from Q divide the convex hull into triangles. Since no three points are collinear, P is inside some $\triangle QRS$, where RS is a side of the boundary. Then P,Q,R,S cannot be the vertices of a convex quadrilateral, a contradiction. So all n points can only be the vertices of the boundary polygon.

Example 2. (*1979 Putnam Exam*) Let A be a set of $2n$ points in the plane, no three of which are collinear, n of them are colored red and the other blue. Prove that there are n line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

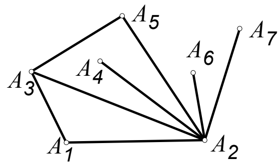
Solution. The case $n = 1$ is true. Suppose all cases less than n are true. For a vertex O on the boundary polygon of the convex hull of these $2n$ points, it is one of the $2n$ points, say its color is red. Let P_1, P_{2n-1} be adjacent vertices to O. If one of them, say P_1, is blue, then draw line segment OP_1 and apply induction to the other $2(n-1)$ points to finish. Otherwise,



let $d = 1$ and rotate the line OP_1 toward line OP_{2n-1} about O hitting the other $2n-3$ points one at a time. When the line hits a red point, increase d by 1 and when it hits a blue point, decrease d by 1. When the line hits P_{2n-1}, $d = (n-1) - n = -1$. So at some time, $d = 0$, say when the line hits P_j. Then $P_1,\ldots,P_j$ are on one side of line OP_j and O, $P_{j+1},\ldots, P_{2n-1}$ are on the other side. The inductive step can be applied to these two sets of points, which leads to the case n being true.

Example 3. (*1985 IMO Longlisted Problem*) Let A and B be finite disjoint sets of points in the plane such that any three distinct points in $A \cup B$ are not collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices are contained in A or in B that does not contained in its interior any point from the other set.

Solution. Suppose A has at least five points. Take a side A_1A_2 of the boundary of the convex hull of A. For any other A_i in A, let $\alpha_i = \angle A_1A_2A_i$, say $\alpha_3 < \alpha_4 < \cdots < 180°$. Then the convex hull H of A_1, A_2, A_3, A_4, A_5 contains no other points of A.



(continued on page 4)

Perpendicular Lines

Kin Yin Li

In geometry, sometimes we are asked to prove two lines are perpendicular. If the given facts are about right angles and lengths of segments, the following theorem is often useful.

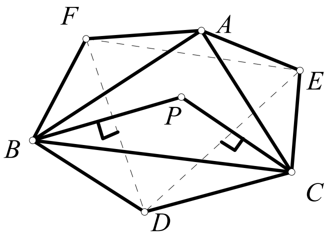
Theorem. On a plane, for distinct points R, S, X, Y, we have $RX^2-SX^2=RY^2-SY^2$ if and only if $RS\perp XY$.

Proof. Let P and Q be the feet of the perpendicular from X and Y to line RS respectively. If $RS\perp XY$, then $P=Q$ and $RX^2-SX^2=RP^2-SP^2=RY^2-SY^2$.

Conversely, if $RX^2-SX^2=RY^2-SY^2=m$, then $m=RP^2-(SR\pm RP)^2$. So $RP=\mp(SR^2+m)/2SR$. Replacing P by Q, we get $RQ=\mp(SR^2+m)/2SR$. Hence, $RP=RQ$. Interchanging R and S, we also get $SP=SQ$. So $P=Q$. Therefore, $RS\perp XY$.

Here are a few illustrative examples.

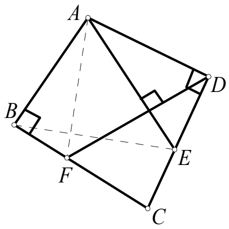
Example 1. (*1997 USA Math Olympiad*) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.



Solution. Let P be the intersection of the perpendicular line from B to FD with the perpendicular line from C to DE. Then $PB\perp FD$ and $PC\perp DE$. By the theorem above, we have $PF^2-PD^2=BF^2-BD^2$ and $PD^2-PE^2=CD^2-CE^2$.

Adding these and using $AF=BF$, $BD=CD$ and $CE=AE$, we get $PF^2-PE^2=AF^2-AE^2$. So $PA\perp EF$ and P is the desired concurrent point.

Example 2. (*1995 Russian Math Olympiad*) $ABCD$ is a quadrilateral such that $AB=AD$ and $\angle ABC$ and $\angle CDA$ are right angles. Points F and E are chosen on BC and CD respectively so that $DF\perp AE$. Prove that $AF\perp BE$.

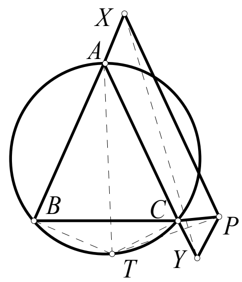


Solution. We have $AE\perp DF$, $AB\perp BF$ and $AD\perp DE$, which are equivalent to

$$AD^2-AF^2=ED^2-EF^2, \qquad \text{(a)}$$
$$AB^2-AF^2=-BF^2, \qquad \text{(b)}$$
$$AD^2-AE^2=-DE^2. \qquad \text{(c)}$$

Doing (a) − (b) + (c) and using $AD=AB$, we get $AB^2-AE^2=BF^2-EF^2$, which implies $AF\perp BE$.

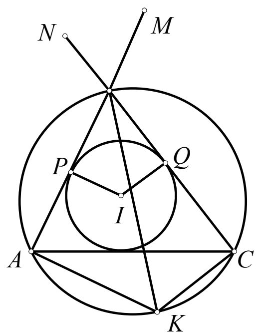
Example 3. In acute $\triangle ABC$, $AB=AC$ and P is a point on ray BC. Points X and Y are on rays BA and AC such that $PX\|AC$ and $PY\|AB$. Point T is the midpoint of minor arc BC on the circumcircle of $\triangle ABC$. Prove that $PT\perp XY$.



Since AT is a diameter, $\angle ABT=90^\circ=\angle ACT$. Then $TX^2=XB^2+BT^2$ and $TY^2=TC^2+CY^2$. So $TX^2-TY^2=BX^2-CY^2$.

Since $PX\parallel AC$, we have $\angle ABC=\angle ACB=\angle XPB$, hence $BX=PX$. Similarly, $CY=PY$. Therefore, $TX^2-TY^2=PX^2-PY^2$, which is equivalent to $PT\perp XY$.

Example 4. (1994 Jiangsu Province Math Competition) For $\triangle ABC$, take a point M by extending side AB beyond B and a point N by extending side CB beyond B such that $AM=CN=s$, where s is the semiperimeter of $\triangle ABC$. Let the inscribed circle of $\triangle ABC$ have center I and the circumcircle of $\triangle ABC$ have diameter BK. Prove that $KI\perp MN$.



Solution. Let the incircle of $\triangle ABC$ touch side AB at P and side BC at Q. We will show $KM^2-KN^2=IM^2-IN^2$.

Now since $\angle MAK=\angle BAK=90^\circ$ and $\angle NCK=\angle BCK=90^\circ$, we get

$$\begin{aligned}KM^2-KN^2&=(KA^2+AM^2)-(KC^2+CN^2)\\&=KA^2-KC^2\\&=(KA^2-KB^2)+(KB^2-KC^2)\\&=BC^2-AB^2.\end{aligned}$$

Also, since $\angle MPI=\angle BPI=90^\circ$ and $\angle NQI=\angle BQI=90^\circ$, we get

$$\begin{aligned}IM^2-IN^2&=(IP^2+PM^2)-(IQ^2+QN^2)\\&=PM^2-QN^2\\&=(AM-AP)^2+(CN-QC)^2.\end{aligned}$$

Now

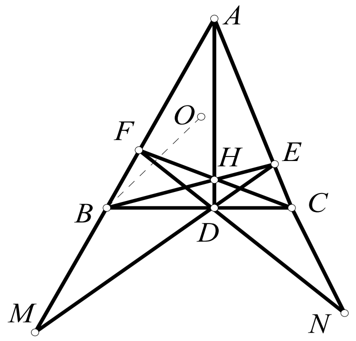
$$AM-AP=s-\frac{AB+CA-BC}{2}=BC$$

and

$$CN-QC=s-\frac{CA+BC-AB}{2}=AB.$$

So $IM^2-IN^2=BC^2-AB^2=KM^2-KN^2$.

Example 5. (*2001 Chinese National Senior High Math Competition*) As in the figure, in $\triangle ABC$, O is the circumcenter. The three altitudes AD, BE and CF intersect at H. Lines ED and AB intersect at M. Lines FD and AC intersect at N. Prove that (1) $OB\perp DF$ and $OC\perp DE$; (2) $OH\perp MN$.



Solution. (1) Since $\angle AFC=90^\circ=\angle ADC$, so A,C,D,F are concyclic. Then $\angle BDF=\angle BAC$. Also,

$$\begin{aligned}\angle OBC&=½(180^\circ-\angle BOC)\\&=90^\circ-\angle BAC=90^\circ-\angle BDF.\end{aligned}$$

So $OB\perp DF$. Similarly, $OC\perp DE$.

(2) Now $CH\perp MA$, $BH\perp NA$, $DA\perp BC$, $OB\perp DF=DN$ and $OC\perp DE=DM$. So

$$MC^2-MH^2=AC^2-AH^2 \qquad \text{(a)}$$
$$NB^2-NH^2=AB^2-AH^2 \qquad \text{(b)}$$
$$DB^2-DC^2=AB^2-AC^2 \qquad \text{(c)}$$
$$BN^2-BD^2=ON^2-OD^2 \qquad \text{(d)}$$
$$CM^2-CD^2=OM^2-OD^2. \qquad \text{(e)}$$

Doing (a) − (b) + (c) + (d) − (e), we get $NH^2-MH^2=ON^2-OM^2$. So $OH\perp MN$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***November 25, 2007.***

Problem 281. Let N be the set of all positive integers. Prove that there exists a function $f : N \to N$ such that $f(f(n)) = n^2$ for all n in N. (*Source: 1978 Romanian Math Olympiad*)

Problem 282. Let a, b, c, A, B, C be real numbers, $a \neq 0$ and $A \neq 0$. For every real number x,

$$|ax^2+bx+c| \le |Ax^2+Bx+C|.$$

Prove that $|b^2-4ac| \le |B^2-4AC|$.

Problem 283. P is a point inside $\triangle ABC$. Lines AC and BP intersect at Q. Lines AB and CP intersect at R. It is known that $AR=RB=CP$ and $CQ=PQ$. Find $\angle BRC$ with proof. (*Source: 2003 Japanese Math Olympiad*)

Problem 284. Let p be a prime number. Integers x, y, z satisfy $0 < x < y < z < p$. If x^3, y^3, z^3 have the same remainder upon dividing by p, then prove that $x^2+y^2+z^2$ is divisible by $x+y+z$. (*Source: 2003 Polish Math Olympiad*)

Problem 285. Determine the largest positive integer N such that for every way of putting all numbers 1 to 400 into a 20×20 table (1 number per cell), one can always find a row or a column having two numbers with difference not less than N. (*Source: 2003 Russian Math Olympiad*)

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Solutions

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Problem 276. Let n be a positive integer. Given a $(2n-1)\times(2n-1)$ square board with exactly one of the following arrows ↑, ↓, →, ← at each of its cells. A beetle sits in one of the cells. Per year the beetle creeps from one cell to another in accordance with the arrow's direction. When the beetle leaves the cell, the arrow at that cell makes a counterclockwise 90-degree turn. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)!-3$ years.
(*Source: 2001 Belarussian Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA), **GRA20 Problem Solving Group** (Roma, Italy), **PUN Ying Anna** (HKU, Math Year 1) and **Fai YUNG**.

Let $a(n)$ be the maximum number of years that the beetle takes to leave the $(2n-1)\times(2n-1)$ board. Then $a(1) = 1$. For $n > 1$, apart from 1 year necessary for the final step, the beetle can stay

(1) in each of the 4 corners for at most 2 years (two directions that do not point outside)

(2) in each of the other $4(2n-3)$ cells of the outer frame for at most 3 years (three directions that do not point outside)

(3) in the inner $(2n-3)\times(2n-3)$ board for at most $a(n-1)$ years (when the starting point is inside the inner board) plus $4(2n-3)a(n-1)$ years (when the arrow in a cell of the outer frame points inward the beetle enters the inner board).

Therefore, $a(n) \le 1 + 4\cdot 2 + 3\cdot 4(2n-3) + (4(2n-3)+1)a(n-1)$. Since $a(n-1) \ge 0$,

$$\begin{aligned} a(n)+3 &\le 8(n-1)(a(n-1)+3) - 3a(n-1) \\ &\le 8(n-1)(a(n-1)+3). \end{aligned}$$

Since $a(1)+3 = 4$, we get $a(n)+3 \le 2^{3n-1}(n-1)!$ and so $a(n) \le 2^{3n-1}(n-1)!-3$.

Problem 277. (*Due to Koopa Koo, Univ. of Washington, Seattle, WA, USA*) Prove that the equation

$$x^2+y^2+z^2+2xyz = 1$$

has infinitely many integer solutions (then try to get all solutions – Editors).

Solution. **Jeff CHEN** (Virginia, USA), **FAN Wai Tong** and **GRA20 Problem Solving Group** (Roma, Italy).

It is readily checked that if n is an integer, then $(x, y, z) = (n, -n, 1)$ is a solution.

Comments: Trying to get all solutions, we can first rewrite the equation as

$$(x^2-1)(y^2-1) = (xy+z)^2.$$

For any solution (x, y, z), we must have $x^2-1 = du^2$, $y^2-1 = dv^2$, $xy+z = \pm duv$ for some integers d, u, v. The cases d is negative, 0 or 1 lead to trivial solutions. For $d > 1$, we may suppose it is square-free (that is, no square divisor greater than 1). Then we can find all solutions of Pell's equation $s^2-dt^2 = 1$ (see *vol.* 6, *no.* 3 of Math Excalibur, *page* 1). Any two solutions (s_0, t_0) and (s_1, t_1) of Pell's equation yield a solution $(x, y, z)=(s_0, s_1, \pm dt_0t_1-s_0s_1)$ of

$$x^2+y^2+z^2+2xyz = 1.$$

Commended solvers: **PUN Ying Anna** (HKU, Math Year 1) and **WONG Kam Wing** (TWGH Chong Ming Thien College).

Problem 278. Line segment SA is perpendicular to the plane of the square $ABCD$. Let E be the foot of the perpendicular from A to line segment SB. Let P, Q, R be the midpoints of SD, BD, CD respectively. Let M, N be on line segments PQ, PR respectively. Prove that AE is perpendicular to MN.

Solution 1. **Stephen KIM** (Toronto, Canada).

Below when we write $XY \perp IJK\ldots$, we mean line XY is perpendicular to the plane containing $I, J, K, \ldots$. Also, we write $XY \perp WZ$ for vectors XY and WZ to mean their dot product is 0.

Since $SA \perp ABCD$, so $SA \perp BC$. Since $AB \perp BC$, so $BC \perp SAB$. Since A,E are in the plane of SAB, $AE \perp BC$. This along with the given fact $AE \perp SB$ imply $AE \perp SBC$.

Since P, Q are midpoints of SD, BD respectively, we get $PQ \| SB$. Similarly, we have $QR \| BC$. Then the planes SBC and PQR are parallel. Since MN is on the plane PQR, so MN is parallel to the plane SBC. Since $AE \perp SBC$ from the last paragraph, so $AE \perp MN$.

Solution 2. **Kelvin LEE** (Winchester College, England) and **PUN Ying Anna** (HKU, Math Year 1).

Let A be the origin, AD be the x-axis, AB be the y-axis and AS be the z-axis. Let $B = (0, a, 0)$ and $S = (0, 0, s)$. Then $C = (a, a, 0)$ and $E = (0, rs, ra)$ for some r. The homothety with center D and ratio 2 sends P to S, Q to B and R to C. Let it send M to M' and N to N'. Then M' is on SB, N' is on SC and $M'N' \| MN$. So $M' = (0, 0, s) + (0, a, -s)u = (0, au, s(1-u))$ for some u and $N' = (0, 0, s) + (a, a, -s)v = (av, av, s(1-v))$ for some v. Now the dot product of AE and $M'N'$ is

$$(0, rs, ra)\cdot(av, a(v-u), s(u-v)) = 0.$$

So $AE \perp M'N'$. Therefore, $AE \perp MN$.

Commended solvers: **WONG Kam Wing** (TWGH Chong Ming Thien College).

Problem 279. Let R be the set of all real numbers. Determine (with proof) all functions $f: R\to R$ such that for all real x and y,

$$f\big(f(x)+y\big)=2x+f\big(f(f(y))-x\big).$$

Solution. **Jeff CHEN** (Virginia, USA), **Salem MALIKIĆ** (Sarajevo College, 3rd Grade, Sarajevo, Bosnia and Herzegovina) and **PUN Ying Anna** (HKU, Math Year 1).

Setting $y = 0$, we get

$$f(f(x)) = 2x + f(f(f(0)) - x). \qquad (1)$$

Then putting $x = 0$ into (1), we get

$$f(f(0)) = f(f(f(0))). \qquad (2)$$

In (1), setting, $x = f(f(0))$, we get

$$f(f(f(f(0)))) = 2f(f(0)) + f(0).$$

Using (2), we get $f(f(0)) = 2f(f(0)) + f(0)$. So $f(f(0)) = -f(0)$. Using (2), we see $f^{(k)}(0) = -f(0)$ for $k = 2,3,4,\ldots$.

In the original equation, setting $x = 0$ and $y = -f(0)$, we get

$$\begin{aligned} f(0) &= -2f(0) + f(f(f(-f(0)))) \\ &= -2f(0) + f^{(5)}(0) \\ &= -2f(0) - f(0) = -3f(0). \end{aligned}$$

So $f(0) = 0$. Then (1) becomes

$$f(f(x)) = 2x + f(-x). \qquad (3)$$

In the original equation, setting $x = 0$, we get $f(y) = f(f(f(y)))$. (4)

Setting $x = f(y)$ in (3), we get

$$f(y) = f(f(f(y))) = 2f(y) + f(-f(y)).$$

So $f(-f(y)) = -f(y)$. Setting $y = -f(x)$ in the original equation, we get

$$0 = 2x + f(f(f(-f(x))) - x).$$

For every real number w, setting $x = -w/2$, we see $w = f(f(f(-f(x))) - x)$. Hence, f is surjective. Then by (4), $w = f(f(w))$ for all w. By (3), setting $x = -w$, we get $f(w) = w$ for all w. Substituting this into the original equation clearly works. So the only solution is $f(w) = w$ for all w.

Commended solvers: **Kelvin LEE** (Winchester College, England),

Problem 280. Let n and k be fixed positive integers. A basket of peanuts is distributed into n piles. We gather the piles and rearrange them into $n + k$ new piles. Prove that at least $k + 1$ peanuts are transferred to smaller piles than the respective original piles that contained them. Also, give an example to show the constant $k + 1$ cannot be improved.

Solution. **Jeff CHEN** (Virginia, USA), **Stephen KIM** (Toronto, Canada) and **PUN Ying Anna** (HKU, Math Year 1).

Before the rearrangement, for each pile, if the pile has m peanuts, then attach a label of $1/m$ to each peanut in the pile. So the total sum of all labels is n.

Assume that only at most k peanuts were put into smaller piles after the rearrangement. Since the number of piles become $n + k$, so there are at least n of these $n + k$ piles, all of its peanuts are now in piles that are larger or as large as piles they were in before the rearrangement. Then the sum of the labels in just these n piles is already at least n. Since there are $k > 0$ more piles, this is a contradiction.

For an example to show $k + 1$ cannot be improved, take the case originally one of the n piles contained $k + 1$ peanuts. Let us rearrange this pile into $k + 1$ piles with 1 peanut each and leave the other $n - 1$ piles alone. Then only these $k + 1$ peanuts go into smaller piles.

Commended solvers: **WONG Kam Wing** (TWGH Chong Ming Thien College).

Olympiad Corner

(continued from page 1)

Problem 3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique in the other room.

Day 2 (*July 26, 2007*)

Problem 4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

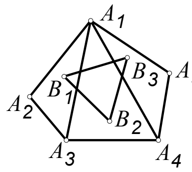
Problem 6. Let n be a positive integer. Consider

$$S=\{(x,y,z): x,y,z\in\{0,1,\ldots,n\}, x+y+z>0\}$$

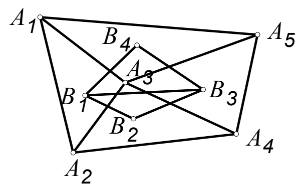
as a set of $(n + 1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Convex Hull

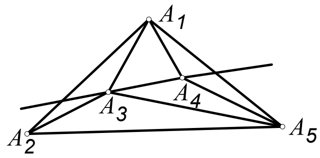
(continued from page 1)



<u>Case 1:</u> (The boundary of H is the pentagon $A_1A_2A_3A_4A_5$.) If $\triangle A_1A_2A_3$ or $\triangle A_1A_3A_4$ or $\triangle A_1A_4A_5$ does not contain any point of B in its interior, then we are done. Otherwise, there exist B_1, B_2, B_3 in their interiors respectively. Then we see $\triangle B_1B_2B_3$ is a desired triangle.



<u>Case 2:</u> (The boundary of H is a quadrilateral, say $A_1A_2A_4A_5$ with A_3 inside.) If $\triangle A_1A_3A_2$ or $\triangle A_2A_3A_4$ or $\triangle A_4A_3A_5$ or $\triangle A_5A_3A_1$ does not contain any point of B in its interior, then we are done. Otherwise, there exist B_1, B_2, B_3, B_4 in their interiors respectively. Then either $\triangle B_1B_2B_3$ or $\triangle B_3B_4B_1$ does not contain A_3 in its interior. That triangle is a desired triangle.



<u>Case 3:</u> (The boundary of H is a triangle, say $A_1A_2A_5$ with A_3, A_4 inside, say A_3 is closer to line A_1A_2 than A_4.) If $\triangle A_1A_2A_3$ or $\triangle A_1A_3A_4$ or $\triangle A_1A_4A_5$ or $\triangle A_2A_3A_5$ or $\triangle A_3A_4A_5$ does not contain any point of B in its interior, then we are done. Otherwise, there exists a point of B in each of their interiors respectively. Then three of these points of B lie on one side of line A_3A_4. The triangle formed by these three points of B is a desired triangle.

Mathematical Excalibur

Volume 12, Number 4 *November 2007 – December 2007*

Olympiad Corner

Below were the problems of the 10th China Hong Kong Math Olympiad, which was held on November 24, 2007. It was a three hour exam.

Problem 1. et D be a point on the side BC of triangle ABC such that $AB+BD = AC+CD$. The line segment AD cut the incircle of triangle ABC at X and Y with X closer to A. Let E be the point of contact of the incircle of triangle ABC on the side BC. Show that

(i) EY is perpendicular to AD,
(ii) XD is $2IA'$, where I is the incentre of the triangle ABC and A' is the midpoint of BC.

Problem 2. Is there a polynomial f of degree 2007 with integer coefficients, such that $f(n)$, $f(f(n))$, $f(f(f(n)))$, … are pairwise relatively prime for every integer n? Justify your claims.

Problem 3. In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 15, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inequalities with Product Condition

Salem Malikić

(4th Grade Student, Sarajevo College, Bosnia and Herzegovina)

There are many inequality problems that have n positive variables $a_1, a_2, \ldots, a_n$ (generally $n = 3$) such that their product is 1. There are several ways to solve this kind of problems. One common method is to change these variables by letting

$$a_1=\left(\frac{x_2}{x_1}\right)^{\alpha}, a_2=\left(\frac{x_3}{x_2}\right)^{\alpha}, \cdots, a_n=\left(\frac{x_1}{x_n}\right)^{\alpha},$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers and generally α=1. Here are some examples on the usage of these substitutions.

Example 1. If a, b, c are positive real numbers such that $abc = 1$, then prove that

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge\frac{3}{2}.$$

Solution. Since $abc = 1$, we can find positive x, y, z such that $a = x/y$, $b = y/z$, $c = z/x$ (for example, x=1=abc, y=bc and z=c). Then

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}$$
$$=\frac{x/y}{(x/z)+1}+\frac{y/z}{(y/x)+1}+\frac{z/x}{(z/y)+1}$$
$$=\frac{zx}{xy+yz}+\frac{xy}{zx+yz}+\frac{yz}{xy+zx}\ge\frac{3}{2},$$

where the inequality follows from *Nesbitt's inequality* applied to zx, xy and yz. (*Editor*—Nesbitt's inequality asserts that if $r,s,t > 0$, then

$$\frac{r}{s+t}+\frac{s}{t+r}+\frac{t}{r+s}\ge\frac{3}{2}.$$

It follows by writing the left side as

$$\frac{r+s+t}{s+t}+\frac{r+s+t}{t+r}+\frac{r+s+t}{r+s}-3$$
$$=\left(\frac{s+t}{2}+\frac{t+r}{2}+\frac{r+s}{2}\right)\left(\frac{1}{s+t}+\frac{1}{t+r}+\frac{1}{r+s}\right)-3$$
$$\ge\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)^2-3=\frac{3}{2},$$

where the inequality sign is due to the Cauchy-Schwarz inequality.)

Equality occurs if and only if the three variables are equal.

Example 2. (*2004 Russian Math Olympiad*) Prove that if $n > 3$ and $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\cdots+\frac{1}{1+x_n+x_nx_1}>1.$$

Solution. Again we use the substitutions $x_1 = a_2/a_1$, $x_2 = a_3/a_2$, …, $x_n = a_1/a_n$ (say a_1=1 and for $i > 1$, a_i=$x_1x_2\cdots x_{i-1}$). Then the inequality is equivalent to

$$\frac{a_1}{a_1+a_2+a_3}+\frac{a_2}{a_2+a_3+a_4}+\cdots+\frac{a_n}{a_n+a_1+a_2}$$
$$>\sum_{i=1}^{n}\frac{a_i}{a_1+a_2+\cdots+a_n}=1,$$

where the inequality sign is because $n > 3$ and $a_i > 0$ for all i so that $a_i+a_{i+1}+a_{i+2} < a_1+a_2+\cdots+a_n$.

Example 3. If $a, b, c > 0$ and $abc = 1$, then prove that

$$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution. Since $abc = 1$, we can find positive x, y, z such that $a = x/y$, $b = z/x$, $c = y/z$ (for example, $x = 1 = abc$, $y = bc$ and z=b). After doing the substitution, the inequality can be rewritten as

$$3+\frac{x^2}{yz}+\frac{z^2}{xy}+\frac{y^2}{zx}\ge\frac{x}{y}+\frac{z}{x}+\frac{y}{z}+\frac{y}{x}+\frac{x}{z}+\frac{z}{y}.$$

Multiplying by xyz on both sides, we get

$$x^3+y^3+z^3+3xyz$$
$$\ge x^2y+xy^2+y^2z+yz^2+z^2x+zx^2,$$

which is just *Schur's inequality* (see *vol.* 10, *no.* 5, *p.* 2 of *Math Excalibur*). Since x,y,z are positive, equality holds if and only if $x = y = z$, that is $a = b = c$.

Example 4. (*Mathlinks Contest*) Prove that if $a,b,c,d > 0$ and $abcd = 1$, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\ge 2.$$

Solution. Let us perform the following substitutions

$$a=\frac{x}{y}, b=\frac{z}{x}, c=\frac{t}{z}, d=\frac{y}{t}$$

with $x,y,z,t>0$ (for example, $x=1=abcd$, $y=bcd$, $z=b$ and $t=bc$). Then after simple transformations, our inequality becomes

$$\frac{y}{x+z}+\frac{x}{z+t}+\frac{z}{y+t}+\frac{t}{x+y}\ge 2.$$

Let I be the left side of this inequality and

$$J=y(x+z)+x(z+t)+z(y+t)+t(x+y).$$

By the Cauchy-Schwarz inequality, we easily get $IJ\ge(x+y+z+t)^2$. Then

$$\frac{y}{x+z}+\frac{x}{z+t}+\frac{z}{y+t}+\frac{t}{x+y}=I$$
$$\ge\frac{(x+y+z+t)^2}{yx+yz+xz+xt+zy+zt+tx+ty}.$$

So it is enough to prove that

$$\frac{(x+y+z+t)^2}{yx+yz+xz+xt+zy+zt+tx+ty}\ge 2,$$

which is equivalent to

$$x^2+y^2+z^2+t^2\ge 2(yz+xt).$$

This one is equivalent to

$$(x-t)^2+(y-z)^2\ge 0,$$

which is obviously true.

For equality case to occur, we must have $x=t$ and $y=z$, which directly imply $a=c$ and $b=d$ so $ab=1$ and therefore $b=d=1/a=1/c$ is the equality case.

Example 5. (*Crux* 3147) Let $n\ge 3$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n=1$. For $n=3$ and $n=4$ prove that

$$\frac{1}{x_1^2+x_1x_2}+\frac{1}{x_2^2+x_2x_3}+\cdots+\frac{1}{x_n^2+x_nx_1}\ge\frac{n}{2}.$$

Solution. We consider the substitutions

$$x_1=\sqrt{\frac{a_2}{a_1}}, x_2=\sqrt{\frac{a_3}{a_2}},\ldots,x_n=\sqrt{\frac{a_1}{a_n}}.$$

The inequality becomes

$$\frac{a_1}{a_2+\sqrt{a_1a_3}}+\frac{a_2}{a_3+\sqrt{a_2a_4}}+\cdots+\frac{a_n}{a_1+\sqrt{a_na_2}}\ge\frac{n}{2}.$$

Since

$$\sqrt{a_1a_3}\le\frac{a_1+a_3}{2},\ldots,\sqrt{a_na_2}\le\frac{a_n+a_2}{2}$$

by the *AM-GM* inequality, it suffices to show that

$$\frac{a_1}{a_1+2a_2+a_3}+\frac{a_2}{a_2+2a_3+a_4}+\cdots+\frac{a_n}{a_n+2a_1+a_2}\ge\frac{n}{4}.$$

Let I be the left side of this inequality and

$$J=a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2).$$

By the Cauchy-Schwarz inequality, we have $IJ\ge(a_1+a_2+\cdots+a_n)^2$. Thus, to prove $I\ge n/4$, it suffices to show that $(a_1+a_2+\cdots+a_n)^2/J\ge n/4$, which is equivalent to

$$4(a_1+a_2+\cdots a_n)^2$$
$$\ge n(a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2)).$$

For $n=4$, by expansion, we can see the inequality is actually an identity. For $n=3$, the inequality is equivalent to

$$a_1^2+a_2^2+a_3^2\ge a_1a_2+a_2a_3+a_3a_1,$$

which is true because

$$2(a_1^2+a_2^2+a_3^2)-2(a_1a_2+a_2a_3+a_3a_1)$$
$$=(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2$$
$$\ge 0.$$

This completes the proof. Equality holds if and only if $x_i=1$ for all i.

NOTE: This problem appeared in the May 2006 issue of the Crux Mathematicorum. It was proposed by Vasile Cîrtoaje and Gabriel Dospinescu. No complete solution was received (except the above solution of the proposers).

Example 6. (*Crux* 2023) Let a,b,c,d,e be positive real numbers such that $abcde=1$. Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}$$
$$+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc}\ge\frac{10}{3}.$$

Solution. Again we consider the standard substitutions

$$a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{t}, d=\frac{t}{u}, e=\frac{u}{x},$$

where $x,y,z,t,u>0$.

Now we have

$$\frac{a+abc}{1+ab+abcd}=\frac{1/y+1/t}{1/x+1/z+1/u}.$$

Writing the other relations and letting

$$a_1=\frac{1}{x}, a_2=\frac{1}{y}, a_3=\frac{1}{z}, a_4=\frac{1}{t}, a_5=\frac{1}{u},$$

we have to show that if $a_1, a_2, a_3, a_4, a_5>0$, then

$$\sum_{cyclic}\frac{a_2+a_4}{a_1+a_3+a_5}\ge\frac{10}{3}. \qquad (*)$$

(*Editor*—The notation

$$\sum_{cyclic}f(a_1,a_2,\ldots,a_n)$$

for n variables $a_1, a_2, \ldots, a_n$ is a shorthand notation for

$$\sum_{i=1}^{n}f(a_i,a_{i+1},\ldots,a_{i+n}),$$

where $a_{i+j}=a_{i+j-n}$ when $i+j>n$.)

Let I be the left side of inequality (*),

$$J=\sum_{cyclic}(a_2+a_4)(a_1+a_3+a_5)$$

and $S=a_1+a_2+a_3+a_4+a_5$. Using the Cauchy-Schwarz inequality, we easily get

$$IJ\ge\left(\sum_{cyclic}(a_2+a_4)\right)^2=(2S)^2=4S^2.$$

So to prove $I\ge 10/3$, it is enough to show

$$\frac{4S^2}{J}\ge\frac{10}{3}. \qquad (**)$$

Now comparing S^2 and J, we can observe that $2S^2-J$ equals

$$T=(a_2+a_4)^2+(a_1+a_4)^2+(a_3+a_5)^2$$
$$+(a_2+a_5)^2+(a_1+a_3)^2.$$

Using this relation, (**) can be rewritten as

$$12S^2\ge 10J=10(2S^2-T)$$
$$=20S^2-10T.$$

This simplifies to $5T\ge 4S^2$. Finally, writing $5=1^2+1^2+1^2+1^2+1^2$, we can get $5T\ge 4S^2$ from the Cauchy-Schwarz inequality easily.

Again equality occurs if and only if all the a_i's are equal, which corresponds to the case $a=b=c=d=e=1$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***January 15, 2008.***

Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number y such that the sum of $\{x_1-y\}, \{x_2-y\}, \ldots, \{x_n-y\}$ is at most $(n-1)/2$. (Here $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x.)

Can y always be chosen to be one of the x_i's ?

Problem 287. Determine (with proof) all nonempty subsets A, B, C of the set of all positive integers $\mathbb{Z}^+$ satisfying

(1) $A\cap B=B\cap C=C\cap A=\varnothing$;
(2) $A\cup B\cup C=\mathbb{Z}^+$;
(3) for every $a\in A$, $b\in B$ and $c\in C$, we have $c+a\in A$, $b+c\in B$ and $a+b\in C$.

Problem 288. Let H be the orthocenter of triangle ABC. Let P be a point in the plane of the triangle such that P is different from A, B, C.

Let L, M, N be the feet of the perpendiculars from H to lines PA, PB, PC respectively. Let X, Y, Z be the intersection points of lines LH, MH, NH with lines BC, CA, AB respectively.

Prove that X, Y, Z are on a line perpendicular to line PH.

Problem 289. Let a and b be positive numbers such that $a+b<1$. Prove that

$$\frac{(a-1)^2+b(2a-b)}{(b-1)^2+a(2b-a)}\geq \min\left\{\frac{a}{b},\frac{b}{a}\right\}.$$

Problem 290. Prove that for every integer a greater than 2, there exist infinitely many positive integers n such that a^n-1 is divisible by n.

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Solutions

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Due to an editorial mistake in the last issue, solution to problems 279 by **Li ZHOU** (Polk Community College, Winter Haven, Florida USA) was overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 281. Let N be the set of all positive integers. Prove that there exists a function $f: N\to N$ such that $f(f(n)) = n^2$ for all n in N. (*Source: 1978 Romanian Math Olympiad*)

***Solution* 1. George Scott ALDA, Jeff CHEN** (Virginia, USA), **NGOO Hung Wing** (HKUST, Math Year 1), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7) and **Fai YUNG**.

Let x_k be the k-th term of the sequence

2,3,5,6,7,8,10,11,12,13,14,15,17,18,19,…

of all positive integers that are not perfect squares in increasing order. By taking square roots (repeatedly) of an integer $n>1$, we will eventually get to one of the x_k's. So every integer $n>1$ is of the 2^m-th power of x_k for some nonnegative integer m and positive integer k.

We define $f(1)=1$. For $n>1$, if n is the 2^m-th power of x_k, then we define $f(n)$ as follow:

<u>case 1</u>: if k is odd, then $f(n)$ is the 2^m-th power of x_{k+1};

<u>case 2</u>: if k is even, then $f(n)$ is the 2^{m+1}-st power of x_{k-1}.

Observe that if n is under case 1, then $f(n)$ will be under case 2. Similarly, if n is under case 2, then $f(n)$ will be under case 1. In computing $f(f(n))$, we have to apply case 2 once so that m increases by 1 and the k value goes up once and down once. Therefore, we have $f(f(n)) = n^2$ for all n in N.

Solution 2. **GRA20 Problem Solving Group** (Roma, Italy) and **Kelvin LEE** (Trinity College, Cambridge, England).

We first define a function $g: N\to N$ such that $g(g(n)) = 2n$. Let p be an odd prime and let $\mathrm{ord}_p(n)$ be the greatest nonnegative integer α such that $p^\alpha \mid n$. If $\mathrm{ord}_p(n)$ is even, then let $g(n)=2pn$, otherwise let $g(n)=n/p$.

Next we will check $g(g(n)) = 2n$. If $\mathrm{ord}_p(n)$ is even, then $\mathrm{ord}_p(g(n)) = \mathrm{ord}_p(2pn)$ is odd and so $g(g(n)) = g(2pn) = 2pn/p = 2n$.

If $\mathrm{ord}_p(n)$ is odd, then $\mathrm{ord}_p(g(n)) = \mathrm{ord}_p(n/p)$ is even and so $g(g(n)) = g(n/p) = 2p(n/p) = 2n$.

Define $f(1)=1$. For an integer $n>1$, let

$$n=\prod_{k=1}^{r} p_k^{\alpha_k}\text{ , where all } \alpha_k>0,$$

be the prime factorization of n, then we define

$$f(n)=\prod_{k=1}^{r} p_k^{g(\alpha_k)}.$$

Finally, we have

$$f(f(n))=\prod_{k=1}^{r} p_k^{g(g(\alpha_k))}=\prod_{k=1}^{n} p_k^{2\alpha_k}=n^2.$$

Problem 282. Let a, b, c, A, B, C be real numbers, $a\neq 0$ and $A\neq 0$. For every real number x,

$$|ax^2+bx+c|\leq |Ax^2+Bx+C|.$$

Prove that $|b^2-4ac|\leq |B^2-4AC|$. (*Source: 2003 Putnam Exam*)

Solution. **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil), **Jeff CHEN** (Virginia, USA), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

We have

$$|a|=\lim_{x\to\infty}\frac{|ax^2+bx+c|}{x^2}\leq\lim_{x\to\infty}\frac{|Ax^2+Bx+C|}{x^2}=|A|.$$

If $B^2-4AC>0$, then $Ax^2+Bx+C=0$ has two distinct real roots x_0 and x_1. By the given inequality, these will also be roots of $ax^2+bx+c=0$. So $b^2-4ac>0$. Then

$$|B^2-4AC|=A^2(x_0-x_1)^2$$
$$\geq a^2(x_0-x_1)^2=|b^2-4ac|.$$

If $B^2-4AC\leq 0$, then by replacing A by $-A$ or a by $-a$ if necessary, we may assume $A\geq a>0$. Since $A>0$ and $B^2-4AC\leq 0$, so for every real number x, $Ax^2+Bx+C\geq 0$. Then the given inequality implies for every real x,

$$Ax^2+Bx+C\geq \pm(ax^2+bx+c)\text{ .}\qquad (*)$$

Then $(A-a)x^2+(B-b)x+(C-c)\geq 0$. This implies

$$(B-b)^2\leq 4(A-a)(C-c).\qquad (**)$$

Similarly,

$$(B+b)^2\leq 4(A+a)(C+c).\qquad (***)$$

Then

$$(B^2-b^2)^2\leq 16(A^2-a^2)(C^2-c^2)$$
$$\leq 16(AC-ac)^2,$$

which implies $B^2-b^2\leq 4|AC-ac|$.

Taking $x=0$ in (*), we get $C\geq |c|$. Since $A\geq a>0$, we get $B^2-b^2\leq$

$4(AC-ac)$. Hence,

$$4ac - b^2 \le 4AC - B^2. \qquad (\dagger)$$

Using (**) and (***), we have

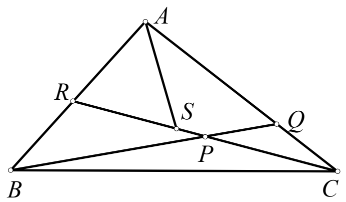
$$\begin{aligned} B^2 + b^2 &= \frac{(B-b)^2 + (B+b)^2}{2} \\ &\le 2\big((A-a)(C-c) + (A+a)(C+c)\big) \\ &= 4(AC + ac). \end{aligned}$$

Then $-(4ac - b^2) \le 4AC - B^2$. Along with (†), we have

$$\begin{aligned} |b^2 - 4ac| &= \pm(4ac - b^2) \\ &\le 4AC - B^2 = |B^2 - 4AC|. \end{aligned}$$

Problem 283. P is a point inside $\triangle ABC$. Lines AC and BP intersect at Q. Lines AB and CP intersect at R. It is known that $AR=RB=CP$ and $CQ=PQ$. Find $\angle BRC$ with proof. (*Source: 2003 Japanese Math Olympiad*)

Solution. **Stephen KIM** (Toronto, Canada)**.**



Let S be the point on segment CR such that $RS=CP=AR$. Since $CQ=PQ$, we have

$$\angle ACS = \angle QPC = \angle BPR.$$

Also, since $RS=CP$, we have

$$SC = CR - RS = CR - CP = RP.$$

Considering line CR cutting $\triangle ABQ$, by Menelaus' theorem, we have

$$\frac{RB}{AR} \cdot \frac{PQ}{BP} \cdot \frac{AC}{CQ} = 1.$$

Since $AR=RB$ and $CQ=PQ$, we get $AC = BP$. Hence, $\triangle ACS \cong \triangle BPR$. Then $AS = BR = AR = CP = RS$ and so $\triangle ARS$ is equilateral. Therefore, $\angle BRC=120°$.

Commended solvers: **FOK Pak Hei** (Pui Ching Middle School, Form 6), **Kelvin LEE** (Trinity College, Cambridge, England), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 5) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

Problem 284. Let p be a prime number. Integers x, y, z satisfy $0 < x < y < z < p$. If x^3, y^3, z^3 have the same remainder upon dividing by p, then prove that $x^2 + y^2 + z^2$ is divisible by $x+y+z$. (*Source: 2003 Polish Math Olympiad*)

Solution. **George Scott ALDA, José Luis DÍAZ-BARRERO** (Universitat Politècnica de Catalunya, Barcelona, Spain), **EZZAKI Mahmoud** (Omar Ibn Abdelaziz, Morocco), **Stephen KIM** (Toronto, Canada), **Kelvin LEE** (Trinity College, Cambridge, England) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Since $x^3 \equiv y^3 \equiv z^3 \pmod p$, so

$$p \mid x^3 - y^3 = (x-y)(x^2+xy+y^2).$$

Since $0 < x < y < z < p$ and p is prime, we have $p \nmid x-y$ and hence

$$p \mid x^2+xy+y^2. \qquad (1)$$

Similarly,

$$p \mid y^2+yz+z^2 \qquad (2)$$

and

$$p \mid z^2+zx+x^2. \qquad (3)$$

By (1) and (2), p divides

$$(x^2+xy+y^2)-(y^2+yz+z^2)=(x-z)(x+y+z).$$

Since $0 < z-x < p$, we have $p \mid x+y+z$.

Also, $0 < x < y < z < p$ implies $x+y+z = p$ or $2p$ and $p > 3$. Now

$$x+y+z \equiv x^2+y^2+z^2 \pmod 2.$$

Thus, it remains to show $p \mid x^2+y^2+z^2$.

Now $x^2+xy+y^2 = x(x+y+z)+ y^2-xz$. From (1), we get

$$p \mid y^2-xz. \qquad (4)$$

Similarly,

$$p \mid x^2-zy \qquad (5)$$

and

$$p \mid z^2-yx. \qquad (6)$$

Adding the right sides of (1) to (6), we get

$$p \mid 3(x^2+y^2+z^2).$$

Since $p > 3$ is prime, we get $p \mid x^2+y^2+z^2$ as desired.

Commended solvers: **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

Problem 285. Determine the largest positive integer N such that for every way of putting all numbers 1 to 400 into a 20×20 table (1 number per cell), one can always find a row or a column having two numbers with difference not less than N. (*Source: 2003 Russian Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **Stephen KIM** (Toronto, Canada).

The answer is 209. We first show $N \le 209$. Divide the table into a left and a right half, each of dimension 20×10. Put 1 to 200 row wise in increasing order into the left half. Similarly, put 201 to 400 row wise in increasing order into the right half. Then the difference of two numbers in the same row is at most 210−1=209 and the difference of two numbers in the same column is at most 191−1=190. So $N \le 209$.

Next we will show $N \ge 209$. Let $M_1 = \{1,2,\ldots,91\}$ and $M_2 = \{300, 301, \ldots, 400\}$.

Color a row or a column <u>red</u> if and only if it contains a number in M_1. Similarly, color a row or a column <u>blue</u> if and only if it contains a number in M_2. We claim that

(1) the number of red rows plus the number of red columns is at least 20 and
(2) the number of blue rows plus the number of blue columns is at least 21.

Hence, there is a row or a column that is colored red and blue. So two of the numbers in that row or column have a difference of at least 300−91=209.

For claim (1), let there be i red rows and j red columns. Since the numbers in M_1 can only be located at the intersections of these red rows and columns, we have $ij \ge 91$. By the *AM-GM* inequality,

$$i + j \ge 2\sqrt{ij} \ge 2\sqrt{91} > 19.$$

Similarly, claim (2) follows from the facts that there are 101 numbers in M_2 and $2\sqrt{101} > 20$.

Olympiad Corner

(continued from page 1)

Problem 3. (*Cont.*) Show that there is a club with at least 11 male and 11 female members.

Problem 4. Determine if there exist positive integer pairs (m,n), such that

(i) the greatest common divisor of m and n is 1, and $m \le 2007$,

(ii) for any k=1,2, …, 2007,

$$\left[\frac{nk}{m}\right] = \left[\sqrt{2}\, k\right].$$

(Here $[x]$ stands for the greatest integer less than or equal to x.)

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2007Estonian IMO Team Selection Contest.

First Day

Problem 1. On the control board of a nuclear station, there are n electric switches ($n > 0$), all in one row. Each switch has two possible positions: up and down. The switches are connected to each other in such a way that, whenever a switch moves down from its upper position, its right neighbor (if it exists) automatically changes position. At the beginning, all switches are down. The operator of the board first changes the position of the leftmost switch once, then the position of the second leftmost switch twice etc., until eventually he changes the position of the rightmost switch n times. How many switches are up after all these operations?

Problem 2. Let D be the foot of the altitude of triangle ABC drawn from vertex A. Let E and F be points symmetric to D with respect to lines AB and AC, respectively. Let R_1 and R_2 be

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

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Square It!

Pham Van Thuan
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Inequalities involving square roots of the form

$$\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{D}\le k$$

can be solved using the Cauchy-Schwarz inequality. However, solving inequalities of the following form

$$\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{D}\ge k$$

is far from straightforward. In this article, we will look at such problems. We will solve them by squaring and making more delicate use of the Cauchy-Schwarz inequality.

Example 1. Three nonnegative real numbers x, y and z satisfy $x^2+y^2+z^2=1$. Prove that

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{z+x}{2}\right)^2}\ge\sqrt{6}.$$

Solution. Squaring both sides of the inequality and simplifying, we get the equivalent inequality

$$\sum_{cyclic}\sqrt{1-\left(\frac{x+y}{2}\right)^2}\sqrt{1-\left(\frac{y+z}{2}\right)^2}\ge\frac{7}{4}+\frac{xy+yz+zx}{4},$$

where

$$\sum_{cyclic}f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y).$$

Notice that

$$1-\left(\frac{x+y}{2}\right)^2=\frac{x^2+y^2+(z^2+1)}{2}-\left(\frac{x+y}{2}\right)^2$$
$$=\frac{(x-y)^2}{4}+\frac{z^2+1}{2}.$$

By the Cauchy-Schwarz inequality,

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}\sqrt{1-\left(\frac{y+z}{2}\right)^2}$$
$$\ge\frac{(x-y)(z-y)}{4}+\frac{\sqrt{(z^2+1)(x^2+1)}}{2}$$
$$\ge\frac{y^2+xz-yz-xy}{4}+\frac{zx+1}{2}.$$

Similarly, we obtain two other such inequalities. Multiplying each of them by 2, adding them together, simplifying and finally using $x^2+y^2+z^2=1$, we get the equivalent inequality in the beginning of this solution.

Example 2. For a, b, $c > 0$, prove that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}$$
$$\ge 2\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}.$$

Solution. Multiplying both sides by $\sqrt{(a+b)(b+c)(c+a)}$, we have to show

$$\sum_{cyclic}\sqrt{a(c+a)(a+b)}$$
$$\ge 2\sqrt{(a+b+c)(ab+bc+ca)}.$$

Squaring both sides, we get the equivalent inequality

$$\sum_{cyclic}a^3+2\sum_{cyclic}(a+b)\sqrt{ab(a+c)(b+c)}$$
$$\ge 3\sum_{cyclic}ab(a+b)+9abc. \qquad (*)$$

By the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$(a+b)\sqrt{ab(a+c)(b+c)}$$
$$\ge(a+b)\sqrt{ab(\sqrt{ab}+c)^2}$$
$$=(a+b)(\sqrt{ab}+c)\sqrt{ab}$$
$$=ab(a+b)+(a+b)c\sqrt{ab}$$
$$\ge ab(a+b)+2abc.$$

Using this, we have

$$\sum_{cyclic}a^3+2\sum_{cyclic}(a+b)\sqrt{ab(c+a)(c+b)}$$
$$\ge\sum_{cyclic}a^3+2\sum_{cyclic}ab(a+b)+12abc.$$

Comparing with (*), we need to show

$$\sum_{cyclic}a^3-\sum_{cyclic}ab(a+b)+3abc\ge 0.$$

This is just Schur's inequality

$$\sum_{cyclic}a(a-b)(a-c)\ge 0.$$

(See <u>*Math. Excalibur*</u>, vol.10, no.5, p.2)

From the last example, we saw that other than the Cauchy-Schwarz inequality, we might need to recall Schur's inequality

$$\sum_{cyclic} x^r(x-y)(x-z) \ge 0.$$

Here we will also point out a common variant of Schur's inequality, namely

$$\sum_{cyclic} x^r(y+z)(x-y)(x-z) \ge 0.$$

This variant can be proved in the same way as Schur's inequality (again see <u>*Math. Excalibur*</u>, vol.10, no.5, p.2). Both inequalities become equality if and only if either the variables are all equal or one of them is zero, while the other two are equal. In the next two examples, we will use these.

<u>Example 3.</u> Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \ge \sqrt{3}.$$

When does equality occur?

<u>Solution.</u> Squaring both sides of the inequality and using

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 1 - 2(ab+bc+ca),$$

we get the equivalent inequality

$$\sum_{cyclic} \sqrt{a+(b-c)^2}\sqrt{b+(c-a)^2} \ge 3(ab+bc+ca).$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\sqrt{a+(b-c)^2}\sqrt{b+(c-a)^2}\\ &= \sqrt{(b-c)^2+(a+b+c)a}\sqrt{(c-a)^2+(a+b+c)b}\\ &\ge |(b-c)(c-a)| + (a+b+c)\sqrt{ab}.\end{aligned}$$

Similarly, we can obtain two other such inequalities. Adding them together, the right side is

$$\sum_{cyclic} |(b-c)(c-a)| + (a+b+c)\sum_{cyclic}\sqrt{ab}.$$

By the triangle inequality and the case $r = 0$ of Schur's inequality, we get

$$\begin{aligned}\sum_{cyclic} |(b-c)(c-a)| &\ge \left|\sum_{cyclic}(b-c)(c-a)\right| \\ &= \sum_{cyclic}(c-b)(c-a) \\ &= (a^2+b^2+c^2) - (ab+bc+ca).\end{aligned} \quad (**)$$

Thus, to finish, it will be enough to show

$$a^2+b^2+c^2+(a+b+c)(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge 4(ab+bc+ca).$$

Now we make the substitutions

$$x = \sqrt{a}, \quad y = \sqrt{b} \quad \text{and} \quad z = \sqrt{c}.$$

In terms of *x, y, z*, the last inequality becomes

$$\sum_{cyclic}(x^4+x^3y+x^3z+x^2yz-4x^2y^2) \ge 0. \quad (***)$$

Since the terms are of degree 4, we consider the case $r = 2$ of Schur's inequality, which is

$$\sum_{cyclic} x^2(x-y)(x-z)$$

$$= \sum_{cyclic}(x^4 - x^3y - x^3z + x^2yz) \ge 0.$$

This is not quite equal to (***). So next (due to degree 4 consideration again), we will look at the case $r = 1$ of the variant

$$\sum_{cyclic} x(y+z)(x-y)(x-z)$$

$$= \sum_{cyclic}(x^3y + x^3z - 2x^2y^2) \ge 0.$$

Readily we see (***) is just the sum of Schur's inequality with twice its variant.

Finally, tracing back, we see equality occurs if and only if $a = b = c = 1/3$ or one of them is 0, while the other two are equal to 1/2.

<u>Example 4.</u> Three nonnegative real numbers *a, b, c* satisfy $a + b + c = 2$. Prove that

$$\sqrt{\frac{a+b}{2}-ab}+\sqrt{\frac{b+c}{2}-bc}+\sqrt{\frac{c+a}{2}-ca} \ge \sqrt{2}.$$

<u>Solution.</u> Squaring both sides of the inequality and using $a + b + c = 2$, we get the equivalent inequality

$$\sum_{cyclic}\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)} \ge \frac{ab+bc+ca}{2}.$$

Note that

$$\begin{aligned}\frac{a+b}{2}-ab &= \frac{2(a+b)-(a+b)^2+(a-b)^2}{4}\\ &= \frac{(a-b)^2+(2-a-b)(a+b)}{4}\\ &= \frac{(a-b)^2}{4}+\frac{c(a+b)}{4}.\end{aligned}$$

Applying twice the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)}\\ &\ge \frac{|(a-b)(b-c)|}{4}+\frac{\sqrt{ca(a+b)(b+c)}}{4}\\ &\ge \frac{1}{4}\left(|(a-b)(b-c)|+\sqrt{ca(b+\sqrt{ca})^2}\right)\\ &= \frac{1}{4}\left(|(a-b)(b-c)|+\sqrt{abc}\sqrt{b}+ca\right).\end{aligned}$$

Similarly, we can obtain two other such inequalities. Adding them together and using (**) in example 3, we get

$$\begin{aligned}&4\sum_{cyclic}\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)}\\ &\ge \sum_{cyclic}\left(|(a-b)(b-c)|\right)+\sqrt{abc}\sum_{cyclic}\sqrt{b}+\sum_{cyclic}ca\\ &\ge a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}).\end{aligned}$$

Substituting

$$x = \sqrt{a}, \quad y = \sqrt{b} \quad \text{and} \quad z = \sqrt{c}$$

and using Schur's inequality and its variant, we have

$$\begin{aligned}&a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})\\ &= x^4+y^4+z^4+x^2yz+xy^2z+xyz^2\\ &\ge \sum_{cyclic}\left(x^3y+x^3z\right)\\ &\ge 2\sum_{cyclic}x^2y^2 = 2(ab+bc+ca).\end{aligned}$$

Combining this with the last displayed inequalities, we can obtain the equivalent inequality in the beginning of this solution.

To conclude this article, we will give two exercises for the readers to practice.

<u>Exercise 1.</u> Three nonnegative real numbers *x*, *y* and *z* satisfy $x^2+y^2+z^2=1$. Prove that

$$\sum_{cyclic}\sqrt{1-xy}\sqrt{1-yz} \ge 2.$$

<u>Exercise 2.</u> Three nonnegative real numbers *x*, *y* and *z* satisfy $x + y + z = 1$. Prove that

$$x\sqrt{1-yz}+y\sqrt{1-zx}+z\sqrt{1-xy} \ge \frac{2\sqrt{2}}{3}.$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***February 25, 2008.***

Problem 291. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

Problem 292. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers with no two of them are consecutive. For every $m = 1, 2, 3, \ldots$, let $S_m = k_1+k_2+\cdots+k_m$. Prove that for every positive integer n, the interval $[S_n, S_{n+1})$ contains at least one perfect square number.
(*Source: 1996 Shanghai Math Contest*)

Problem 293. Let CH be the altitude of triangle ABC with $\angle ACB = 90°$. The bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the midpoints of PM and QN is parallel to line AB.

Problem 294. For three nonnegative real numbers x, y, z satisfying the condition $xy + yz + zx = 3$, prove that

$$x^2 + y^2 + z^2 + 3xyz \geq 6.$$

Problem 295. There are $2n$ distinct points in space, where $n \geq 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are at least n distinct triangles formed.
(*Source: 1989 Chinese IMO team training problem*)

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Solutions

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Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number y such that the sum of $\{x_1-y\}, \{x_2-y\}, \ldots, \{x_n-y\}$ is at most $(n-1)/2$.

(Here $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x.)

Can y always be chosen to be one of the x_i's ?

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **HO Kin Fai** (HKUST, Math Year 3), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

For $i = 1, 2, \ldots, n$, let

$$S_i = \sum_{j=1}^{n} \{x_j - x_i\}.$$

For all real x, $\{x\} + \{-x\} \leq 1$ (since the left side equals 0 if x is an integer and equals 1 otherwise). Using this, we have

$$\sum_{i=1}^{n} S_i = \sum_{1\le i<j\le n} (\{x_j - x_i\} + \{x_i - x_j\})$$

$$\leq \sum_{1\le i<j\le n} 1 = \frac{n(n-1)}{2}.$$

So the average value of S_i is at most $(n-1)/2$. Therefore, there exists some $y = x_i$ such that S_i is at most $(n-1)/2$.

Problem 287. Determine (with proof) all nonempty subsets A, B, C of the set of all positive integers $\mathbb{Z}^+$ satisfying

(1) $A\cap B = B\cap C = C\cap A = \varnothing$;
(2) $A\cup B\cup C = \mathbb{Z}^+$;
(3) for every $a\in A$, $b\in B$ and $c\in C$, we have $c + a \in A$, $b + c \in B$ and $a + b \in C$.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **HO Kin Fai** (HKUST, Math Year 3), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

Let the minimal element of C be x. Then $\{1, 2, \ldots, x-1\}\subseteq A\cup B$. Since for every $a\in A$, $b\in B$, we have $x + a\in A$, $b + x\in B$. So all numbers not divisible by x are in $A\cup B$. Then every $c\in C$ is a multiple of x. By (3), the sum of every $a\in A$ and $b\in B$ is a multiple of x.

Assume $x = 1$. Then $a\in A$, $b\in B$ imply $a+1\in A$, $b + 1\in B$, which lead to $a + b \in A\cap B$ contradicting (1).

Assume $x = 2$. We may suppose $1\in A$. Then by (3), all odd positive integers are in A. For $b \in B$, we get $1 + b \in C$. Then b is odd, which lead to $b\in A\cap B$ contradicting (1).

Assume $x \geq 4$. Then $\{1,2,3\}\subseteq A\cup B$, say $y, z\in \{1,2,3\}\cap A$. Taking a $b \in B$, we get $y+b$, $z+b \in C$ by (3). Then $(y+b) - (z+b) = y - z$ is a multiple of x. But $|y - z| < x$ leads to a contradiction.

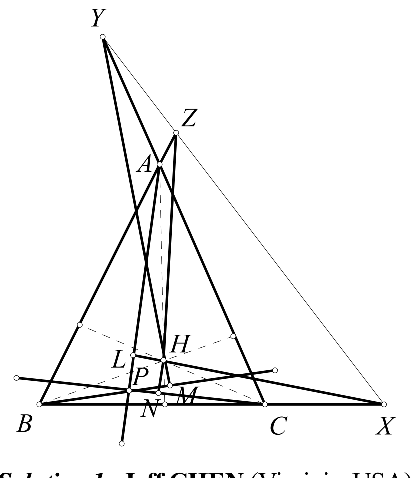
Therefore, $x = 3$. We claim 1 and 2 cannot both be in A (or both in B). If $1, 2\in A$, then (3) implies $3k + 1$, $3k + 2\in A$ for all $k\in \mathbb{Z}^+$. Taking a $b\in B$, we get $1 + b\in C$, which implies $b = 3k + 2\in A$. Then $b\in A\cap B$ contradicts (1).

Therefore, either $1\in A$ and $2\in B$ (which lead to $A = \{1,4,7,\ldots\}$, $B = \{2,5,8,\ldots\}$, $C = \{3,6,9,\ldots\}$) or $2\in A$ and $1\in B$ (which similarly lead to $A = \{2,5,8,\ldots\}$, $B = \{1,4,7,\ldots\}$, $C = \{3,6,9,\ldots\}$).

Problem 288. Let H be the orthocenter of triangle ABC. Let P be a point in the plane of the triangle such that P is different from A, B, C.

Let L, M, N be the feet of the perpendiculars from H to lines PA, PB, PC respectively. Let X, Y, Z be the intersection points of lines LH, MH, NH with lines BC, CA, AB respectively.

Prove that X, Y, Z are on a line perpendicular to line PH.



Solution 1. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England).

Since $XH = LH \perp PA$, $AH \perp CB = XB$, $BH\perp AC=AY$ and $YH=MH \perp BP$, we have respectively (see Math. Excalibur, vol.12, no.3, p.2)

$$XP^2-XA^2=HP^2-HA^2 \quad (1)$$
$$AX^2-AB^2=HX^2-HB^2 \quad (2)$$
$$BA^2-BY^2=HA^2-HY^2 \quad (3)$$
$$YB^2-YP^2=HB^2-HP^2 \quad (4)$$

Doing (1)+(2)+(3)+(4), we get

$$XP^2-YP^2=XH^2-YH^2,$$

which implies $XY \perp PH$. Similarly, $ZY \perp PH$. So, X, Y, Z are on a line perpendicular to line PH.

Solution 2. **Anna Ying PUN** (HKU, Math Year 2) and **Stephen KIM** (Toronto, Canada)**.**

Set the origin of the coordinate plane at H. For a point J, let (x_J, y_J) denote its coordinates. Since the slope of line PA is $(y_P - y_A)/(x_P - x_A)$, the equation of line HL is

$$(x_P - x_A)x + (y_P - y_A)y = 0. \qquad (1)$$

Since the slope of line HA is y_A/x_A, the equation of line BC is

$$x_A x + y_A y = x_A x_B + y_A y_B. \qquad (2)$$

Let $t = x_A x_B + y_A y_B$. Since point C is on line BC, we get $x_A x_C + y_A y_C = x_A x_B + y_A y_B = t$. Similarly, $x_B x_C + y_B y_C = t$.

Since X is the intersection of lines BC and HL, so the coordinates of X satisfy the sum of equations (1) and (2), that is

$$x_P x + y_P y = t.$$

(Since the slope of line PH is y_P/x_P, this is the equation of a line that is perpendicular to line PH.) Similarly, the coordinates of Y and Z satisfy $x_P x + y_P y = t$. Therefore, X, Y, Z lie on a line perpendicular to line PH.

Commended solvers: **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Problem 289. Let a and b be positive numbers such that $a + b < 1$. Prove that

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \ge \min\left\{\frac{a}{b}, \frac{b}{a}\right\}.$$

Solution. **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil), **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **Simon YAU Chi Keung** (City University of Hong Kong) and **Fai YUNG**.

Since $0 < a, b < a + b < 1$, we have

$$(b-1)^2 + a(2b-a) = b^2 + 2(a-1)b - a^2 + 1$$
$$= (b+a-1)^2 + 2a(1-a) > 0.$$

In case $a \ge b > 0$, we have

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \ge \min\left\{\frac{a}{b}, \frac{b}{a}\right\} = \frac{b}{a}$$

$$\Leftrightarrow \; a(a-1)^2 + ab(2a-b) \ge b(b-1)^2 + ab(2b-a)$$

$$\Leftrightarrow \; (a-b)[(a+b-1)^2 + 2ab] \ge 0,$$

which is true. In case $b > a > 0$, we have

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \ge \min\left\{\frac{a}{b}, \frac{b}{a}\right\} = \frac{a}{b}$$

$$\Leftrightarrow \; b(a-1)^2 + b^2(2a-b) \ge a(b-1)^2 + a^2(2b-a)$$

$$\Leftrightarrow \; (b-a)(1 - a^2 - b^2) \ge 0,$$

which is also true as $a^2 + b^2 < a + b < 1$.

Problem 290. Prove that for every integer a greater than 2, there exist infinitely many positive integers n such that $a^n - 1$ is divisible by n.

Solution 1. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **GRA20 Problem Solving Group** (Roma, Italy) and **HO Kin Fai** (HKUST, Math Year 3).

We will show by math induction that $n = (a-1)^k$ for $k = 1, 2, 3, \ldots$ satisfy the requirement. For $k = 1$, since $a - 1 > 1$ and $a \equiv 1 \pmod{a-1}$, so

$$a^{a-1} - 1 \equiv 1^{a-1} - 1 = 0 \pmod{a-1}.$$

Next, suppose case k is true. Then $a^{(a-1)^k} - 1$ is divisible by $(a-1)^k$. For the case $k + 1$, all we need to show is

$$\frac{a^{(a-1)^{k+1}} - 1}{a^{(a-1)^k} - 1} \equiv 0 \pmod{a-1}.$$

Note $b = a^{(a-1)^k} \equiv 1 \pmod{a-1}$. The left side of the above displayed congruence is

$$\frac{b^{a-1} - 1}{b - 1} = \sum_{k=0}^{a-2} b^k \equiv \sum_{k=0}^{a-2} 1 = a - 1 \equiv 0 \pmod{a-1}.$$

This completes the induction.

Solution 2. **Anna Ying PUN** (HKU, Math Year 2) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Note $n = 1$ works. We will show if n works, then $a^n - 1 (> 2^n - 1 \ge n)$ also works. If n works, then $a^n - 1 = nk$ for some positive integer k. Then

$$a^{a^n - 1} - 1 = a^{nk} - 1 = (a^n - 1)\sum_{j=0}^{k-1} a^{nj},$$

which shows $a^n - 1$ works.

Comments: Cheung Wang Chi pointed out that interestingly $n = 1$ is the only positive integer such that $2^n - 1$ is divisible by n (denote this by $n \mid 2^n - 1$). [This fact appeared in the 1972 Putnam Exam.-*Ed*.] To see this, he considered a minimal $n > 1$ such that $n \mid 2^n - 1$. He showed if $a, b, q \in \mathbb{Z}^+$ and $a = bq + r$ with $0 \le r < b$, then $2^a - 1 = ((2^b)^q - 1)2^r + (2^r - 1) = (2^b - 1)N + (2^r - 1)$ for some $N \in \mathbb{Z}^+$. Hence,

$$\gcd(2^a - 1, 2^b - 1) = \gcd(2^b - 1, 2^r - 1)$$
$$= \cdots = 2^{\gcd(a,b)} - 1$$

by the Euclidean algorithm. Since $n \mid 2^n - 1$ and $n \mid 2^{\varphi(n)} - 1$ by Euler's theorem, so $n \mid 2^d - 1$, where $d = \gcd(n, \varphi(n)) \le \varphi(n) < n$. Then $n \mid 2^d - 1$ implies $d > 1$ and $d \mid 2^d - 1$, contradicting minimality of n.

Commended solvers: **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil) and **Fai YUNG.**

Olympiad Corner

(continued from page 1)

Problem 2. (*Cont.*) the circumradii of triangles BDE and CDF, respectively, and r_1 and r_2 be the inradii of the same triangles. Prove that

$$|S_{ABD} - S_{ACD}| \ge |R_1 r_1 - R_2 r_2|,$$

where S_K is the area of figure K.

Problem 3. Let n be a natural number, $n \ge 2$. Prove that if $(b^n - 1)/(b - 1)$ is a prime power for some positive integer b, then n is prime.

Second Day

Problem 4. In square $ABCD$, points E and F are chosen in the interior of sides BC and CD, respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and diagonal BD. A point K is chosen on FG such that $AK = EF$. Find $\angle EKF$.

Problem 5. Find all continuous functions f: $\mathbb{R} \to \mathbb{R}$ such that for all reals x and y, $f(x + f(y)) = y + f(x+1)$.

Problem 6. Consider a 10×10 grid. On every move, we color 4 unit squares that lie in the intersection of some two rows and two columns. A move is allowed if at least one of the 4 squares is previously uncolored. What is the *largest* possible number of moves that can be taken to color the whole grid?

Mathematical Excalibur

Volume 13, Number 1 *February-April, 2008*

Olympiad Corner

The 2008 APMO was held in March. Here are the problems.

Problem 1. Let ABC be a triangle with $\angle A < 60°$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA+AX = CB+BX$ and $BA+AY = BC+CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120°$.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

Problem 3. Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 20, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Point Set Combinatorics

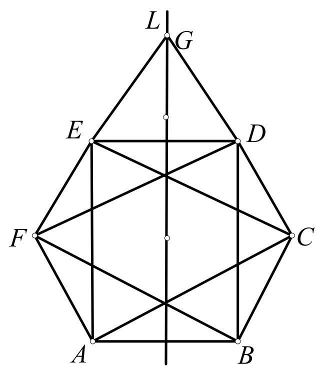
Kin Y. Li

Problems involving sets of points in the plane or in space often appear in math competitions. We will look at some typical examples. The solutions of these problems provide us the basic ideas to attack similar problems.

The following are some interesting examples.

Example 1. (*2001 USA Math Olympiad*) Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solution. Let A, B be arbitrary distinct points and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and EF intersect at G. Let L be the line through G perpendicular to line DE.



Observe that ΔCEG and ΔDFG are symmetric with respect to L and hence they have the same incenter. So $c+e+g = d+f+g$. Also, ΔACE and ΔBDF are symmetric with respect to L and have the same incenter. So $a+c+e=b+d+f$. Subtracting these two equations, we see $a=b$.

Comments: This outstanding elegant solution was due to Michael Hamburg, who was given a handsome cash prize as a Clay Math Institute award by the USAMO Committee.

Example 2. (*1987 IMO Shortlisted Problem*) In space, is there an infinite set M of points such that the intersection of M with every plane is nonempty and finite?

Solution. Yes, there is such a set M. For example, let

$$M = \{(t^5, t^3, t) : t \in \mathbb{R}\}.$$

Then, for every plane with equation $Ax + By + Cz + D = 0$, the intersection points are found by solving

$$At^5 + Bt^3 + Ct + D = 0,$$

which has at least one solution (since A or B or C is nonzero) and at most five solutions (since the degree is at most five).

Example 3. (*1963 Beijing Mathematics Competition*) There are $2n + 3$ ($n \geq 1$) given points on a plane such that no three of them are collinear and no four of them are concyclic.

Is it always possible to draw a circle through three of them so that half of the other $2n$ points are inside and half are outside the circle?

Solution. Yes, it is always possible.

Take the *convex hull* of these points, i.e. the smallest convex set containing them. The boundary is a polygon with vertices from the given points.

Let AB be a side of the polygon. Since no three are collinear, no other given points are on AB. By convexity, the other points $C_1, C_2, \cdots, C_{2n+1}$ are on the same side of line AB. Since no four are collinear, angles AC_iB are all distinct, say

$$\angle AC_1B < \angle AC_2B < \cdots < \angle AC_{2n+1}B.$$

Then $C_1, C_2, \cdots, C_n$ are inside the circle through A, B and C_{n+1} and $C_{n+2}, C_{n+3}, \cdots, C_{2n+1}$ are outside.

Example 4. (*1941 Moscow Math. Olympiad*) On a plane are given n points such that every three of them is inside some circle of radius 1. Prove that all these points are inside some circle of radius 1.

Solution. For every three of the n given points, consider the triangle they formed. If the triangle is an acute triangle, then draw their circumcircle, otherwise take the longest side and draw the circle having that side as the diameter. By the given condition, all these circles have radius less than 1.

Let S be one of these circles with minimum radius, say S arose from considering points A, B, C.

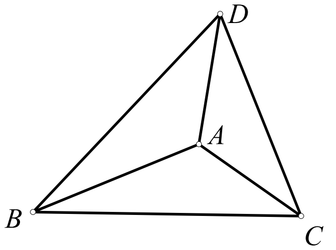
Assume one of the given points D is not inside S.

If ΔABC is acute, then D is on the same side as one of A,B,C with respect to the line through the other two points, say D and A are on the same side of line BC. Then the circle drawn for B,C,D would be their circumcircle and would have a radius greater than the radius of S, a contradiction.

If ΔABC is not acute and S is the circle with diameter AB, then the circle drawn for A, B, D would have AB as a chord and not as a diameter, which implies that circle has a radius greater than the radius of S, a contradiction.

Therefore, all n points are inside or on S. Since the radius of S was less than 1, we can take the circle of radius 1 at the same center as S to contain all n points.

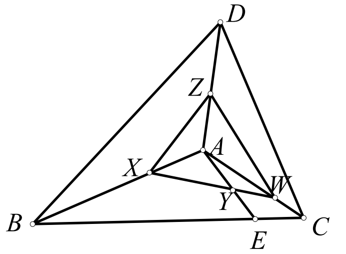
In the next example, we will consider a problem in space and the solution will involve a basic fact from solid geometry. Namely,



about vertex A of a tetrahedron $ABCD$, we have

$$\angle BAC \leq \angle BAD + \angle DAC \leq 360^\circ.$$

Nowadays, very little solid geometry is taught in school. So let's recall Euclid's proofs in Book XI, Problems 20 and 21 of his *Elements*.



For the left inequality, we may assume that $\angle BAC$ is the largest of the three angles about vertex A. Let E be on side BC so that $\angle BAD = \angle BAE$. Let X, Y, Z be on rays AB, AC, AD respectively, and $AX=AY=AZ$. Then $\Delta AXZ \cong \Delta AXY$ and we have $XZ=XY$. Let line XY intersect line AC at W. Since $XZ + ZW > XW$, cancelling $XZ = XY$ from both sides, we have $ZW > YW$. Comparing triangles WAZ and WAY, we have $WA=WA$, $AZ=AY$, so $ZW > YW$ implies $\angle ZAW > \angle YAW$. Then

$$\begin{aligned}\angle BAC &= \angle XAY + \angle YAW \\ &< \angle XAZ + \angle ZAW \\ &= \angle BAD + \angle DAC.\end{aligned}$$

For the right inequality, by the left inequality, we have

$$\begin{aligned}\angle DBC &\leq \angle DBA + \angle ABC, \\ \angle BCD &\leq \angle BCA + \angle ACD, \\ \angle CDB &\leq \angle CDA + \angle ADB.\end{aligned}$$

Adding them, we get 180˚ is less than or equal to the sum of the six angles on the right. Now the sum of these six angles and the three angles about A is 3×180˚. So the sum of the three angles about A is less than or equal to 360˚.

Example 5. (*1969 All Soviet Math. Olympiad*) There are n given points in space with no three collinear. For every three of them, they form a triangle having an angle greater than 120°. Prove that there is a way to order the points as $A_1, A_2, \ldots, A_n$ such that whenever $1 \leq i < j < k \leq n$, we have

$$\angle A_iA_jA_k > 120°.$$

Solution. Take two furthest points among these n points and call them A_1 and A_n.

For every two points X, Y among the other $n-2$ points, since A_1A_n is the longest side in both ΔA_1XA_n and ΔA_1YA_n, we have $\angle XA_1A_n < 60°$ and $\angle YA_1A_n < 60°$. About vertex A_1 of the tetrahedron A_1A_nXY, we have

$$\begin{aligned}\angle XA_1Y &\leq \angle XA_1A_n + \angle YA_1A_n \\ &< 60° + 60° = 120°.\end{aligned}$$

Similarly, $\angle XA_nY < 120°$.

Also, $A_1X \neq A_1Y$ (since otherwise, the two equal angles in ΔXA_1Y cannot be greater than 90˚ and so only $\angle XA_1Y$ can be greater than 120°, which will contradict the inequality above). Now order the points by its distance to A_1 so that $A_1A_2 < A_1A_3 < \cdots < A_1A_n$.

For $1 < j < k \leq n$, taking $X=A_j$ and $Y=A_k$ in the inequality above, we get $\angle A_jA_1A_k < 120°$. Since $A_1A_k > A_1A_j$, so in $\Delta A_1A_jA_k$, $\angle A_1A_jA_k > 120°$.

For $1 < i < j < k \leq n$, we have $\angle A_1A_iA_j > 120°$ and $\angle A_1A_iA_k > 120°$ by the last paragraph. Then, about vertex A_i of the tetrahedron $A_iA_jA_kA_1$, we have $\angle A_jA_iA_k < 120°$. Next since $A_1A_k > A_1A_j > A_1A_i$, about vertex A_k of the tetrahedron $A_kA_jA_iA_1$, we have

$$\begin{aligned}\angle A_iA_kA_j &\leq \angle A_iA_kA_1 + \angle A_jA_kA_1 \\ &< 60° + 60° = 120°.\end{aligned}$$

Hence, in $\Delta A_iA_jA_k$, we have $\angle A_iA_jA_k > 120°$.

Example 6. (*1994 All Russian Math. Olympiad*) There are k points, $2 \leq k \leq 50$, inside a convex 100-sided polygon. Prove that we can choose at most $2k$ vertices from this 100-sided polygon so that the k points are inside the polygon with the chosen points as vertices.

Solution. Let $M = A_1A_2 \cdots A_n$ be the boundary of the convex hull of the k points. Hence, $n \leq k$. Let O be a point inside M. From $i=1$ to n, let ray OA_i intersect the 100-sided polygon at B_i. Let M' be the boundary of the convex hull of $B_1, B_2, \cdots, B_n$.

For every point P on or inside M, the line OP intersects M at two sides, say A_iA_{i+1} and A_jA_{j+1}. By the definition of the points B_i's, we see the line OP intersects B_iB_{i+1} and B_jB_{j+1}, say at points S and T respectively. Since B_i, B_{i+1}, B_j and B_{j+1} are in M', so S, T are in M'. Then O and P are in M'. Thus M' contains M.

Let $M' = C_1C_2 \cdots C_m$. Then $m \leq n \leq k$. Observe that all C_i's are on the 100-sided polygon. Now each C_i is a vertex or between two consecutive vertices of the 100-sided polygon. Let G be the set of all these vertices. Then G has at most $2k$ points and the polygon with vertices from G contains the k points.

(Continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 20, 2008.***

Problem 296. Let $n > 1$ be an integer. From a $n\times n$ square, one 1×1 corner square is removed. Determine (with proof) the least positive integer k such that the remaining areas can be partitioned into k triangles with equal areas.
(*Source 1992 Shanghai Math Contest*)

Problem 297. Prove that for every pair of positive integers p and q, there exist an integer-coefficient polynomial $f(x)$ and an open interval with length $1/q$ on the real axis such that for every x in the interval, $|f(x) - p/q| < 1/q^2$.
(*Source:1983 Finnish Math Olympiad*)

Problem 298. The diagonals of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1M_2 \perp H_1H_2$.

Problem 299. Determine (with proof) the least positive integer n such that in every way of partitioning $S = \{1,2,\ldots,n\}$ into two subsets, one of the subsets will contain two distinct numbers a and b such that ab is divisible by $a+b$.

Problem 300. Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

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Solutions

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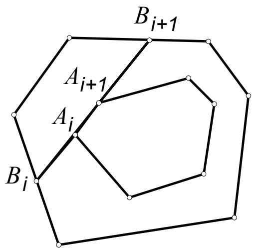
Problem 291. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

Solution. **Jeff CHEN** (Virginia, USA), **HO Kin Fai** (HKUST, Math Year 3) and **Fai YUNG**.



We will define a sequence of convex polygons $P_0, P_1, \ldots, P_{n-1}$. Let the outer convex polygon be P_0 and the inner convex polygon be $A_1A_2\ldots A_n$. For $i = 1$ to $n-1$, let the line A_iA_{i+1} intersect P_{i-1} at B_i, B_{i+1}. The line A_iA_{i+1} divides P_{i-1} into two parts with one part enclosing $A_1A_2\ldots A_n$. Let P_i be the polygon formed by putting the segment B_iB_{i+1} together with the part of P_{i-1} enclosing $A_1A_2\ldots A_n$. Note P_{n-1} is $A_1A_2\ldots A_n$. Finally, the perimeter of P_i is less than the perimeter of P_{i-1} because the length of B_iB_{i+1}, being the shortest distance between B_i and B_{i+1}, is less than the length of the part of P_{i-1} removed to form P_i.

Commended solvers: **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **Raúl A. SIMON** (Santiago, Chile) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 292. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers with no two of them are consecutive. For every $m = 1, 2, 3, \ldots$, let $S_m = k_1+k_2+\cdots+k_m$. Prove that for every positive integer n, the interval $[S_n, S_{n+1})$ contains at least one perfect square number.
(*Source: 1996 Shanghai Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **HO Kin Fai** (HKUST, Math Year 3), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Raúl A. SIMON** (Santiago, Chile).

There is a nonnegative integer a such that $a^2 < S_n \le (a+1)^2$. We have

$$\begin{aligned} S_n &= k_n + k_{n-1} + \cdots + k_1 \\ &< k_n + (k_n - 2) + \cdots + (k_n - 2n + 2) \\ &= n(k_n - n + 1). \end{aligned}$$

By the *AM-GM* inequality,

$$a < \sqrt{S_n} < \frac{n + (k_n - n + 1)}{2} = \frac{k_n + 1}{2}.$$

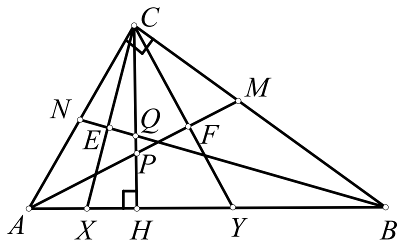
Then

$$\begin{aligned} (a+1)^2 &= a^2 + 2a + 1 < S_n + (k_n+1) + 1 \\ &\le S_n + k_{n+1} = S_{n+1}. \end{aligned}$$

Commended solvers: **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 293. Let CH be the altitude of triangle ABC with $\angle ACB = 90°$. The bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the midpoints of PM and QN is parallel to line AB.
(*Source: 52nd Belorussian Math. Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).



Let E, F be the midpoints of QN, PM respectively. Let X, Y be the intersection of CE, CF with AB respectively. Now

$$\begin{aligned} \angle CMP &= 90° - \angle CAM \\ &= 90° - \angle BAM \\ &= \angle APH = \angle CPM. \end{aligned}$$

So $CM=CP$. Then $CF\perp AF$. Since AF bisects $\angle CAY$, by ASA, $\Delta CAF \cong \Delta YAF$. So $CF=FY$. Similarly, $CE=EX$. By the midpoint theorem, we have EF parallel to line XY, which is the same as line AB.

Commended solvers: **Konstantine ZELATOR** (University of Toledo, Toledo, Ohio, USA).

Problem 294. For three nonnegative real numbers x, y, z satisfying the condition $xy+yz+zx = 3$, prove that

$$x^2 + y^2 + z^2 + 3xyz \ge 6.$$

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Ovidiu FURDUI** (Cimpia - Turzii, Cluj, Romania), **MA Ka Hei** (Wah Yan College, Kowloon) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Let $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz$. Now

$$p^2-9=x^2+y^2+z^2-xy-yz-zx$$
$$=\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2}\ge 0,$$

So $p \ge 3$. By Schur's inequality (see Math Excalibur, vol. 10, no. 5, p. 2, column 2), $12p=4pq\le p^3+9r$. Since

$$p^2=x^2+y^2+z^2+2(xy+yz+zx)$$
$$=x^2+y^2+z^2+6,$$

we get

$$3xyz=3r\ge 9r/p$$
$$\ge 12-p^2$$
$$=6-(x^2+y^2+z^2).$$

Problem 295. There are $2n$ distinct points in space, where $n\ge 2$. No four of them are on the same plane. If n^2+1 pairs of them are connected by line segments, then prove that there are at least n distinct triangles formed.
(*Source: 1989 Chinese IMO team training problem*)

Solution. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England).

We prove by induction on n. For n=2, say the points are A,B,C,D. For five segments connecting them, only one pair of them is not connected, say they are A and B. Then triangles ACD and BCD are formed.

Suppose the case n=k is true. Consider the case n=k+1. We first claim there is at least one triangle. Suppose AB is one such connected segment. Let α, β be the number of segments connecting A, B to the other $2n-2=2k$ points respectively.

If $\alpha+\beta>2k+1$, then A, B are both connected to one of the other $2k$ points, hence a triangle is formed.

If $\alpha+\beta\le 2k$, then the other $2k$ points have at least $(k+1)^2+1-(2k+1)=k^2+1$ segments connecting them. By the case n=k, there is a triangle in these $2k$ points.

So the claim is established. Now take one such triangle, say ABC. Let α, β, γ be the number of segments connecting A, B, C to the other $2k-1$ points respectively.

If $\alpha+\beta+\gamma\ge 3k-1$, then let D_1, D_2, …, D_m ($m\le 2k-1$) be all the points among the other $2k-1$ points connecting to at least one of A or B or C. The number of segments to D_i from A or B or C is n_i = 1 or 2 or 3. Checking each of these three cases, we see there are at least n_i-1 triangles having D_i as a vertex and the two other vertices from A, B, C. So there are

$$\sum_{i=1}^{m}(n_i-1)\ge 3k-1-m\ge k$$

triangles, each having one D_i vertex, plus triangle ABC, resulting in at least k+1 triangles.

If $\alpha+\beta+\gamma\le 3k-2$, then the sum of $\alpha+\beta$, $\gamma+\alpha$, $\beta+\gamma$ is at most $6k-4$. Hence the least of them cannot be $2k-1$ or more, say $\alpha+\beta\le 2k-2$. Then removing A and B and all segments connected to at least one of them, we have at least $(k+1)^2+1-(2k+1)=k^2+1$ segments left for the remaining $2k$ points. By the case n=k, we have k triangles. These plus triangle ABC result in at least k+1 triangles. The induction is complete.

Commended solvers: **Raúl A. SIMON** (Santiago, Chile) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Olympiad Corner

(continued from page 1)

Problem 4. Consider the function $f:\mathbb{N}_0\to\mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions:

(i) $f(0)=0$, (ii) $f(2n)=2f(n)$ and (iii) $f(2n+1)=n+2f(n)$ for all $n\ge 0$.

(a) Determine the three sets L:=$\{n\mid f(n)<f(n+1)\}$, E:=$\{n\mid f(n)=f(n+1)\}$, and G:=$\{n\mid f(n)>f(n+1)\}$.

(b) For each $k\ge 0$, find a formula for a_k:= $\max\{f(n)\mid 0\le n\le 2^k\}$ in terms of k.

Problem 5. Let a, b, c be integers satisfying $0<a<c-1$ and $1<b<c$. For each k, $0\le k\le a$, let r_k, $0\le r_k<c$, be the remainder of kb when dived by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, \ldots, a\}$ are different.

Point Set Combinatorics

(continued from page 2)

Example 7. (*1987 Chinese IMO Team Selection Test*) There are $2n$ distinct points in space, where $n\ge 2$. No four of them are on the same plane. If n^2+1 pairs of them are connected by line segments, then prove that there are two triangles sharing a common side.

Solution. We prove by induction on n. For n=2, say the points are A,B,C,D. For five segments connecting them, only one pair of them is not connected, say they are A and B. Then triangles ACD and BCD are formed and the side CD is common to them.

Suppose the case n=k is true. Consider the case n=k+1. Suppose AB is one such connected segment. Let α, β be the number of segments connecting A, B to the other $2n-2=2k$ points respectively.

Case 1. If $\alpha+\beta\ge 2k+2$, then there are points C, D among the other $2k$ points such that AC, BC, AD, BD are connected. Then triangles ABC and ABD are formed and the side AB is common to them.

Case 2. If $\alpha+\beta\le 2k$, then removing A, B and all segments connecting to at least one of them, there would still be at least $(k+1)^2+1-(2k+1)=k^2+1$ segments left for the remaining $2k$ points. By the case $n=k$, there would exist two triangles sharing a common side among them.

Case 3. Assume cases 1 and 2 do not occur for all the connected segments. Then take any connected segment AB and we have $\alpha+\beta=2k+1$. There would then be a point C among the other $2k$ points such that triangle ABC is formed.

Let γ be the number of segments connecting C to the other $2k-1$ points respectively. Since cases 1 and 2 do not occur, we have

$$\beta+\gamma=2k+1 \text{ and } \gamma+\alpha=2k+1,$$

too. However, this would lead to

$$(\alpha+\beta)+(\beta+\gamma)+(\gamma+\alpha)=6k+3,$$

which is contradictory as the left side is even and the right side is odd.

One cannot help to notice the similarity between the last example and problem 295 in the problem corner. Naturally this raise the question: when n is large, again if n^2+1 pairs of the points are connected by line segments, would we be able to get more pairs of triangles sharing common sides? Any information or contribution for this question from the readers will be appreciated.

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Olympiad Corner

The following are the four problems of the 2008 Balkan Mathematical Olympiad.

Problem 1. An acute-angled scalene triangle ABC is given, with $AC > BC$. Let O be its circumcenter, H its orthocenter and F the foot of the altitude from C. Let P be the point (other than A) on the line AB such that $AF=PF$ and M be the midpoint of AC. We denote the intersection of PH and BC by X, the intersection of OM and FX by Y and the intersection of OF and AC by Z. Prove that the points F, M, Y and Z are concyclic.

Problem 2. Does there exist a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of positive real numbers satisfying both of the following conditions:

(i) $\sum_{i=1}^{n} a_i \le n^2$, for every positive integer n;

(ii) $\sum_{i=1}^{n} \frac{1}{a_i} \le 2008$, for every positive integer n ?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 20, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometric Transformations I

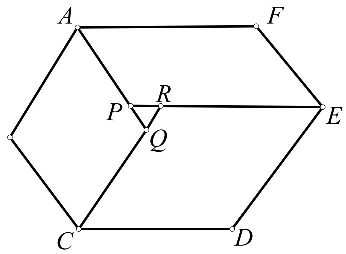
Kin Y. Li

Too often we <u>*stare*</u> at a figure in solving a geometry problem. In this article, we will <u>*move*</u> parts of the figure to better positions to facilitate the way to a solution.

Below we shall denote the vector from X to Y by the boldface italics $\boldsymbol{XY}$. On a plane, a <u>*translation*</u> by a vector v moves every point X to a point Y such that $\boldsymbol{XY} = v$. We denote this translation by $T(v)$.

<u>Example 1.</u> The opposite sides of a hexagon $ABCDEF$ are parallel. If $BC-EF = ED-AB = AF-CD > 0$, show that all angles of $ABCDEF$ are equal.

<u>Solution.</u> One idea is to move the side lengths closer to do the subtractions. Let $T(\boldsymbol{FA})$ move E to P, $T(\boldsymbol{BC})$ move A to Q and $T(\boldsymbol{DE})$ move C to R.



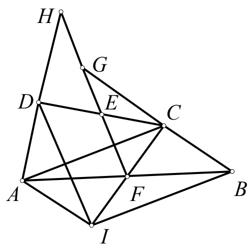
Hence, $EFAP$, $ABCQ$, $CDER$ are parallelograms. Since the opposite sides of the hexagon are parallel, P is on AQ, Q is on CR and R is on EP. Then, we get $BC - EF = AQ - AP = PQ$. Similarly, $ED - AB = QR$ and $AF - CD = RP$. Hence, ΔPQR is equilateral.

Now, $\angle ABC = \angle AQC = 120^\circ$. Also, $\angle BCD = \angle BCQ + \angle DCQ = 60^\circ + 60^\circ = 120^\circ$. Similarly, $\angle CDE = \angle DEF = \angle EFA = \angle FAB = 120^\circ$.

<u>Example 2.</u> $ABCD$ is a convex quadrilateral with $AD = BC$. Let E, F be midpoints of CD, AB respectively. Suppose rays AD, FE intersect at H and rays BC, FE intersect at G. Show that

$$\angle AHF = \angle BGF.$$

<u>Solution.</u> One idea is to move BC closer to AD. Let $T(\boldsymbol{CB})$ move A to I.



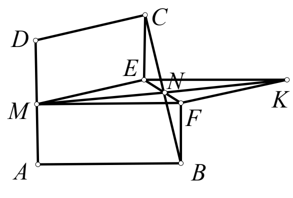
Then $BCAI$ is a parallelogram. Since F is the midpoint of AB, so F is also the midpoint of CI. Applying the midpoint theorem to ΔCDI, we get $EF||DI$. Using this and $CB||AI$, we get $\angle BGF = \angle AID$. From $AI = BC = AD$, we get $\angle AID = \angle ADI$. Since $EF \,||\, DI$, $\angle AHF = \angle ADI = \angle AID = \angle BGF$.

<u>Example 3.</u> Let M and N be the midpoints of sides AD and BC of quadrilateral $ABCD$ respectively. If

$$2MN = AB+CD,$$

then prove that $AB||CD$.

<u>Solution.</u> One idea is to move AB, CD closer to MN. Let $T(\boldsymbol{DC})$ move M to E and $T(\boldsymbol{AB})$ move M to F.



Then we can see $CDME$ and $BAMF$ are parallelograms. Since $\boldsymbol{EC} = \frac{1}{2}\boldsymbol{AD} = \boldsymbol{BF}$, $BFCE$ is a parallelogram. Since N is the midpoint of BC, so N is also the midpoint of EF.

Next, let $T(\boldsymbol{ME})$ move F to K. Then $EMFK$ is a parallelogram and

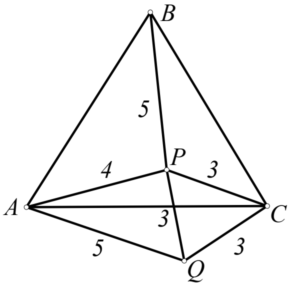
$$MK = 2MN = AB+CD = MF+EM = MF+FK.$$

So F, M, K, N are collinear and $AB||MN$. Similarly, $CD||MN$. Therefore, $AB||CD$.

On a plane, a <u>rotation</u> about a center O by angle α moves every point X to a point Y such that $OX = OY$ and $\angle XOY = \alpha$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$). We denote this rotation by $R(O,\alpha)$.

Example 4. Inside an equilateral triangle ABC, there is a point P such that PC=3, PA=4 and PB=5. Find the perimeter of ΔABC.

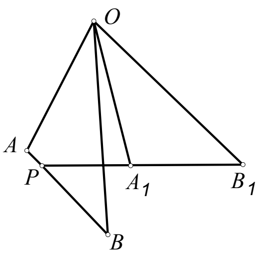
Solution. One idea is to move PC, PA, PB to form a triangle. Let $R(C,60°)$ move ΔCBP to ΔCAQ.



Now $CP=CQ$ and $\angle PCQ = 60°$ imply ΔPCQ is equilateral. As $AQ = BP = 5$, $AP = 4$ and $PQ = PC = 3$, so $\angle APQ = 90°$. Then $\angle APC = \angle APQ + \angle QPC = 90°+60° = 150°$. So the perimeter of ΔABC is

$$3AC = 3\sqrt{3^2+4^2-12\cos 150^\circ}$$
$$= 3\sqrt{25+12\sqrt{3}}.$$

For our next example, we will point out a property of rotation, namely



if $R(O,\alpha)$ moves a line AB to the line A_1B_1 and P is the intersection of the two lines, then these lines intersect at an angle α.

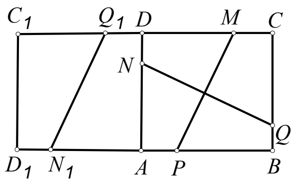
This is because $\angle OAB = \angle OA_1B_1$ implies O,A,P,A_1 are concyclic so that $\angle BPB_1 = \angle AOA_1 = \alpha$.

Example 5. $ABCD$ is a unit square. Points P,Q,M,N are on sides AB, BC, CD, DA respectively such that

$$AP + AN + CQ + CM = 2.$$

Prove that $PM \perp QN$.

Solution. One idea is to move AP, AN together and CQ, CM together. Let $R(A,90°)$ map $B\to D$, $C\to C_1$, $D\to D_1$, $Q\to Q_1$, $N\to N_1$ as shown below.



Then $AN=AN_1$ and $CQ=C_1Q_1$. So

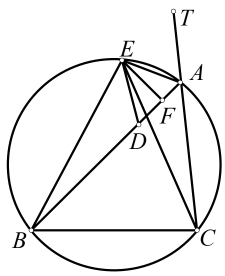
$$PN_1 = AP+AN_1 = AP+AN = 2-(CM+CQ)$$
$$= CC_1-(CM+C_1Q_1) = MQ_1.$$

Hence, PMQ_1N_1 is a parallelogram and $MP \| Q_1N_1$. By the property before the example, lines QN and Q_1N_1 intersect at 90°. Therefore, $PM \perp QN$.

Example 6. (*1989 Chinese National Senoir High Math Competition*) In ΔABC, $AB > AC$. An external bisector of $\angle BAC$ intersects the circumcircle of ΔABC at E. Let F be the foot of perpendicular from E to line AB. Prove that

$$2AF = AB-AC.$$

Solution. One idea is to move AC to coincide with a part of AB. To do that, consider $R(E,\angle CEB)$.



Observe that $\angle EBC = \angle EAT = \angle EAB = \angle ECB$ implies $EC=EB$. So $R(E,\angle CEB)$ move C to B. Let $R(E,\angle CEB)$ move A to D. Since $\angle CAB = \angle CEB$, by the property above and $AB > AC$, D is on segment AB.

So $R(E,\angle CEB)$ moves ΔAEC to ΔDEB. Then $\angle DAE = \angle EAT = \angle EDA$ implies ΔAED is isosceles. Since $EF \perp AD$,

$$2AF = AD = AB - BD = AB - AC.$$

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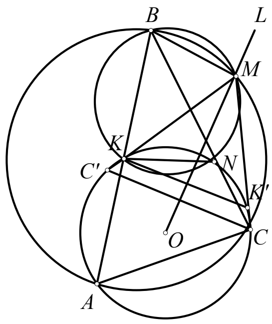
On a plane, a <u>reflection</u> across a line moves every point X to a point Y such that the line is the perpendicular bisector of segment XY. We say Y is the <u>mirror image</u> of X with respect to the line.

Example 7. (*1985 IMO*) A circle with center O passes through vertices A and C of ΔABC and cuts sides AB, BC at K, N respectively. The circumcircles of ΔABC and ΔKBN intersect at B and M. Prove that $\angle OMB = 90°$.

Solution. Let L be the line through O perpendicular to line BM. We are done if we can show M is on L.



Let the reflection across L maps $C \to C'$ and $K \to K'$. Then $CC' \perp L$ and $KK' \perp L$, which imply lines CC', KK', BM are parallel. We have

$$\angle KC'C = \angle KAC = \angle BNK = \angle BMK,$$

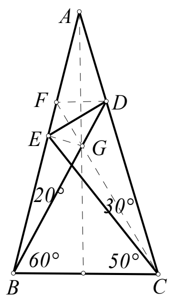
which implies C',K,M collinear. Now

$$\angle C'CK' = \angle CC'K = \angle CAK$$
$$= \angle CAB = 180° - \angle BMC$$
$$= \angle C'CM,$$

which implies C,K',M collinear. Then lines $C'K$ and CK' intersect at M. Since lines $C'K$ and CK' are symmetric with respect to L, so M is on L.

Example 8. Points D and E are on sides AB and AC of ΔABC respectively with $\angle ABD = 20°$, $\angle DBC = 60°$, $\angle ACE = 30°$ and $\angle ECB = 50°$. Find $\angle EDB$.

Solution. Note $\angle ABC = \angle ACB$. Consider the reflection across the perpendicular bisector of side BC. Let the mirror image of D be F. Let BD intersect CF at G. Since $BG = CG$, lines BD, CF intersect at 60° so that ΔBGC and ΔDGF are equilateral. Then $DF=DG$.



We claim $EF = EG$ (which implies $\Delta EFD \cong \Delta EGD$. So $\angle EDB = ½\angle FDG = 30°$). For the claim, we have $\angle EFG = \angle CDG = 40°$ and $\angle FGB = 120°$.

Next $\angle BEC = 50°$. So $BE=BC$. As ΔBGC is equilateral, so $BE = BC = BG$ .This gives $\angle EGB = 80°$. Then

$$\angle EGF = \angle FGB - \angle EGB$$
$$= 40° = \angle EFG,$$

which implies the claim.

(Continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 20, 2008.***

Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Problem 302. Let $\mathbb{Z}$ denotes the set of all integers. Determine (with proof) all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all x, y in $\mathbb{Z}$, we have $f(x+f(y)) = f(x) - y$.

Problem 303. In base 10, let N be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of N, forming numbers that are different (integral) powers of two.

Problem 304. Let M be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the x-y coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in M and whose sides are parallel to the x-axis or the y-axis.

Problem 305. A circle Γ_2 is internally tangent to the circumcircle Γ_1 of ΔPAB at P and side AB at C. Let E, F be the intersection of Γ_2 with sides PA, PB respectively. Let EF intersect PC at D. Lines PD, AD intersect Γ_1 again at G, H respectively. Prove that F, G, H are collinear.

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Solutions

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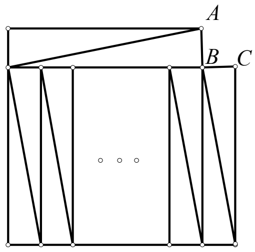
Problem 296. Let $n > 1$ be an integer. From a $n\times n$ square, one 1×1 corner square is removed. Determine (with proof) the least positive integer k such that the remaining areas can be partitioned into k triangles with equal areas.

(*Source 1992 Shanghai Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **O Kin Chit Alex** (GT Ellen Yeung College), **PUN Ying Anna** (HKU Math Year 2), **Simon YAU Chi-Keung** (City University of Hong Kong) and **Fai YUNG**.



The figure above shows the least k is at most $2n$+2. Conversely, suppose the required partition is possible for some k. Then one of the triangles must have a side lying in part of segment AB or in part of segment BC. Then the length of that side is at most 1. Next, the altitude perpendicular to that side is at most $n - 1$. Hence, that triangle has an area at most $(n-1)/2$. That is $(n^2 - 1)/k \le (n-1)/2$. So $k \ge 2n + 2$. Therefore, the least k is $2n$+2.

Problem 297. Prove that for every pair of positive integers p and q, there exist an integer-coefficient polynomial $f(x)$ and an open interval with length $1/q$ on the real axis such that for every x in the interval, $|f(x) - p/q| < 1/q^2$.

(*Source:1983 Finnish Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **PUN Ying Anna** (HKU Math Year 2).

If q = 1, then take $f(x) = p$ works for any interval of length $1/q$. If $q > 1$, then define the interval $I = \left(\frac{1}{2q}, \frac{3}{2q}\right)$.

Choosing a positive integer m greater than $(\log q)/(\log 2q/3)$, we get $[3/(2q)]^m < 1/q$. Let $a = 1-[1/(2q)]^m$. Then for all x in I, we have $0 < 1 - qx^m < a < 1$.

Choosing a positive integer n greater than $-(\log pq)/(\log a)$, we get $a^n < 1/(pq)$. Let

$$f(x) = \frac{p}{q}[1-(1-qx^m)^n].$$

Now

$$f(x) = \frac{p}{q}[1-(1-qx^m)]\sum_{k=0}^{n-1}(1-qx^m)^k$$

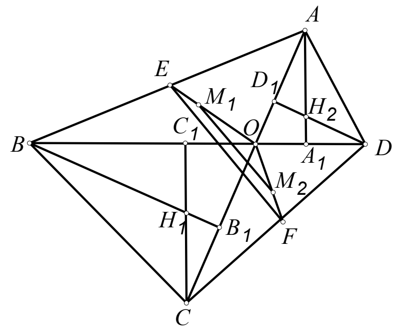
$$= px^m \sum_{k=0}^{n-1}(1-qx^m)^k$$

has integer coefficients. For x in I, we have

$$\left|f(x) - \frac{p}{q}\right| = \frac{p}{q}\left|(1-qx^m)^n\right| < \frac{p}{q}a^n < \frac{1}{q^2}.$$

Problem 298. The diagonals of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1M_2 \perp H_1H_2$.

Solution. **Jeff CHEN** (Virginia, USA).



Let A_1, C_1 be the feet of the perpendiculars from A, C to line BD respectively. Let B_1, D_1 be the feet of the perpendiculars from B, D to line AC respectively. Let E, F be the midpoints of sides AB, CD respectively. Since

$$OM_1/OE = 2/3 = OM_2/OF,$$

we get $EF \parallel M_1M_2$. Thus, it suffices to show $H_1H_2 \perp EF$.

Now the angles AA_1B and BB_1A are right angles. So A, A_1, B, B_1 lie on a circle Γ_1 with E as center. Similarly, C, C_1, D, D_1 lie on a circle Γ_2 with F as center.

Next, since the angles AA_1D and DD_1A are right angles, points A,D,A_1,D_1 are concyclic. By the intersecting chord theorem, $AH_2\cdot H_2A_1 = DH_2\cdot H_2D_1$.

This implies H_2 has equal power with respect to Γ_1 and Γ_2. Similarly, H_1 has equal power with respect to Γ_1 and Γ_2. Hence, line H_1H_2 is the radical axis of Γ_1 and Γ_2. Since the radical axis is perpendicular to the line joining the centers of the circles, we get $H_1H_2 \perp EF$.

Comments: For those who are not familiar with the concepts of power and radical axis of circles, please see *Math. Excalibur*, *vol.* 4, *no.* 3, *pp.* 2,4.

Commended solvers: **PUN Ying Anna** (HKU Math Year 2) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 299. Determine (with proof) the least positive integer n such that in every way of partitioning $S = \{1,2,\ldots,n\}$ into two subsets, one of the subsets will contain two distinct numbers a and b such that ab is divisible by $a+b$.

Solution. **Jeff CHEN** (Virginia, USA),

PUN Ying Anna (HKU Math Year 2).

Call a pair (a,b) of distinct positive integers a <u>*good*</u> pair if and only if ab is divisible by $a+b$. Here is a list of good pairs with $1 < a < b < 50$:

(3,6), (4,12), (5,20), (6,12), (6,30), (7,42), (8,24), (9,18), (10,15), (10,40), (12,24), (12,36), (14,35), (15,30), (16,48), (18,36), (20,30), (21,28), (21,42), (24,40), (24,48), (30,45), (36,45).

Now we try to put the positive integers from 1 to 39 into one of two sets S_1, S_2 so that no good pair is in the same set. If a positive integer is not in any good pair, then it does not matter which set it is in, say we put it in S_1. Then we get

S_1={1, 2, 3, 5, 8, 10, 12, 13, 14, 18, 19, 21, 22, 23, 30, 31, 32, 33, 34, 36} and S_2={4, 6, 7, 9, 11,15, 17, 20, 24, 25, 26, 27, 28, 29, 35, 37, 38, 39}.

So 1 to 39 do not have the property.

Next, for $n = 40$, we observe that any two consecutive terms of the sequence 6, 30, 15, 10, 40, 24, 12, 6 forms a good pair. So no matter how we divide the numbers 6, 30, 15, 10, 40, 24, 12 into two sets, there will be a good pair in one of them. So, $n = 40$ is the least case.

Problem 300. Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

Solution. **Jeff CHEN** (Virginia, USA) and **G.R.A. 20 Problem Solving Group** (Roma, Italy), **PUN Ying Anna** (HKU Math Year 2).

We first show by induction that <u>for every positive integer k, there is a k-digit number n_k whose digits are all odd and n_k is a multiple of 5^k</u>. We can take n_1=5. Suppose this is true for k. We will consider the case $k+1$. If n_k is a multiple of 5^{k+1}, then take n_{k+1} to be $n_k + 5\times 10^k$. Otherwise, n_k is of the form $5^k(5i+j)$, where i is a nonnegative integer and j = 1, 2, 3 or 4. Since $\gcd(5,2^k) = 1$, one of the numbers 10^k+n_k, $3\times10^k+n_k$, $7\times10^k+n_k$, $9\times10^k+n_k$ is a multiple of 5^{k+1}. Hence we may take it to be n_{k+1}, which completes the induction.

Now for the problem, let m be an odd number. Let $N(a,b)$ denote the number whose digits are those of a written b times in a row. For example, $N(27,3)$ = 272727.

Observe that m is of the form 5^kM, where k is a nonnegative integer and $\gcd(M,5) = 1$. Let $n_0 = 1$ and for $k > 0$, let n_k be as in the underlined statement above. Consider the numbers $N(n_k,1)$, $N(n_k,2)$, …, $N(n_k, M+1)$. By the pigeonhole principle, two of these numbers, say $N(n_k, i)$ and $N(n_k, j)$ with $1 \le i < j \le M+1$, have the same remainder when dividing by M. Then $N(n_k, j) - N(n_k, i) = N(n_k, j-i)\times 10^{ik}$ is a multiple of M and 5^k.

Finally, since $\gcd(M, 10) = 1$, $N(n_k, j-i)$ is also a multiple of M and 5^k. Therefore, it is a multiple of m and it has only odd digits.

Olympiad Corner

(continued from page 1)

Problem 3. Let n be a positive integer. The rectangle $ABCD$ with side lengths AB=90n+1 and BC=90n+5 is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let S be the set of all points which are vertices of these unit squares. Prove that the number of lines which pass through at least two points from S is divisible by 4.

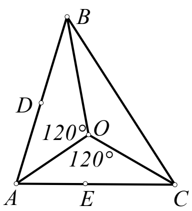
Problem 4. Let c be a positive integer. The sequence a_1, a_2, …, a_n, … is defined by $a_1=c$ and $a_{n+1}=a_n^2+a_n+c$ for every positive integer n. Find all values of c for which there exist some integers $k \ge 1$ and $m \ge 2$ such that $a_k^2+c^3$ is the m^{th} power of some positive integer.

Geometric Transformations I

(continued from page 2)

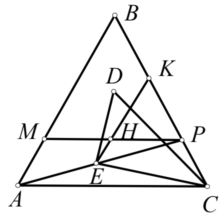
On a plane, a <u>*spiral similarity*</u> with center O, angle α and ratio k moves every point X to a point Y such that $\angle XOY = \alpha$ and $OY/OX = k$, i.e. it is a rotation with a homothety. We denote it by $S(O,\alpha, k)$.

<u>***Example 9.***</u> (*1996 St. Petersburg Math Olympiad*) In ΔABC, $\angle BAC$=60°. A point O is inside the triangle such that $\angle AOB = \angle BOC = \angle COA$. Points D and E are the midpoints of sides AB and AC, respectively. Prove that A, D, O, E are concyclic.



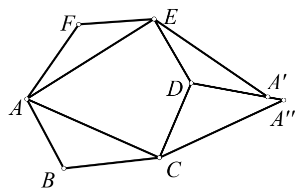
<u>***Solution.***</u> Since $\angle AOB = \angle COA$=120° and $\angle OBA$=60°$-\angle OAB = \angle OAC$, we see $\Delta AOB \sim \Delta COA$. Then the spiral similarity $S(O,120°,OC/OA)$ maps $\Delta AOB \to \Delta COA$ and also $D \to E$. Then $\angle DOE$ = 120° = 180°$-\angle BAC$, which implies A, D, O, E concyclic.

<u>***Example 10.***</u> (*1980 All Soviet Math Olympiad*) ΔABC is equilateral. M is on side AB and P is on side CB such that $MP||AC$. D is the centroid of ΔMBP and E is the midpoint of PA. Find the angles of ΔDEC.



<u>***Solution.***</u> Let H and K be the midpoints of PM and PB respectively. Observe that $S(D,-60°,1/2)$ maps $P \to H$, $B \to K$ and so $PB \to HK$. Now H, K, E are collinear as they are midpoints of PM, PB, PA. Note $BC/BP = BA/BM = KE/KH$, which implies $S(D,-60°,1/2)$ maps $C \to E$. Then $\angle EDC$ = 60° and DE=½DC. So we have $\angle DEC$ = 90° and $\angle DCE$ = 30°.

<u>***Example 11.***</u> (*1998 IMO Proposal by Poland*) Let $ABCDEF$ be a convex hexagon such that $\angle B+\angle D+\angle F$ = 360° and $(AB/BC)(CD/DE)(EF/FA)$=1. Prove $(BC/CA)(AE/EF)(FD/DB)$=1.



<u>***Solution.***</u> Since $\angle B+\angle D+\angle F$ =360°, $S(E, \angle FED, ED/EF)$ maps $\Delta FEA \to \Delta DEA'$ and $S(C, \angle BCD, CD/CB)$ maps $\Delta BCA \to \Delta DCA''$. So $\Delta FEA \sim \Delta DEA'$ and $\Delta BCA \sim \Delta DCA''$. These yield $BC/CA = DC/CA''$, $DE/EF = DA'/FA$ and using the given equation, we get

$$\frac{A''D}{DC} = \frac{AB}{BC} = \frac{DE}{CD}\frac{FA}{EF} = \frac{DA'}{CD},$$

which implies $A' = A''$. Next $\angle AEF = \angle A'ED$ implies $\angle DEF = \angle A'EA$. As $DE/FE = A'E/AE$, so $\Delta DEF \sim \Delta A'EA$ and $AE/FE = AA'/FD$. Similarly, we get $\Delta DCB \sim \Delta A'CA$ and $DC/A'C = DB/A'A$. Therefore,

$$\frac{BC}{CA}\frac{AE}{EF}\frac{FD}{DB} = \frac{DC}{CA''}\frac{AA'}{DB} = 1.$$

Mathematical Excalibur

Volume 13, Number 3 *July-October, 2008*

Olympiad Corner

The following are the problems of the 2008 IMO held at Madrid in July.

Problem 1. An acute-angled triangle ABC has orthocenter H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with the centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Problem 2. (a) Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\ge 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor which is greater than $2n+\sqrt{2n}$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 31, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometric Transformations II

Kin Y. Li

Below the vector from X to Y will be denoted as $\boldsymbol{XY}$. The notation $\measuredangle ABC = \alpha$ means the ray BA after rotated an angle $|\alpha|$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$) will coincide with the ray BC.

On a plane, a *translation* by a vector v (denoted as T(v)) moves every point X to a point Y such that $\boldsymbol{XY}$ = v. On the complex plane $\mathbb{C}$, if the vector v corresponds to the vector from 0 to v, then T(v) has the same effect as the function $f:\mathbb{C}\to\mathbb{C}$ given by $f(w)= w+v$.

A *homothety* about a center C and ratio r (denoted as $H(C,r)$) moves every point X to a point Y such that $\boldsymbol{CY} = r\,\boldsymbol{CX}$. If C corresponds to the complex number c in $\mathbb{C}$, then $H(C,r)$ has the same effect as $f(w) = r(w-c)+c = rw + (1-r)c$.

A *rotation* about a center C by angle α (denoted as $R(C,\alpha)$) moves every point X to a point Y such that $CX = CY$ and $\measuredangle XCY = \alpha$. In $\mathbb{C}$, if C corresponds to the complex number c, then $R(C,\alpha)$ has the same effect as $f(w) = e^{i\alpha}(w-c)+c = e^{i\alpha}w + (1-e^{i\alpha})c$.

A *reflection* across a line ℓ (denoted as $S(\ell)$) moves every point X to a point Y such that the line ℓ is the perpendicular bisector of segment XY. In $\mathbb{C}$, let $S(\ell)$ send 0 to b. If $b = 0$ and ℓ is the line through 0 and $e^{i\theta/2}$, then $S(\ell)$ has the same effect as $f(w)=e^{i\theta}\overline{w}$. If $b \ne 0$, then let $b = |b|\,e^{i\beta}$, $e^{i\theta} = -e^{2i\beta}$ and L be the vertical line through $|b|/2$. In $\mathbb{C}$, $S(L)$ sends w to $|b|-\overline{w}$. Using that, $S(\ell)$ is

$$f(w)=e^{i\beta}(|b|-\overline{we^{-i\beta}})=e^{i\theta}\overline{w}+b.$$

We have the following useful facts:

Fact 1. If $\ell_1 \parallel \ell_2$, then

$$S(\ell_2)\circ S(\ell_1) = T(2\boldsymbol{A_1A_2}),$$

where A_1 is on ℓ_1 and A_2 is on ℓ_2 such that the length of $\boldsymbol{A_1A_2}$ is the distance d from ℓ_1 to ℓ_2.

(*Reason*: Say ℓ_1, ℓ_2 are vertical lines through $A_1 = 0$, $A_2 = d$. Then $S(\ell_1)$, $S(\ell_2)$ are $f_1(w)=-\overline{w}$ and $f_2(w)=-\overline{w}+2d$. So $S(\ell_2)\circ S(\ell_1)$ is

$$f_2(f_1(w)) = -\overline{\left(-\overline{w}\right)}+2d = w+2d,$$

which is $T(2\boldsymbol{A_1A_2})$.)

Fact 2. If $\ell_1 \nparallel \ell_2$, then

$$S(\ell_2)\circ S(\ell_1) = R(O,\alpha),$$

where ℓ_1 intersects ℓ_2 at O and α is twice the angle from ℓ_1 to ℓ_2 in the anticlockwise direction.

(*Reason:* Say O is the origin, ℓ_1 is the x-axis. Then $S(\ell_1)$ and $S(\ell_2)$ are

$$f_1(w)=\overline{w} \text{ and } f_2(w)=e^{i\alpha}\overline{w},$$

so $S(\ell_2)\circ S(\ell_1)$ is $f_2(f_1(w)) = e^{i\alpha}w$, which is $R(O,\alpha)$.)

Fact 3. If $\alpha+\beta$ is not a multiple of 360°, then

$$R(O_2,\beta)\circ R(O_1,\alpha) = R(O,\alpha+\beta),$$

where $\measuredangle OO_1O_2 = \alpha/2$, $\measuredangle O_1O_2O = \beta/2$. If $\alpha+\beta$ is a multiple of 360°, then

$$R(O_2,\beta)\circ R(O_1,\alpha) = T(\boldsymbol{O_1O_3}),$$

where $R(O_2,\beta)$ sends O_1 to O_3.

(*Reason*: Say O_1 is 0, O_2 is -1. Then $R(O_1,\alpha)$, $R(O_2,\beta)$ are $f_1(w) = e^{i\alpha}w$, $f_2(w) = e^{i\beta}w+(e^{i\beta}-1)$, so $f_2(f_1(w)) = e^{i(\alpha+\beta)}w + (e^{i\beta}-1)$. If $e^{i(\alpha+\beta)} \ne 1$, this is a rotation about $c' = (e^{i\beta}-1)/(1-e^{i(\alpha+\beta)})$ by angle $\alpha+\beta$. We have

$$c'=\frac{\sin(\beta/2)}{\sin((\alpha+\beta)/2)}e^{i(\pi-\alpha/2)},$$

$$c'-1=\frac{\sin(\alpha/2)}{\sin((\alpha+\beta)/2)}e^{i\beta/2}.$$

If $e^{i(\alpha+\beta)} = 1$, this is a translation by $e^{i\beta}-1 = f_2(0)$.)

Fact 4. If O_1, O_2, O_3 are noncollinear, $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1+\alpha_2+\alpha_3=360°$ and

$$R(O_3,\alpha_3)\circ R(O_2,\alpha_2)\circ R(O_1,\alpha_1) = I,$$

where I is the identity transformation, then $\measuredangle O_3O_1O_2 = \alpha_1/2$, $\measuredangle O_1O_2O_3 = \alpha_2/2$ and $\measuredangle O_2O_3O_1 = \alpha_3/2$.

(This is just the case $\alpha_3=360°-(\alpha_1+\alpha_2)$ of fact 3.)

Fact 5. Let $O_1 \neq O_2$. For $r_1r_2 \neq 1$,

$$H(O_2,r_2) \circ H(O_1,r_1) = H(O,r_1r_2)$$

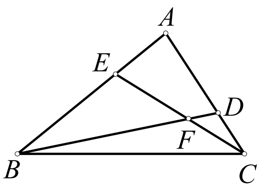
for some O on line O_1O_2. For $r_1r_2 = 1$,

$$H(O_2,r_2) \circ H(O_1,r_1) = T((1-r_2)\boldsymbol{O_1O_2}).$$

(*Reason*: Say O_1 is 0, O_2 is c. Then $H(O_1,r_1)$, $H(O_2,r_2)$ are $f_1(w) = r_1w$, $f_2(w) = r_2w+(1-r_2)c$, so $f_2(f_1(w)) = r_1r_2w+(1-r_2)c$. For $r_1r_2 \neq 1$, this is a homothety about $c' = (1-r_2)c/(1-r_1r_2)$ and ratio r_1r_2. For $r_1r_2 = 1$, this is a translation by $(1-r_2)c$.)

Next we will present some examples.

Example 1. In ΔABC, let E be onside AB such that AE:EB=1:2 and D be on side AC such that AD:DC = 2:1. Let F be the intersection of BD and CE. Determine FD:FB and FE:FC.



Solution. We have $H(E, -1/2)$ sends B to A and $H(C, 1/3)$ sends A to D. Since $(1/3)\times(-1/2) \neq 1$, by fact 5,

$$H(C, 1/3) \circ H(E, -1/2) = H(O, -1/6),$$

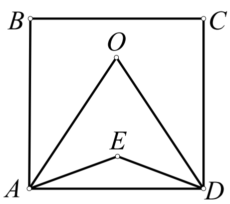
where the center O is on line CE. However, the composition on the left side sends B to D. So O is also on line BD. Hence, O must be F. Then we have FD: FB = OD: OB =1:6.

Similarly, we have

$$H(B, 2/3) \circ H(D, -2) = H(F, -4/3)$$

sends C to E, so FE:FC = 4:3.

Example 2. Let E be inside square $ABCD$ such that $\angle EAD = \angle EDA$ =15°. Show that ΔEBC is equilateral.

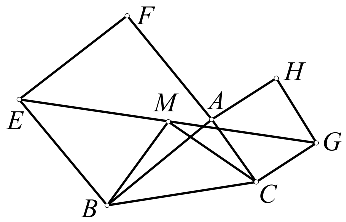


Solution. Let O be inside the square such that ΔADO is equilateral. Then $R(D, 30°)$ sends C to O and $R(A, 30°)$ sends O to B. Since $\measuredangle EDA$ = 15° $= \measuredangle DAE$, by fact 3,

$$R(A, 30°) \circ R(D, 30°) = R(E, 60°),$$

So $R(E, 60°)$ sends C to B. Therefore, ΔEBC is equilateral.

Example 3. Let $ABEF$ and $ACGH$ be squares outside ΔABC. Let M be the midpoint of EG. Show that $MB = MC$ and $MB \perp MC$.

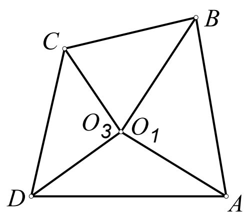


Solution. Since $GC = AC$ and $\measuredangle GCA$ = 90°, so $R(C,90°)$ sends G to A. Also, $R(B, 90°)$ sends A to E. Then $R(B, 90°) \circ R(C,90°)$ sends G to E. By fact 3,

$$R(B, 90°) \circ R(C, 90°) = R(O, 180°),$$

where O satisfies $\measuredangle OCB$ = 45° and $\measuredangle CBO$ = 45°. Since the composition on the left side sends G to E, O must be M. Now $\measuredangle BOC$ = 90°. So $MB \perp MC$.

Example 4. On the edges of a convex quadrilateral $ABCD$, construct the isosceles right triangles ABO_1, BCO_2, CDO_3, DAO_4 with right angles at O_1, O_2, O_3, O_4 overlapping with the interior of the quadrilateral. Prove that if $O_1 = O_3$, then $O_2 = O_4$.



Solution. Now $R(O_1, 90°)$ sends A to B, $R(O_2, 90°)$ sends B to C, $R(O_3, 90°)$ sends C to D and $R(O_4, 90°)$ sends D to A. By fact 3,

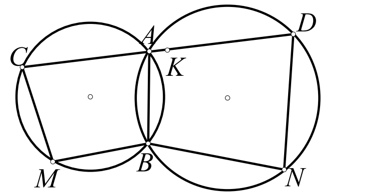
$$R(O_2, 90°) \circ R(O_1, 90°) = R(O, 180°),$$

where O satisfies $\angle OO_1O_2$ = 45° and $\angle O_1O_2O$ = 45° (so $\angle O_2OO_1$ = 90°). Now the composition on the left side sends A to C, which implies O must be the midpoint of AC. Similarly, we have

$$R(O_4, 90°) \circ R(O_3, 90°) = R(O, 180°).$$

By fact 3, $\measuredangle O_4OO_3$ = 90° and $\measuredangle OO_3O_4$ = 45°=$\measuredangle O_3O_4O$. Hence, $R(O, 90°)$ sends O_4O_2 to O_3O_1. Therefore, if $O_1 = O_3$, then $O_2 = O_4$.

Example 4. (*1999-2000 Iranian Math Olympiad, Round 2*) Two circles intersect in points A and B. A line ℓ that contains the point A intersects again the circles in the points C and D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN$ = 90°.



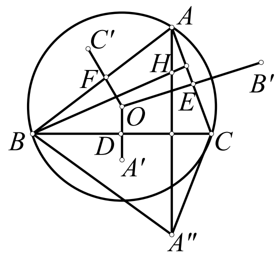
Solution. Since $\angle CAB + \angle BAD$ = 180°, it follows that $\angle BMC + \angle DNB$ =180°.

Now $R(M, \measuredangle BMC)$ sends B to C, $R(K, 180°)$ sends C to D and $R(N, \measuredangle DNB)$ sends D to B. However, by fact 3,

$$R(N,\measuredangle DNB) \circ R(K,180°) \circ R(M,\measuredangle BMC)$$

is a translation and since it sends B to B, it must be the identity transformation I. By fact 4, $\measuredangle MKN$ = 90°.

Example 6. Let H be the orthocenter of ΔABC and lie inside it. Let A', B', C' be the circumcenters of ΔBHC, ΔCHA, ΔAHB respectively. Show that AA', BB', CC' are concurrent and identify the point of concurrency.



Solution. For ΔABC, let O be its circumcenter and G be its centroid. Let the reflection across line BC sends A to A''. Then $\angle BAC = \angle BA''C$. Now

$$\begin{aligned} &\angle BHC \\ &= \angle ABH + \angle BAC + \angle ACH \\ &= (90° - \angle BAC) + \angle BAC + (90° - \angle BAC) \\ &= 180° - \angle BA''C. \end{aligned}$$

So A'' is on the circumcircle of ΔHBC.

Now the reflection across line BC sends O to A', the reflection across line CA sends O to B' and the reflection across line AB sends O to C'. Let D, E, F be the midpoints of sides BC, CA, AB respectively. Then $H(G, -1/2)$ sends ΔABC to ΔDEF and $H(O, 2)$ sends ΔDEF to $\Delta A'B'C'$. Since $(-1/2)\times 2 \neq 1$, by fact 5,

$$H(O, 2) \circ H(G, -1/2) = H(X, -1)$$

for some point X. Since the composition on the left side sends ΔABC to $\Delta A'B'C'$, segments AA', BB', CC' concur at X and in fact X is their common midpoint.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 31, 2008.***

Problem 306. Prove that for every integer $n \geq 48$, every cube can be decomposed into n smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Problem 307. Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

be a polynomial with real coefficients such that $a_0 \neq 0$ and for all real x,

$$f(x)\, f(2x^2) = f(2x^3 + x).$$

Prove that $f(x)$ has no real root.

Problem 308. Determine (with proof) the greatest positive integer $n > 1$ such that the system of equations

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2.$$

has an integral solution $(x, y_1, y_2, \cdots, y_n)$.

Problem 309. In acute triangle ABC, $AB > AC$. Let H be the foot of the perpendicular from A to BC and M be the midpoint of AH. Let D be the point where the incircle of ΔABC is tangent to side BC. Let line DM intersect the incircle again at N. Prove that $\angle BND = \angle CND$.

Problem 310. (*Due to Pham Van Thuan*) Prove that if p, q are positive real numbers such that $p + q = 2$, then

$$3p^q q^p + p^p q^q \leq 4.$$

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Solutions

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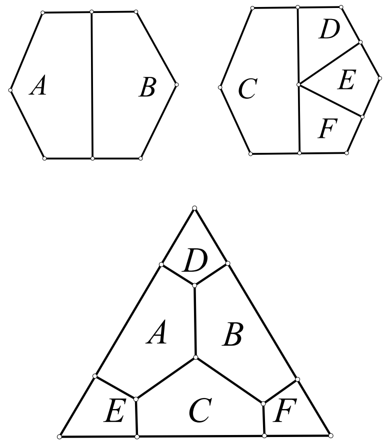
Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy).



Commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil), **Glenier L. BELLO- BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **KONG Catherine Wing Yan** (G.T. Ellen Yeung College, Grade 9), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **PUN Ying Anna** (HKU Math Year 3).

Problem 302. Let $\mathbb{Z}$ denotes the set of all integers. Determine (with proof) all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that for all x, y in $\mathbb{Z}$, we have $f(x+f(y)) = f(x) - y$.
(*Source: 2004 Spanish Math Olympiad*)

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **Ozgur KIRCAK** (Jahja Kemal College, Teacher, Skopje, Macedonia), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education), **PUN Ying Anna** (HKU Math Year 3), **Salem MALIKIĆ** (Sarajevo College, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

Assume there is a function f satisfying

$$f(x+f(y)) = f(x) - y. \qquad (*)$$

If $f(a) = f(b)$, then

$$f(x) - a = f(x+f(a)) = f(x+f(b)) = f(x) - b,$$

which implies $a = b$, i.e. f is injective. Taking $y = 0$ in (*), $f(x+f(0)) = f(x)$. By injectivity, we see $f(0) = 0$. Taking $x=0$ in (*), we get

$$f(f(y)) = -y. \qquad (**)$$

Applying f to both sides of (*) and using (**), we have

$$f(f(x) - y) = f(f(x + f(y)) = -x - f(y).$$

Taking $x = 0$ in this equation, we get

$$f(-y) = -f(y). \qquad (***)$$

Using (**), (*) and (***), we get

$$f(x+y) = f(x + f(f(-y)) = f(x) - f(-y) = f(x) + f(y).$$

Thus, f satisfies the Cauchy equation. By mathematical induction and (***), $f(n) = n f(1)$ for every integer n. Taking $n = f(1)$ in the last equation and $y = 1$ into (**), we get $f(1)^2 = -1$. This yields a contradiction.

Problem 303. In base 10, let N be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of N, forming numbers that are different (integral) powers of two.
(*Source: 2004 Spanish Math Olympiad*)

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education) and **PUN Ying Anna** (HKU Math Year 3).

Assume there exist two permutations of the digits of N, forming the numbers 2^k and 2^m for some positive integers k and m with $k > m$. Then $2^k < 10 \times 2^m$. So $k \leq m+3$.

Since every number is congruent to its sum of digits (mod 9), we get $2^k \equiv 2^m$ (mod 9). Since 2^m and 9 are relatively prime, we get $2^{k-m} \equiv 1$ (mod 9). However, $k - m = 1$, 2 or 3, which contradicts $2^{k-m} \equiv 1$ (mod 9).

Problem 304. Let M be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the x-y coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in M and whose sides are parallel to the x-axis or the y-axis.
(*Source: 2003 Chinese IMO Team Training Test*)

Solution 1. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain) and **PUN Ying Anna** (HKU Math Year 3).

Let O be a point in M. We say a rectangle is good if all its sides are parallel to the x or y-axis and all its vertices are in M, one of which is O. We claim there are at most 81 good rectangles. (Once the claim is proved, we see there can only be at most $(81 \times 100)/4 = 2025$ desired rectangles.

The division by 4 is due to such rectangle has 4 vertices, hence counted 4 times).

For the proof of the claim, we may assume O is the origin of the plane. Suppose the x-axis contains m points in M other than O and the y-axis contains n points in M other than O. For a point P in M not on either axis, it can only be a vertex of at most one good rectangle. There are at most $99-m-n$ such point P and every good rectangle has such a vertex.

If $m+n \ge 18$, then there are at most $99-m-n \le 81$ good rectangles. Otherwise, $m+n \le 17$. Now every good rectangle has a vertex on the x-axis and a vertex on the y-axis other than O. So there are at most $mn \le (m+n)^2/4 < 81$ rectangles by the *AM-GM* inequality. The claim follows.

Solution 2. **G.R.A. 20 Problem Solving Group** (Roma, Italy).

Let $f(x) = x(x-1)/2$. We will prove that if there are N lattice points, there are at most $[f(N^{1/2})]^2$ such rectangles. For $N=100$, we have $[f(10)]^2 = 45^2 = 2025$ (this bound is attained when the 100 points form a 10×10 square).

Suppose the N points are distributed on m lines parallel to an axis. Say the number of points in the m lines are $r_1, r_2, \ldots, r_m$, arranged in increasing order. Now the two lines with r_i and r_j points can form no more than $f(\min\{r_i,r_j\})$ rectangles. Hence, the number of rectangles is at most

$$\sum_{1\le i<j\le m} f(\min\{r_i,r_j\}) = \sum_{i=1}^{m-1}(m-i)f(r_i)$$
$$\le \sum_{i=1}^{m-1}(m-i)f\left(\frac{N}{m}\right) = f(m)f\left(\frac{N}{m}\right)$$
$$\le \left(f(\sqrt{N})\right)^2.$$

The second inequality follows by expansion and usage of the *AM-GM* inequality. The first one can be proved by expanding and simplifying it to

$$2m\sum_{i=1}^{m-1}(m-i)r_i(r_i-1)\le(m-1)\sum_{i=1}^{m}r_i\sum_{i=1}^{m}(r_i-1). \quad (*)$$

We will prove this by induction on m. For $m=2$, $4r_1(r_1-1)\le(r_1+r_2)(r_1-1+r_2-1)$ follows from $1 \le r_1 \le r_2$. For the inductive step, we suppose (*) is true. To do the $(m+1)$-st case of (*), observe that $r_i \le r_{m+1}$ implies

$$m\sum_{i=1}^{m} r_i(r_i-1)\le m(r_{m+1}-1)\sum_{i=1}^{m} r_i,$$

$$m\sum_{i=1}^{m} r_i(r_i-1)\le mr_{m+1}\sum_{i=1}^{m} (r_i-1),$$

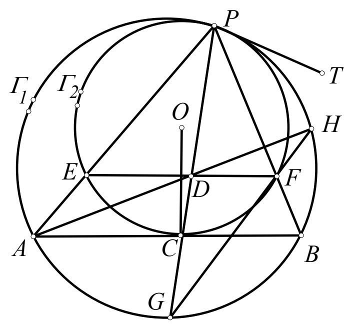
$$2\sum_{i=1}^{m}(m+1-i)r_i(r_i-1)$$
$$\le mr_{m+1}(r_{m+1}-1)+\sum_{i=1}^{m} r_i\sum_{i=1}^{m}(r_i-1).$$

Let $L(m)$ and $R(m)$ denote the left and right sides of (*) respectively. Adding the last three inequalities, it turns out we get $L(m+1) - L(m) \le R(m+1) - R(m)$. Now (*) holds, so $L(m) \le R(m)$. Adding these, we get $L(m+1) \le R(m+1)$.

Commended solvers: **Victor FONG** (CUHK Math Year 2), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **Raúl A. SIMON** (Santiago, Chile).

Problem 305. A circle Γ_2 is internally tangent to the circumcircle Γ_1 of ΔPAB at P and side AB at C. Let E, F be the intersection of Γ_2 with sides PA, PB respectively. Let EF intersect PC at D. Lines PD, AD intersect Γ_1 again at G, H respectively. Prove that F, G, H are collinear.

Solution. **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education) and **PUN Ying Anna** (HKU Math Year 3).



Let PT be the external tangent to both circles at P. We have

$$\angle PAB = \angle BPT = \angle PEF,$$

which implies $EF||AB$. Let O be the center of Γ_2. Since $OC \perp AB$ (because AB is tangent to Γ_2 at C), we deduce that $OC\perp EF$ and therefore OC is the perpendicular bisector of EF. Hence C is the midpoint of arc ECF. Then PC bisects $\angle EPF$. On the other hand,

$$\angle HDF = \angle HAB = \angle HPB = \angle HPF,$$

which implies H, P, D, F are concyclic. Therefore,

$$\angle DHF = \angle DPF = \angle EPD$$
$$= \angle APG = \angle AHG = \angle DHG,$$

which implies F, G, H are collinear.

Remarks. A few solvers got $EF||AB$ by observing there is a homothety with center P sending Γ_2 to Γ_1 so that E goes to A and F goes to B.

Commended solvers: **Victor FONG** (CUHK Math Year 2) and **Salem MALIKIĆ** (Sarajevo College, Sarajevo, Bosnia and Herzegovina).

Olympiad Corner

(continued from page 1)

Problem 4. Find all functions f: $(0,\infty) \to (0,\infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{\left(f(w)\right)^2+\left(f(x)\right)^2}{f(y^2)+f(z^2)} = \frac{w^2+x^2}{y^2+z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \ge n$ and $k-n$ an even number. Let $2n$ lamps labeled 1, 2, …, $2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamp 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ and $2n$ is ever switched on.

Determine the ratio N/M.

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \ne |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

Mathematical Excalibur

Volume 13, Number 4 *November-December, 2008*

Olympiad Corner

The following were the problems of the Hong Kong Team Selection Test 2, which was held on November 8, 2008 for the 2009 IMO.

Problem 1. Let f:$Z\to Z$ (Z is the set of all integers) be such that $f(1) = 1$, $f(2) = 20$, $f(-4) = -4$ and

$$f(x+y) = f(x) + f(y) + axy(x+y) + bxy + c(x+y) + 4$$

for all $x,y \in Z$, where a, b and c are certain constants.

(a) Find a formula for $f(x)$, where x is any integer.

(b) If $f(x) \geq mx^2 + (5m+1)x + 4m$ for all non-negative integers x, find the greatest possible value of m.

Problem 2. Define a *k-clique* to be a set of k people such that every pair of them know each other (knowing is mutual). At a certain party, there are two or more 3-cliques, but no 5-clique. Every pair of 3-cliques has at least one person in common. Prove that there exist at least one, and not more than two persons at the party, whose departure (or simultaneous departure) leaves no 3-clique remaining.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 10, 2009***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Double Counting

Law Ka Ho, Leung Tat Wing and Li Kin Yin

There are often different ways to count a quantity. By counting it in two ways (i.e. double counting), we thus obtain the same quantity in different forms. This often yields interesting equalities and inequalities. We begin with some simple examples.

Below we will use the notation $C_r^n = n!/(r!(n-r)!)$.

Example 1. (*IMO HK Prelim 2003*) Fifteen students join a summer course. Every day, three students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

Solution. Let the answer be k. We count the total number of pairs of students were on duty together in the k days. Since every pair of students was on duty together exactly once, this is equal to $C_2^{15} \times 1 = 105$. On the other hand, since 3 students were on duty per day, this is also equal to $C_2^3 \times k = 3k$. Hence $3k = 105$ and so $k = 35$.

Example 2. (*IMO 1987*) Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(*Remark*: A *permutation* f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a *fixed point* of the permutation f if $f(i) = i$.)

Solution. Note that the left hand side of the equality is the total number of fixed points in all permutations of $\{1,2,\ldots,n\}$. To show that this number is equal to $n!$, note that there are $(n-1)!$ permutations of $\{1, 2, \ldots, n\}$ fixing 1, $(n-1)!$ permutations fixing 2, and so on, and $(n-1)!$ permutations fixing n. It follows that the total number of fixed points in all permutations is equal to $n \cdot (n-1)! = n!$.

The simplest combinatorial identity is perhaps $C_r^n = C_{n-r}^n$. While this can be verified algebraically, we can give a proof in a more combinatorial flavour: to choose r objects out of n, it is equivalent to choosing $n-r$ objects out of n to be discarded. There are C_r^n ways to do the former and C_{n-r}^n ways to do the latter. So the two quantities must be equal.

Example 3. Interpret the following equalities from a combinatorial point of view:

(a) $C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$

(b) $C_1^n + 2C_2^n + \cdots + nC_n^n = n \cdot 2^{n-1}$

Solution. (a) On one hand, the number of ways to choose k objects out of n objects is C_k^n. On the other hand, we may count by including the first object or not. If we include the first object, we need to choose $k-1$ objects from the remaining $n-1$ objects and there are C_{k-1}^{n-1} ways to do so.

If we do not include the first object, we need to choose k objects from the remaining $n-1$ objects and there are C_k^{n-1} ways to do so. Hence

$$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}.$$

(b) Suppose that from a set of n people, we want to form a committee with a chairman of the committee. On one hand, there are n ways to choose a chairman, and for each of the remaining $n-1$ persons we may or may not include him in the committee. Hence there are $n \cdot 2^{n-1}$ ways to finish the task.

On the other hand, we may choose k people to form a committee ($1\leq k \leq n$), which can be done in C_k^n ways, and for each of these ways there are k ways to select the chairman. Hence the number of ways to finish the task is also equal to

$$C_1^n + 2C_2^n + \cdots + nC_n^n.$$

Example 4. (*IMO 1989*) Let n and k be positive integers and let S be a set of n points in the plane such that:

(i) no three points of S are collinear, and

(ii) for every point P of S, there are at least k points of S equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.

Solution. Solving for n, the desired inequality is equivalent to $n > k(k-1)/2 + 1/8$. Since n and k are positive integers, this is equivalent to $n - 1 \geq C_2^k$. Now we join any two vertices of S by an *edge* and count the number of edges in two ways.

On one hand, we have C_2^n edges. On the other hand, from any point of S there are at least k points equidistant from it. Hence if we draw a circle with the point as centre and with the distance as radius then there are at least C_2^k chords as edges. The total number of such chords, counted with multiplicities, is at least nC_2^k. Any two circles can have at most one common chord and hence there could be a maximum C_2^n chords (for every possible pairs of circles) counted twice. Therefore,

$$nC_2^k - C_2^n \leq C_2^n,$$

which simplifies to $n - 1 \geq C_2^k$. (Note that collinearity was not needed.)

Example 5. (*IMO 1998*) In a competition, there are m contestants and n judges, where $n \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

Solution. We begin by considering pairs of judges who agree on certain contestants. We study this from two perspectives.

For contestant i, $1 \leq i \leq m$, suppose there are x_i judges who pass him, and y_i judges who fail him. On one hand, the number of pairs of judges who agree on him is

$$C_2^{x_i} + C_2^{y_i} = \frac{x_i^2 - x_i + y_i^2 - y_i}{2}$$

$$\geq \frac{(x_i + y_i)^2/2}{2} - \frac{x_i + y_i}{2}$$

$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[(n-1)^2 - 1\right].$$

Since n is odd and $C_2^{x_i} + C_2^{y_i}$ is an integer, it is at least $(n-1)^2/4$.

On the other hand, there are n judges and each pair of judges agree on at most k contestants. Hence the number of pairs of judges who agree on a certain contestant is at most kC_2^n. Thus,

$$kC_2^n \geq \sum_{i=1}^{m} (C_2^{x_i} + C_2^{y_i}) \geq \frac{m(n-1)^2}{4},$$

which can be simplified to obtain the desired result.

Some combinatorial problems in mathematical competitions can be solved by double counting certain ordered triples. The following are two such examples.

Example 6. (*CHKMO 2007*) In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club. Show that there is a club with at least 11 male and 11 female members.

Solution. Assume on the contrary that every club either has at most 10 male members or at most 10 female members. We shall get a contradiction via double counting certain ordered triples.

Let S be the number of ordered triples of the form (m, f, c), where m denotes a male student, f denotes a female student and c denotes a club. On one hand, since any two students of opposite genders have joined at least one common club, we have

$$S \geq 2007^2 = 4028049.$$

On the other hand, we can consider two types of clubs: let X be the set of clubs with at most 10 male members, and Y be the set of clubs with at least 11 male members (and hence at most 10 female members). Note that there are at most $10\times2007\times100=2007000$ triples (m, f, c) with $c \in X$, because there are 2007 choices for f, then at most 100 choices for c (each student joins at most 100 clubs), and then at most 10 choices for m (each club $c \in X$ has at most 10 male members). In exactly the same way, we can show that there are at most 2007000 triples (m, f, c) with $c \in Y$. This gives

$$S \leq 2007000+2007000=4014000,$$

a contradiction.

Example 7. (*2004 IMO Shortlisted Problem*) There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose the following conditions hold:

(i) Each pair of students is in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

Solution. An ordered triple (a, C, S) will be called <u>*acceptable*</u> if a is a student, C is a club and S is a society such that $a \in C$ and $C \in S$. We will count the number of acceptable ordered triples in two ways.

On one hand, for every student a and society S, by (ii), there is a unique club C such that (a, C, S) is acceptable. Hence, there are $10001k$ acceptable ordered triples.

On the other hand, for every club C, let the number of members in C be denoted by $|C|$. By (iii), C is in exactly $(|C|-1)/2$ societies. So there are $|C|(|C|-1)/2$ acceptable ordered triples with C as the second coordinates. Let Γ be the set of all clubs. Hence, there are

$$\sum_{C\in\Gamma} \frac{|C|(|C|-2)}{2}$$

acceptable ordered triples. By (i), this is equal to the number of pairs of students, which is 10001×5000. Therefore,

$$10001k = \sum_{C\in\Gamma} \frac{|C|(|C|-2)}{2}$$

$$= 10001\times5000,$$

which implies $k = 5000$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***January 10, 2009.***

Problem 311. Let $S = \{1,2,\ldots,2008\}$. Prove that there exists a function $f : S \to$ {red, white, blue, green} such that there does not exist a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$.

Problem 312. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2}+\frac{y^4}{(z-1)^2}+\frac{z^4}{(x-1)^2}\ge 48.$$

Problem 313. In ΔABC, $AB < AC$ and O is its circumcenter. Let the tangent at A to the circumcircle cut line BC at D. Let the perpendicular lines to line BC at B and C cut the perpendicular bisectors of sides AB and AC at E and F respectively. Prove that D, E, F are collinear.

Problem 314. Determine all positive integers x, y, z satisfying $x^3 - y^3 = z^2$, where y is a prime, z is not divisible by 3 and z is not divisible by y.

Problem 315. Each face of 8 unit cubes is painted white or black. Let n be the total number of black faces. Determine the values of n such that in every way of coloring n faces of the 8 unit cubes black, there always exists a way of stacking the 8 unit cubes into a 2×2×2 cube C so the numbers of black squares and white squares on the surface of C are the same.

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Solutions

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Problem 306. Prove that for every integer $n \ge 48$, every cube can be decomposed into n smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

For such an integer n, we will say cubes are *n-decomposable*. Let *r-cube* mean a cube with sidelength r. If a r-cube C is n-decomposable, then we can first decompose C into 8 $r/2$-cubes and then decompose one of these $r/2$-cubes into n cubes to get a total of $n+7$ cubes so that C is $(n+7)$-decomposable.

Let C be a 1-cube. All we need to show is C is n-decomposable for $48 \le n \le 54$.

For $n=48$, decompose C to 27 1/3-cubes and then decompose 3 of these, each into 8 1/6-cubes.

For $n=49$, cut C by two planes parallel to the bottom at height 1/2 and 1/6 from the bottom, which can produce 4 1/2-cubes at the top layer, 9 1/3-cubes in the middle layer and 36 1/6-cubes at the bottom layer.

For $n=50$, decompose C to 8 1/2-cubes and then decompose 6 of these, each into 8 1/4-cubes.

For $n=51$, decompose C into 8 1/2-cubes, then take 3 of these 1/2-cubes on the top half to form a L-shaped prism and cut out 5 1/3-cubes and 41 1/6-cubes.

For $n=52$, decompose C into 1 3/4-cube and 37 1/4-cubes, then decompose 2 1/4-cubes, each into 8 1/8-cubes.

For $n=53$, decompose C to 27 1/3-cubes and then decompose 1 of these into 27 1/9-cubes.

For $n=54$, decompose C into 8 1/2-cubes, then take 2 of the adjacent 1/2-cubes, which form a 1×1/2×1/2 box, from which we can cut 2 3/8-cubes, 4 1/4-cubes and 42 1/8-cubes.

Comments: Interested readers may find more information on this problem by visiting mathworld.wolfram.com and by searching for *Cube Dissection* .

Problem 307. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a polynomial with real coefficients such that $a_0 \ne 0$ and for all real x,

$$f(x)f(2x^2) = f(2x^3+x).$$

Prove that $f(x)$ has no real root.

Solution. **José Luis DÍAZ-BARRERO** (Universitat Politècnica de Catalunya, Barcelona, Spain), **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **Ozgur KIRCAK and Bojan JOVESKI** (Jahja Kemal College, Skopje, Macedonia), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **Fai YUNG**.

For such polynomial $f(x)$, let k be largest such that $a_k\ne 0$. Then

$$f(x)f(2x^2)=a_0^2 2^n x^{3n}+\cdots+a_k^2 2^{n-k}x^{3(n-k)},$$

$$f(2x^3+x)=a_0 2^n x^{3n}+\cdots+a_k x^{n-k},$$

where the terms are ordered by decreasing degrees. This can happen only if $n - k = 0$. So $f(0) = a_n \ne 0$. Assume $f(x)$ has a real root $x_0 \ne 0$. The equation $f(x)f(2x^2) = f(2x^3+x)$ implies that if x_n is a real root, then $x_{n+1} = 2x_n^3 + x_n$ is also a real root. Since this sequence is strictly monotone, this implies $f(x)$ has infinitely many real roots, which is a contradiction.

Commended solvers: **Simon YAU Chi Keung** (City U).

Problem 308. Determine (with proof) the greatest positive integer $n > 1$ such that the system of equations

$$(x+1)^2+y_1^2=(x+2)^2+y_2^2=\cdots=(x+n)^2+y_n^2.$$

has an integral solution $(x, y_1, y_2, \cdots, y_n)$.

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **Ozgur KIRCAK and Bojan JOVESKI** (Jahja Kemal College, Skopje, Macedonia) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

We will show the greatest such n is 3. For $n = 3$, $(x, y_1, y_2, y_3) = (-2, 0, 1, 0)$ is a solution. For $n \ge 4$, assume the system has an integral solution. Since $x+1$, $x+2$, …, $x+n$ are of alternate parity, so $y_1, y_2, \ldots, y_n$ are also of alternate parity. Since $n \ge 4$, y_k is even for $k = 2$ or 3. Consider

$$(x+k-1)^2+y_{k-1}^2=(x+k)^2+y_k^2=(x+k+1)^2+y_{k+1}^2.$$

The double of the middle expression equals the sum of the left and right expressions. Eliminating common terms in that equation, we get

$$2y_k^2 = y_{k-1}^2 + y_{k+1}^2 + 2. \qquad (*)$$

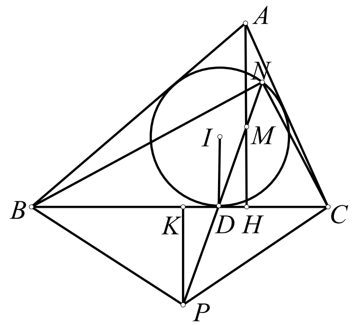
Now y_{k-1} and y_{k+1} are odd. Then the left side of (*) is 0 (mod 8), but the right side is 4 (mod 8), a contradiction.

Commended solvers: **O Kin Chit Alex** (G.T. (Ellen Yeung) College), **Raúl A. SIMON** (Santiago, Chile) and **Simon**

YAU Chi Keung (City U).

Problem 309. In acute triangle ABC, $AB > AC$. Let H be the foot of the perpendicular from A to BC and M be the midpoint of AH. Let D be the point where the incircle of ΔABC is tangent to side BC. Let line DM intersect the incircle again at N. Prove that $\angle BND = \angle CND$.

Solution.



Let I be the center of the incircle. Let the perpendicular bisector of segment BC cut BC at K and cut line DM at P. To get the conclusion, it is enough to show $DN{\cdot}DP=DB{\cdot}DC$ (which implies B,P,C,N are concyclic and since $PB = PC$, that will imply $\angle BND = \angle CND$).

Let sides $BC=a$, $CA=b$ and $AB=c$. Let $s = (a+b+c)/2$, then $DB = s-b$ and $DC = s-c$. Let r be the radius of the incircle and $[ABC]$ be the area of triangle ABC. Let $\alpha = \angle CDN$ and $AH = h_a$. Then $[ABC]$ equals

$$ah_a/2 = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

Now

$$DK = DB - KB = \frac{a+c-b}{2} - \frac{a}{2} = \frac{c-b}{2},$$

$$DH = DC - HC = \frac{a+b-c}{2} - b\cos\angle ACB$$

$$= \frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a}$$

$$= \frac{(c-b)(b+c-a)}{2a} = \frac{(c-b)(s-a)}{a}.$$

Moreover, $DN = 2r\sin\alpha$, $DP = DK/(\cos\alpha) = (c-b)/(2\cos\alpha)$. So

$$DN\cdot DP = r(c-b)\tan\alpha = r(c-b)\frac{MH}{DH}$$

$$= r(c-b)\frac{h_a/2}{(c-b)(s-a)/a}$$

$$= r\frac{ah_a/2}{s-a} = \frac{rsrs}{s(s-a)} = \frac{[ABC]^2}{s(s-a)}$$

$$= (s-b)(s-c) = DB\cdot DC.$$

Problem 310. (*Due to Pham Van Thuan*) Prove that if p, q are positive real numbers such that $p + q = 2$, then

$$3p^q q^p + p^p q^q \le 4.$$

***Solution* 1. Proposer's Solution.**

As $p, q > 0$ and $p + q = 2$, we may assume $2 > p \ge 1 \ge q > 0$. Applying Bernoulli's inequality, which asserts that if $x > -1$ and $r \in [0,1]$, then $1+rx \ge (1+x)^r$, we have

$$p^p = pp^{p-1} \ge p(1+(p-1)^2) = p(p^2-2p+2),$$
$$q^q \le 1+q(q-1) = 1+(2-p)(1-p) = p^2-3p+3,$$
$$p^q \le 1+q(p-1) = 1+(2-p)(p-1) = -p^2+3p-1,$$
$$q^p = qq^{p-1} \le q(1+(p-1)(q-1)) = p(2-p)^2.$$

Then

$$\begin{aligned} &3p^q q^p + p^p q^q - 4 \\ &\le 3(-p^2+3p-1)p(2-p)^2 \\ &\quad +p(p^2-2p+2)(p^2-3p+3) - 4 \\ &= -2p^5+16p^4-40p^3+36p^2-6p-4 \\ &= -2(p-1)^2(p-2)((p-2)^2-5) \le 0. \end{aligned}$$

(To factor with $p-1$ and $p-2$ was suggested by the observation that $(p,q) = (1,1)$ and $(p,q) \to (2,0)$ lead to equality cases.)

Comments: The case $r = m/n \in \mathbb{Q}\cap[0,1]$ of Bernoulli's inequality follows by applying the *AM-GM* inequality to $a_1,\ldots,a_n$, where $a_1 = \cdots = a_m = 1+x$ and $a_{m+1} = \cdots = a_n = 1$. The case $r \in [0,1]\backslash\mathbb{Q}$ follows by taking rational m/n converging to r.

***Solution* 2. LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Suppose $2 > p \ge 1 \ge q > 0$. Applying Bernoulli's inequality with $1+x = p/q$ and $r = p/2$, we have

$$\left(\frac{p}{q}\right)^{p/2} \le 1 + \frac{p}{2}\left(\frac{p}{q} - 1\right) = \frac{p^2+q^2}{2q}.$$

Multiplying both sides by q and squaring both sides, we have

$$p^p q^q \le (p^2+q^2)^2/4.$$

Similarly, applying Bernoulli's inequality with $1+x = q/p$ and $r = p/2$, we can get $p^p q^q \le p^2q^2$. So

$$\begin{aligned} 3p^q q^p + p^p q^q &\le (p^4+14p^2q^2+q^4)/4 \\ &= (p^4+6p^2q^2+q^4+4pq(2pq))/4 \\ &\le (p^4+6p^2q^2+q^4+4pq(p^2+q^2))/4 \\ &= (p+q)^4/4 = 4. \end{aligned}$$

Commended solvers: **Paolo Perfetti** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that there are infinitely many primes p such that $N_p = p^2$, where N_p is the total number of solutions to the equation

$$3x^3+4y^3+5z^3-y^4z \equiv 0 \pmod{p}.$$

Problem 4. Two circles C_1, C_2 with different radii are given in the plane, they touch each other externally at T. Consider any points $A \in C_1$ and $B \in C_2$, both different from T, such that $\angle ATB = 90°$.

(a) Show that all such lines AB are concurrent.

(b) Find the locus of midpoints of all such segments AB.

Double Counting

(continued from page 2)

Example 8. (*2003 IMO Shortlisted Problem*) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \ge 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n{\times}n$ matrix with entries 0 or 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A=B$.

Solution. Let $A = (a_{ij})_{1\le i,j\le n}$. Define

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i + y_j)(a_{ij} - b_{ij}).$$

On one hand, we have

$$S = \sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} b_{ij}\right) + \sum_{j=1}^{n} y_j\left(\sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} b_{ij}\right)$$
$$= 0.$$

On the other hand, if $x_i+y_j \ge 0$, then $a_{ij} = 1$, which implies $a_{ij}-b_{ij} \ge 0$; if $x_i+y_j < 0$, then $a_{ij} = 0$, which implies $a_{ij}-b_{ij} \le 0$. Hence, $(x_i+y_j)(a_{ij}-b_{ij}) \ge 0$ for all i,j. Since $S = 0$, all $(x_i+y_j)(a_{ij}-b_{ij}) = 0$.

In particular, if $a_{ij}=0$, then $x_i+y_j < 0$ and so $b_{ij} = 0$. Since a_{ij}, b_{ij} are 0 or 1, so $a_{ij} \ge b_{ij}$ for all i,j. Finally, since the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A, so $a_{ij} = b_{ij}$ for all i,j.

Mathematical Excalibur

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Olympiad Corner

The following were the problems of the Final Round (Part 2) of the Austrian Mathematical Olympiad 2008.

<u>First Day: June 6th, 2008</u>

Problem 1. Prove the inequality

$$\sqrt{a^{1-a}b^{1-b}c^{1-c}} \le \frac{1}{3}$$

holds for all positive real numbers a, b and c with $a+b+c = 1$.

Problem 2. (a) Does there exist a polynomial $P(x)$ with coefficients in integers, such that $P(d) = 2008/d$ holds for all positive divisors of 2008?

(b) For which positive integers n does a polynomial $P(x)$ with coefficients in integers exists, such that $P(d) = n/d$ holds for all positive divisors of n?

Problem 3. We are given a line g with four successive points P, Q, R, S, reading from left to right. Describe a straight-edge and compass construction yielding a square $ABCD$ such that P lies on the line AD, Q on the line BC, R on the line AB and S on the line CD.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 7, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Generating Functions

Kin Yin Li

In some combinatorial problems, we may be asked to determine a certain sequence of numbers a_0, a_1, a_2, a_3, …. We can associate such a sequence with the following series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

This is called the *<u>generating function</u>* of the sequence. Often the geometric series $1/(1-t) = 1+t+t^2+t^3+\cdots$ for $|t| < 1$ and its square

$$\begin{aligned}1/(1-t)^2 &= (1+t+t^2+t^3+\cdots)^2\\ &= 1+2t+3t^2+4t^3+5t^4+\cdots\end{aligned}$$

will be involved in our discussions.

Below we will provide examples to illustrate how generating functions can solve some combinatorial problems.

<u>Example 1.</u> Let $a_0=1$, $a_1=1$ and

$$a_n = 4a_{n-1} - 4a_n \quad \text{for } n \ge 2.$$

Find a formula for a_n in terms of n.

<u>Solution.</u> Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then we have

$$\begin{aligned}f(x) - 1 - x &= a_2x^2 + a_3x^3 + a_4x^4 + \cdots\\ &= (4a_1-4a_0)x^2 + (4a_2-4a_1)x^3 + \cdots\\ &= (4a_1x^2+4a_2x^3+\cdots)-(4a_0x^2+4a_1x^3+\cdots)\\ &= 4x(f(x)-1) - 4x^2f(x).\end{aligned}$$

Solving for $f(x)$ and taking $|x| < ½$,

$$\begin{aligned}f(x) &= (1-3x)/(1-2x)^2\\ &= 1/(1-2x)-x/(1-2x)^2\\ &= \sum_{n=0}^{\infty}(2x)^n - x\sum_{n=1}^{\infty}n(2x)^{n-1}\\ &= \sum_{n=0}^{\infty}(2^n - n2^{n-1})x^n.\end{aligned}$$

Therefore, $a_n = 2^n - n\,2^{n-1}$.

<u>Example 2.</u> Find the number a_n of ways n dollars can be changed into 1 or 2 dollar coins (regardless of order). For example, when $n = 3$, there are 2 ways, namely three 1 dollar coins or one 1 dollar coin and one 2 dollar coin.

<u>Solution.</u> Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. To study this infinite series, let $|x| < 1$.

For each way of changing n dollars into r 1 dollar and s 2 dollar coins, we can record it as $x^r x^{2s} = x^n$. Now r and s may be any nonnegative integers. Adding all the recorded terms for all nonnegative integers n, then factoring, we get

$$\sum_{r=0}^{\infty}x^r\sum_{s=0}^{\infty}x^{2s} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}x^{r+2s} = \sum_{n=0}^{\infty}a_nx^n = f(x).$$

On the other hand,

$$\begin{aligned}\sum_{r=0}^{\infty}x^r\sum_{s=0}^{\infty}x^{2s} &= \frac{1}{1-x}\cdot\frac{1}{1-x^2} = \frac{1}{(1-x)^2(1-x)}\\ &= \frac{1}{2}\left(\frac{1}{(1-x)^2}+\frac{1}{1-x^2}\right)\\ &= \tfrac{1}{2}\left((1+2x+3x^2+\cdots)+(1+x^2+x^4+\cdots)\right)\\ &= 1+x+2x^2+2x^3+3x^4+3x^5+\cdots.\\ &= \sum_{n=0}^{\infty}([n/2]+1)x^n.\end{aligned}$$

Therefore, $a_n = [n/2] + 1$.

<u>Example 3.</u> Let n be a positive integer. Find the number a_n of polynomials $P(x)$ with coefficients in $\{0,1,2,3\}$ such that $P(2) = n$.

<u>Solution.</u> Let $f(t)$ be the generating function of the sequence a_0, a_1, a_2, a_3, …. Let $P(x) = c_0 + c_1x + \cdots + c_kx^k$ with $c_i \in \{0,1,2,3\}$. Now $P(2) = n$ if and only if $c_0 + 2c_1 + \cdots + 2^kc_k = n$. Taking $t \in (-1,1)$, we can record this as

$$t^n = t^{c_0}t^{2c_1}\cdots t^{2^kc_k}.$$

Note 2^ic_i is one of the four numbers 0, 2^i, 2^{i+1}, $3\cdot 2^i$. Adding all the recorded terms for all nonnegative integers n and all possible c_0, c_1, …, $c_k \in \{0,1,2,3\}$, then factoring on the right, we have

$$f(t) = \sum_{n=0}^{\infty}a_nt^n = \prod_{i=0}^{\infty}(1+t^{2^i}+t^{2^{i+1}}+t^{3\cdot 2^i}).$$

Using $1+s+s^2+s^3=(1-s^4)/(1-s)$, we see

$$\begin{aligned}f(t) &= \frac{1-t^4}{1-t}\cdot\frac{1-t^8}{1-t^2}\cdot\frac{1-t^{16}}{1-t^4}\cdot\frac{1-t^{32}}{1-t^8}\cdots\\ &= \frac{1}{1-t}\cdot\frac{1}{1-t^2}.\end{aligned}$$

As in example 2, we get $a_n = [n/2] + 1$.

For certain problems, instead of using the generating function of a_0, a_1, a_2, a_3, …, we may consider the series

$$x^{a_0}+x^{a_1}+x^{a_2}+x^{a_3}+\cdots.$$

Example 4. (*1998 IMO Shortlisted Problem*) Let a_0, a_1, a_2, … be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i+2a_j+4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

Solution. For $|x|<1$, let $f(x)=\sum_{i=0}^{\infty}x^{a_i}$.

The given condition implies

$$f(x)f(x^2)f(x^4)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}.$$

Replacing x by x^2, we get

$$f(x^2)f(x^4)f(x^8)=\frac{1}{1-x^2}.$$

From these two equations, we get $f(x)=(1+x)f(x^8)$. Repeating this recursively, we get

$$f(x)=(1+x)(1+x^8)(1+x^{8^2})(1+x^{8^3})\cdots.$$

In expanding the right side, we see the exponents a_0, a_1, a_2, … are precisely the nonnegative integers whose base 8 representations have only digit 0 or 1. Since $1998=2+2^2+2^3+2^6+2^7+2^8+2^9+2^{10}$, so $a_{1998}=8+8^2+8^3+8^6+8^7+8^8+8^9+8^{10}$.

For our next examples, we need some identities involving p-th roots of unity, where p is a positive integer. These are complex numbers λ, which are all the solutions of the equation $z^p=1$. For a real θ, we will use the common notation $e^{i\theta}=\cos\theta+i\sin\theta$. Since the equation is of degree p, there are exactly p p-th roots of unity. We can easily check that they are $e^{i\theta}$ with $\theta=0$, $2\pi/p$, $4\pi/p$, …, $2(p-1)\pi/p$.

Below let λ be any p-th root of unity, other than 1. When we have a series

$$B(z)=b_0+b_1z+b_2z^2+b_3z^3+\cdots,$$

sometimes we need to find the value of $b_p+b_{2p}+b_{3p}+\cdots$. We can use the fact

$$1+\lambda^j+\lambda^{2j}+\cdots+\lambda^{(p-1)j}=\frac{1-\lambda^{pj}}{1-\lambda^j}=0$$

(for any j not divisible by p) to get

$$\frac{1}{p}\sum_{j=0}^{p-1}B(\lambda^j)=b_p+b_{2p}+b_{3p}+\cdots. \quad (*)$$

For p odd, we have the factorization

$$1+t^p=(1+t)(1+\lambda t)\cdots(1+\lambda^{p-1}t) \quad (**)$$

since both sides have $-1/\lambda^i$ ($i=0,1,\ldots,p-1$) as roots and are monic of degree p.

Example 5. Can the set $\mathbb{N}$ of all positive integers be partitioned into more than one, but still a finite number of arithmetic progressions with no two having the same common differences?

Solution. (*Due to Donald J. Newman*) Assume the set $\mathbb{N}$ can be partitioned into sets $S_1, S_2,\ldots,S_k$, where $S_i=\{a_i+nd_i: n\in\mathbb{N}\}$ with $d_1>d_2>\cdots>d_k$. Then for $|z|<1$,

$$\sum_{n=1}^{\infty}z^n=\sum_{n=1}^{\infty}z^{a_1+nd_1}+\sum_{n=1}^{\infty}z^{a_2+nd_2}+\cdots+\sum_{n=1}^{\infty}z^{a_k+nd_k}.$$

Summing the geometric series, this gives

$$\frac{z}{1-z}=\frac{z^{a_1}}{1-z^{d_1}}+\frac{z^{a_2}}{1-z^{d_2}}+\cdots+\frac{z^{a_k}}{1-z^{d_k}}.$$

Letting z tend to $e^{2\pi i/d_1}$, we see the left side has a finite limit, but the right side goes to infinity. That gives a contradiction.

Example 6. (*1995 IMO*) Let p be an odd prime number. Find the number of subsets A of the set $\{1,2,\ldots,2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

Solution. Consider the polynomial

$$F_a(x)=(1+ax)(1+a^2x)(1+a^3x)\cdots(1+a^{2p}x)$$

When the right side is expanded, let $c_{n,k}$ count the number of terms of the form $(a^{i_1}x)(a^{i_2}x)\cdots(a^{i_k}x)$, where $i_1, i_2, \ldots, i_k$ are integers such that $1\le i_1<i_2<\cdots<i_k\le 2p$ and $i_1+i_2+\cdots+i_k=n$. Then

$$F_a(x)=1+\sum_{k=1}^{2p}\left(\sum_{n=1}^{\infty}c_{n,k}a^n\right)x^k.$$

Now, in terms of $c_{n,k}$, the answer to the problem is $C=c_{p,p}+c_{2p,p}+c_{3p,p}+\cdots$.

To get C, note the coefficient of x^p in $F_a(x)$ is $\sum_{n=1}^{\infty}c_{n,p}a^n$. By (*) above, we see

$$C=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty}c_{n,p}\omega^{nj}.$$

Now the right side is the coefficient of x^p in $\frac{1}{p}\sum_{j=0}^{p-1}F_{\omega^j}(x)$, which equals

$$\frac{1}{p}\sum_{j=0}^{p-1}(1+\omega^jx)(1+\omega^{2j}x)\cdots(1+\omega^{2pj}x).$$

For $j=0$, the term is $(1+x)^{2p}$. For $1\le j\le p-1$, using (**) with $\lambda=\omega^j$ and $t=\lambda x$, we see the j-th term is $(1+x^p)^2$. Using these, we have

$$\frac{1}{p}\sum_{j=0}^{p-1}F_{\omega^j}(x)=\frac{1}{p}[(1+x)^{2p}+(p-1)(1+x^p)^2].$$

Therefore, the coefficient of x^p is

$$C=\frac{1}{p}\left[\binom{2p}{p}+2(p-1)\right].$$

So far all generating functions were in one variable. For the curious mind, next we will look at an example involving a two variable generating function

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}a_{i,j}x^iy^j$$

of the simplest kind.

Example 7. An $a\times b$ rectangle can be tiled by a number of $p\times1$ and $1\times q$ types of rectangles, where a, b, p, q are fixed positive integers. Prove that a is divisible by p or b is divisible by q. (Here a $k\times1$ and a $1\times k$ rectangles are considered to be different types.)

Solution. Inside the (i,j) cell of the $a\times b$ rectangle, let us put the term x^iy^j for $i=1,2,\ldots,a$ and $j=1,2,\ldots,b$. The sum of the terms inside a $p\times1$ rectangle is

$$x^iy^j+\cdots+x^{i+p-1}y^j=(1+x+\cdots+x^{p-1})\,x^iy^j,$$

if the top cell is at (i,j), while the sum of the terms inside a $1\times q$ rectangle is

$$x^iy^j+\cdots+x^iy^{j+q-1}=x^iy^j\,(1+y+\cdots+y^{q-1}),$$

if the leftmost cell is at (i,j). Now take

$$x=e^{2\pi i/p} \text{ and } y=e^{2\pi i/q}.$$

Then both sums become 0. If the desired tiling is possible, then the total sum of all terms in the $a\times b$ rectangle would be

$$0=\sum_{i=1}^{a}\sum_{j=1}^{b}x^iy^j=xy\frac{(1-x^a)(1-y^b)}{(1-x)(1-y)}.$$

This implies that a is divisible by p or b is divisible by q.

For the readers who like to know more about generating functions, we recommend two excellent references:

T. Andreescu and Z. Feng, <u>A Path to Combinatorics for Undergraduates</u>, Birkhäuser, Boston, 2004.

M. Novaković, <u>Generating Functions</u>, The IMO Compendium Group, 2007 (www.imomath.com)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***March 7, 2009.***

Problem 316. For every positive integer $n > 6$, prove that in every n-sided convex polygon $A_1A_2\ldots A_n$, there exist $i \neq j$ such that

$$|\cos\angle A_i - \cos\angle A_j| < \frac{1}{2(n-6)}.$$

Problem 317. Find all polynomial $P(x)$ with integer coefficients such that for every positive integer n, 2^n-1 is divisible by $P(n)$.

Problem 318. In ΔABC, side BC has length equal to the average of the two other sides. Draw a circle passing through A and the midpoints of AB, AC. Draw the tangent lines from the centroid of the triangle to the circle. Prove that one of the points of tangency is the incenter of ΔABC.
(*Source: 2000 Chinese Team Training Test*)

Problem 319. For a positive integer n, let S be the set of all integers m such that $|m| < 2n$. Prove that whenever $2n+1$ elements are chosen from S, there exist three of them whose sum is 0.
(*Source: 1990 Chinese Team Training Test*)

Problem 320. For every positive integer $k > 1$, prove that there exists a positive integer m such that among the rightmost k digits of 2^m in base 10, at least half of them are 9's.
(*Source: 2005 Chinese Team Training Test*)

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Solutions

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Problem 311. Let $S = \{1,2,\ldots,2008\}$. Prove that there exists a function $f: S \to$ {red, white, blue, green} such that there does not exist a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$.

Solution 1. **Kipp JOHNSON** (Valley Catholic School, teacher, Beaverton, Oregon, USA) and **PUN Ying Anna** (HKU Math, Year 3).

The number of 10-term arithmetic progressions in S is the same as the number of ordered pairs (a,d) such that a, d are in S and $a+9d \le 2008$. Since $d \le 2007/9=223$ and for each such d, a goes from 1 to $2008-9d$, so there are at most

$$4^{(2008-10)} \times 4 \times \sum_{d=1}^{223}(2008-9d)$$

$$= 4^{1999} \times 223000$$

functions $f: S \to$ {red, white, blue, green} such that there exists a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$, while there are more (namely 4^{2008}) functions from S to {red, white, blue, green}. So the desired function exists.

Solution 2. **G.R.A. 20 Problem Solving Group** (Roma, Italy).

Replace red, white, blue, green by 0, 1, 2, 3 respectively. It can be seen by a direct checking that $f: \{1,2,\ldots,2048\} \to \{0,1,2,3\}$ given by

$$f(n) = \left[\frac{n-1}{8}\right]_{\text{mod } 2} + 2\left[\frac{n-1}{128}\right]_{\text{mod } 2}$$

avoids any 9-term arithmetic progression having the same value (where $k_{\text{mod } 2}$ is 0 if k is even and 1 if k is odd). The range of f is $((0^8 1^8)^8 (2^8 3^8)^8)^8$, where for any string x, x^8 denotes the string obtained by putting eight copies of the string x one after another in a row and $f(n)$ is the n-th digit in the specified string.

Commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Problem 312. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2} \ge 48.$$

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **Kipp JOHNSON** (Valley Catholic School, teacher, Beaverton, Oregon, USA), **Kelvin LEE** (Trinity College, University of Cambridge, Year 2), **LEUNG Kai Chung** (HKUST Math, Year 2), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery), **MA Ka Hei** (Wah Yan College, Kowloon), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **PUN Ying Anna** (HKU Math, Year 3).

Let $x = a + 1$, $y = b + 1$ and $z = c + 1$. Applying the *AM-GM* inequality twice, we have

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2}$$

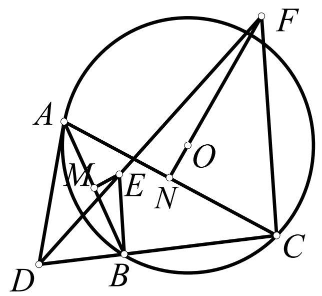
$$= \frac{(a+1)^4}{b^2} + \frac{(b+1)^4}{c^2} + \frac{(c+1)^4}{a^2}$$

$$\ge 3\left(\frac{(a+1)^4(b+1)^4(c+1)^4}{a^2b^2c^2}\right)^{1/3}$$

$$\ge 3\left(\frac{(2\sqrt{a})^4(2\sqrt{b})^4(2\sqrt{c})^4}{a^2b^2c^2}\right)^{1/3} = 48.$$

Commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 5), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6), **Paolo PERFETTI** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Dimitar TRENEVSKI** (Yahya Kemal College, Skopje, Macedonia) and **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College, Form 6).

Problem 313. In ΔABC, $AB < AC$ and O is its circumcenter. Let the tangent at A to the circumcircle cut line BC at D. Let the perpendicular lines to line BC at B and C cut the perpendicular bisectors of sides AB and AC at E and F respectively. Prove that D, E, F are collinear.



Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **CHUNG Ping Ngai** (La Salle College, Form 5), **Kelvin LEE** (Trinity College, University of Cambridge, Year 2), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6) and **PUN Ying Anna** (HKU Math, Year 3).

Let M be the midpoint of AB and N be the midpoint of AC. Using $\angle ABE = \angle ABC - 90°$, $\angle FCA = 90° - \angle ABC$ and the sine law, we have

$$\frac{BE}{CF}=\frac{BM/\cos\angle ABE}{CN/\cos\angle FCA}$$

$$=\frac{\frac{1}{2}AB/\sin\angle ABC}{\frac{1}{2}AC/\sin\angle BCA}=\frac{AB^2}{AC^2}.$$

From $\Delta DCA \sim \Delta DAB$, we see

$$\frac{DA}{DC}=\frac{DB}{DA}=\frac{\sin\angle DAB}{\sin\angle DBA}=\frac{\sin\angle ACB}{\sin\angle ABC}=\frac{AB}{AC}.$$

So

$$\frac{BE}{CF}=\frac{AB^2}{AC^2}=\frac{DA}{DC}\cdot\frac{DB}{DA}=\frac{DB}{DC}.$$

Then $\angle BDE=\angle CDF$. Therefore D,E,F are collinear.

Commended solvers: **Stefan LOZANOVSKI and Bojan JOVESKI** (Private Yahya Kemal College, Skopje, Macedonia).

Problem 314. Determine all positive integers x, y, z satisfying $x^3-y^3=z^2$, where y is a prime, z is not divisible by 3 and z is not divisible by y.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **PUN Ying Anna** (HKU Math, Year 3).

Suppose there is such a solution. Then

$$z^2=x^3-y^3=(x-y)(x^2+xy+y^2)$$
$$=(x-y)\,((x-y)^2+3xy). \qquad (*)$$

Since y is a prime, z is not divisible by 3 and z is not divisible by y, (*) implies $(x,y)=1$ and $(x-y,3)=1$. Then

$$(x^2+xy+y^2,\,x-y)=(3xy,\,x-y)=1. \qquad (**)$$

Now (*) and (**) imply

$$x-y=m^2,\ x^2+xy+y^2=n^2 \text{ and } z=mn$$

for some positive integers m and n. Consequently,

$$4n^2=4x^2+4xy+4y^2=(2x+y)^2+3y^2.$$

Then $3y^2=(2n+2x+y)(2n-2x-y)$. Since y is prime, there are 3 possibilities:

(1) $2n+2x+y=3y^2,\ 2n-2x-y=1$
(2) $2n+2x+y=3y,\ 2n-2x-y=y$
(3) $2n+2x+y=y^2,\ 2n-2x-y=3$.

In (1), subtracting the equations leads to $3y^2-1=2(2x+y)=2(2m^2+3y)$. Then

$$m^2+1=3y^2-6y-3m^2\equiv 0 \pmod 3).$$

However, $m^2+1\equiv 1$ or $2 \pmod 3$. We get a contradiction.

In (2), subtracting the equations leads to $x=0$, contradiction.

In (3), subtracting the equations leads to $y^2-3=2(2x+y)=2(2m^2+3y)$, which can be rearranged as $(y-3)^2-4m^2=12$. This leads to $y=7$ and $m=1$. Then $x=8$ and $z=13$. Since $8^3-7^3=13^2$, this gives the only solution.

Commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Problem 315. Each face of 8 unit cubes is painted white or black. Let n be the total number of black faces. Determine the values of n such that in every way of coloring n faces of the 8 unit cubes black, there always exists a way of stacking the 8 unit cubes into a $2\times2\times2$ cube C so the numbers of black squares and white squares on the surface of C are the same.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **PUN Ying Anna** (HKU Math, Year 3).

The answer is $n=23$ or 24 or 25. First notice that if n is a possible value, then so is $48-n$. This is because we can interchange all the black and white coloring and the condition can still be met by symmetry. Hence, without loss of generality, we may assume $n\le 24$.

For the 8 unit cubes, there are totally 24 pairs of opposite faces. In each pair, no matter how the cubes are stacked, there is one face on the surface of C and one face hidden.

If $n\le 22$, there is a coloring that has $[n/2]$ pairs with both opposite faces black. Then at least $[n/2]$ black faces will be hidden so that there can be at most $n-[n/2]\le 11$ black faces on the surface of C. This contradicts the existence of a stacking with 12 black and 12 white squares on the surface of C. So only $n=23$ or 24 is possible.

Next, start with an arbitrary stacking. Let b be the number of black squares on the surface of C. For each of the 8 unit cubes, take an axis formed by the centers of a pair of opposite faces and rotate the cube about that axis by 90°. Then take an axis formed by the centers of another pair of opposite faces of the same cube and rotate the cube about the axis by 90° twice. These three 90° rotations switch the three exposed faces with the three hidden faces. So after doing the twenty-four 90° rotations for the 8 unit cubes, there will be $n-b$ black squares on the surface of C.

For $n=23$ or 24 and $b\le n$, the average of b and $n-b$ is 11.5 or 12, hence 12 is between b and $n-b$ inclusive.

Finally, observe that after each of the twenty-four 90° rotations, one exposed square will be hidden and one hidden square will be exposed. So the number of black squares on the surface of C can only increase by one, stay the same or decrease by one.

Therefore, at a certain moment, there will be exactly 12 black squares (and 12 white squares) on the surface of C.

Commended solvers: **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Olympiad Corner

(continued from page 1)

Second Day: June 7th, 2008

Problem 4. Determine all functions f mapping the set of positive integers to the set of non-negative integers satisfying the following conditions:

(1) $f(mn)=f(m)+f(n)$,
(2) $f(2008)=0$, and
(3) $f(n)=0$ for all $n\equiv 39 \pmod{2008}$.

Problem 5. Which positive integers are missing in the sequence $\{a_n\}$, with

$$a_n=n+[\sqrt{n}]+[\sqrt[3]{n}]$$

for all $n\ge 1$? ($[x]$ denotes the largest integer less than or equal to x, i.e. g with $g\le x<g+1$.)

Problem 6. We are given a square $ABCD$. Let P be a point not equal to a corner of the square or to its center M. For any such P, we let E denote the common point of the lines PD and AC, if such a point exists. Furthermore, we let F denote the common point of the lines PC and BD, if such a point exists.

All such points P, for which E and F exist are called acceptable points. Determine the set of all acceptable points, for which the line EF is parallel to AD.

Mathematical Excalibur

Volume 14, Number 1 *March - April, 2009*

Olympiad Corner

The following were the problems of the 2009 Asia-Pacific Math Olympiad.

Problem 1. Consider the following operation on positive real numbers written on a blackboard: *Choose a number r written on the blackboard, erase that number, and then write a pair of real numbers a and b satisfying the condition* $2r^2 = ab$ *on the board.*

Assume that you start out with just one positive real number r on the blackboard, and apply this operation k^2-1 times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2+1}+\frac{a_2}{k^2+2}+\frac{a_3}{k^2+3}+\frac{a_4}{k^2+4}+\frac{a_5}{k^2+5}=\frac{1}{k^2}$$

for $k = 1,2,3,4,5$. Find the value of

$$\frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}.$$

(Express the value in a single fraction.)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 7, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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A Nice Identity

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There are many methods to prove inequalities. In this paper, we would like to introduce to the readers some applications of a nice identity for solving inequalities.

Theorem 0. Let a, b, c be real numbers. Then

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc.$$

Proof. This follows immediately by expanding both sides.

Corollary 1. Let a, b, c be real numbers. If $abc = 1$, then

$$(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-1.$$

Corollary 2. Let a, b, c be real numbers. If $ab + bc + ca = 1$, then

$$(a+b)(b+c)(c+a)=a+b+c-abc\ .$$

Next we will give some applications of these facts. The first example is a useful well-known inequality.

Example 1. Let a, b, c be nonnegative real numbers. Prove that

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca).$$

Solution. By the *AM-GM* inequality,

$$\frac{1}{9}(a+b+c)(ab+bc+ca)-abc$$
$$\ge\frac{1}{9}(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2})-abc=0.$$

Using Theorem 0, we have

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca).$$

The next example was a problem on the third team selection test of Romania for the Balkan Mathematical Olympiad 2005. Subsequently, it also appeared in the Croatian Team Selection Test 2006.

Example 2. (*Cezar Lupu, Romania 2005; Croatia TST 2006*) Let a, b, c be positive real numbers satisfying $(a+b)(b+c)(c+a) = 1$. Prove that

$$ab+bc+ca \le 3/4.$$

Solution. By the *AM-GM* inequality,

$$a+b+c=\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$
$$\ge 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}=\frac{3}{2}$$

and

$$abc=\sqrt{ab}\sqrt{bc}\sqrt{ca}$$
$$\le\frac{(a+b)(b+c)(c+a)}{8}=\frac{1}{8}.$$

Using Theorem 0, we get

$$1=(a+b)(b+c)(c+a)$$
$$=(a+b+c)(ab+bc+ca)-abc$$
$$\ge\frac{3}{2}(ab+bc+ca)-\frac{1}{8}.$$

Hence $ab+bc+ca\le\dfrac{3}{4}.$

The following example was taken from the Vietnamese magazine, Mathematics and Youth Magazine.

Example 3. (*Proposed by Tran Xuan Dang*) Let a, b, c be nonnegative real numbers satisfying $abc = 1$. Prove that

$$(a+b)(b+c)(c+a)\ge 2(1+a+b+c).$$

Solution. Using Corollary 1, this is equivalent to

$$(a+b+c)(ab+bc+ca-2)\ge 3.$$

We can obtain this by the *AM-GM* inequality as follows:

$$(a+b+c)(ab+bc+ca-2)$$
$$\ge(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2}-2)=3.$$

The inequality in the next example is very hard. It was a problem in the Korean Mathematical Olympiad.

Example 4. (*KMO Winter Program Test 2001*) Let a, b, c be positive real numbers. Prove that

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)}$$
$$\ge abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}.$$

Solution. Dividing by abc, the given inequality becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)}$$
$$\geq 1+\sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x = a/b$, $y = b/c$ and $z = c/a$, we get $xyz = 1$. It takes the form

$$\sqrt{(x+y+z)(xy+yz+zx)}$$
$$\geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

Using Corollary 1, the previous inequality becomes

$$\sqrt{(x+y)(y+z)(z+x)+1}$$
$$\geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

Setting $t=\sqrt[3]{(x+y)(y+z)(z+x)}$, we need to prove that

$$\sqrt{t^3+1}\geq 1+t.$$

By the *AM-GM* inequality, we have

$$t=\sqrt[3]{(x+y)(y+z)(z+x)}$$
$$\geq \sqrt[3]{2\sqrt{xy}\,2\sqrt{yz}\,2\sqrt{zx}}=2.$$

Therefore,

$$\sqrt{t^3+1}=\sqrt{(t+1)(t^2-t+1)}$$
$$\geq \sqrt{(t+1)(2t-t+1)}=1+t.$$

In the next example, we will see a nice inequality. It was from a problem in the 2001 USA Math Olympiad Summer Program.

Example 5. (*MOSP 2001*) Let a, b, c be positive real numbers satisfying abc=1. Prove that

$$(a+b)(b+c)(c+a)\geq 4(a+b+c-1).$$

Solution. Using Corollary 1, it suffices to prove that

$$(a+b+c)(ab+bc+ca)-1$$
$$\geq 4(a+b+c-1)$$

or $$ab+bc+ca+\frac{3}{a+b+c}\geq 4.$$

We will use the inequality

$$(x+y+z)^2\geq 3(xy+yz+zx), \qquad (*)$$

which after expansion and cancelling common terms amounts to

$$x^2+y^2+z^2-xy-yz-zx$$
$$=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)\geq 0.$$

Using (*), it is easy to see that

$$(ab+bc+ca)^2\geq 3(ab\cdot bc+bc\cdot ca+ca\cdot ab)$$
$$=3(a+b+c). \qquad (**)$$

By the *AM-GM* inequality and (**),

$$ab+bc+ca+\frac{3}{a+b+c}$$
$$=3\left(\frac{ab+bc+ca}{3}\right)+\frac{3}{a+b+c}$$
$$\geq 4\sqrt[4]{\frac{3(ab+bc+ca)^3}{3^3(a+b+c)}}$$
$$\geq 4\sqrt[4]{\frac{3(3\sqrt[3]{a^2b^2c^2})(3(a+b+c))}{3^3(a+b+c)}}=4.$$

Next, we will show some nice trigonometric inequalities can also be proved using Theorem 0.

Example 6. For a triangle ABC, prove that

(i) $\sin A+\sin B+\sin C\leq 3\sqrt{3}/2.$

(ii) $\cos A+\cos B+\cos C\leq 3/2.$

Solution. By the substitutions

$a = \tan(A/2)$, $b = \tan(B/2)$, $c = \tan(C/2)$,

we get $ab+bc+ca = 1$.

Using the facts $\sin 2x= (2\tan x) / (1+\tan^2 x)$ and $1 + a^2 = a^2 + ab + bc + ca = (a+b)(a+c)$, inequality (i) becomes

$$\frac{a}{1+a^2}+\frac{b}{1+b^2}+\frac{c}{1+c^2}\leq\frac{3\sqrt{3}}{4},$$

which is the same as

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\leq\frac{3\sqrt{3}}{4}.$$

Clearing the denominators, this simplifies to $(a+b)(b+c)(c+a)\geq 8\sqrt{3}/9$.

To prove this, use the *AM-GM* inequality to get

$$1=ab+bc+ca\geq 3\sqrt[3]{a^2b^2c^2},$$

which is

$$abc\leq\sqrt{3}/9. \qquad (***)$$

Next, by (*),

$$a+b+c\geq\sqrt{3(ab+bc+ca)}=\sqrt{3}. \qquad (****)$$

Finally, by Corollary 2,

$$(a+b)(b+c)(c+a)=a+b+c-abc$$
$$\geq\sqrt{3}-\frac{\sqrt{3}}{9}=\frac{8\sqrt{3}}{9}.$$

Next, using $\cos 2x=(1-\tan^2 x)/(1+\tan^2 x)$, inequality (ii) becomes

$$\frac{1-a^2}{1+a^2}+\frac{1-b^2}{1+b^2}+\frac{1-c^2}{1+c^2}\leq\frac{3}{2}.$$

Using $1 + a^2 = a^2 + ab + bc + ca = (a+b)(a+c)$ in the denominators, doing the addition on the left and applying Corollary 2 in the common denominator, we can see the above inequality is the same as

$$\frac{2(a+b+c)-[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{a+b+c-abc}\leq\frac{3}{2}.$$

Observe that $a^2(b+c)+b^2(c+a)+c^2(a+b) = (a+b+c)(ab+bc+ca)-3abc = a+b+c - 3abc$. So the inequality becomes

$$\frac{2(a+b+c)-(a+b+c-3abc)}{a+b+c-abc}\leq\frac{3}{2},$$

which simplifies to $a+b+c \geq 9abc$. This follows easily from (***) and (****).

Finally, we have some exercises for the readers.

Exercise 1. (*Due to Nguyen Van Ngoc*) Let a, b, c be positive real numbers. Prove that

$$abc(a+b+c)\leq\frac{3\left((a+b)(b+c)(c+a)\right)^{4/3}}{16}.$$

Exercise 2. (*Due to Vedula N. Murty*) Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+c}{3}\leq\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}}.$$

Exercise 3. (*Carlson's inequality*) Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}\geq\sqrt{\frac{ab+bc+ca}{3}}.$$

Exercise 4. Let ABC be a triangle. Prove that

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\sqrt{3}.$$

References

[1] Hojoo Lee, Topics in Inequalities -Theorems and Techniques, 2007.

[2] Pham Kin Hung, Secrets in Inequalities (in Vietnames), 2006.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***May 7, 2009.***

Problem 321. Let *AA', BB'* and *CC'* be three non-coplanar chords of a sphere and let them all pass through a common point *P* inside the sphere. There is a (unique) sphere S_1 passing through *A, B, C, P* and a (unique) sphere S_2 passing through *A', B', C', P*.

If S_1 and S_2 are externally tangent at *P*, then prove that *AA'=BB'=CC'*.

Problem 322. (*Due to Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam*) Let *a, b, c* be positive real numbers satisfying the condition $a+b+c=3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2.$$

Problem 323. Prove that there are infinitely many positive integers *n* such that 2^n+2 is divisible by *n*.

Problem 324. *ADPE* is a convex quadrilateral such that $\angle ADP = \angle AEP$. Extend side *AD* beyond *D* to a point *B* and extend side *AE* beyond *E* to a point *C* so that $\angle DPB = \angle EPC$. Let O_1 be the circumcenter of ΔADE and let O_2 be the circumcenter of ΔABC.

If the circumcircles of ΔADE and ΔABC are not tangent to each other, then prove that line O_1O_2 bisects line segment *AP*.

Problem 325. On a plane, *n* distinct lines are drawn. A point on the plane is called a *k-point* if and only if there are exactly *k* of the *n* lines passing through the point. Let $k_2, k_3, \ldots, k_n$ be the numbers of 2-points, 3-points, …, *n*-points on the plane, respectively.

Determine the number of regions the *n* lines divided the plane into in terms of $n, k_2, k_3, \ldots, k_n$.

(*Source: 1998 Jiangsu Province Math Competition*)

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Solutions

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Problem 316. For every positive integer $n>6$, prove that in every *n*-sided convex polygon $A_1A_2\ldots A_n$, there exist $i\neq j$ such that

$$|\cos\angle A_i-\cos\angle A_j|<\frac{1}{2(n-6)}.$$

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).

Note the sum of all angles is

$$(n-2)180^\circ = 6\times 120^\circ+(n-6)180^\circ.$$

So there are at most five angles less than 120°. The remaining angles are in [120°, 180°) and their cosines are in (−1,−1/2]. Divide (−1,−1/2] into *n*−6 left open, right closed intervals with equal length. By the pigeonhole principle, there exist two of the cosines in the same interval, which has length equal to $1/(2n-12)$. The desired inequality follows.

Problem 317. Find all polynomial $P(x)$ with integer coefficients such that for every positive integer *n*, 2^n-1 is divisible by $P(n)$.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).

First we prove a fact: for all integers *p* and *n* and all polynomials $P(x)$ with integer coefficients, *p* divides $P(n+p)-P(n)$. To see this, let $P(x)=a_kx^k+\cdots+a_0$. Then

$$P(n+p)-P(n)=\sum_{i=1}^{k}a_i\left[(n+p)^i-n^i\right]$$

$$=\sum_{i=1}^{k}a_i p\left[\sum_{j=0}^{i-1}(n+p)^j n^{i-1-j}\right].$$

Now we claim that the only polynomials $P(x)$ solving the problem are the constant polynomials 1 and −1.

Assume $P(x)$ is such a polynomial and $P(n)\neq\pm1$ for some integer $n>1$. Let *p* be a prime which divides $P(n)$, then *p* divides 2^n-1. So *p* is odd and $2^n\equiv 1 \pmod p$.

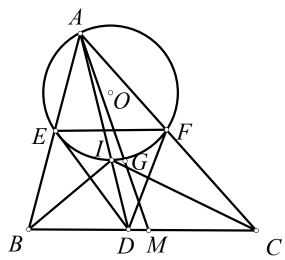
By the fact above, *p* also divides $P(n+p)-P(n)$. Hence, *p* divides $P(n+p)$. Since $P(n+p)$ divides $2^{n+p}-1$, *p* also divides $2^{n+p}-1$. Then $2^p\equiv 2^n2^p=2^{n+p}\equiv 1$ (mod *p*).

By Fermat's little theorem, $2^p\equiv 2$ (mod *p*). Thus, $1\equiv 2$ (mod *p*). This leads to *p* divides 2−1=1, which is a contradiction. Hence, $P(n)=1$ or −1 for every integer $n>1$. Then $P(x)-1$ or $P(x)+1$ has infinitely many roots, i.e. $P(x)\equiv 1$ or −1.

Comments: Two readers pointed out that this problem appeared earlier as Problem 252 in vol. 11, no. 2.

Problem 318. In ΔABC, side *BC* has length equal to the average of the two other sides. Draw a circle passing through *A* and the midpoints of *AB*, *AC*. Draw the tangent lines from the centroid of the triangle to the circle. Prove that one of the points of tangency is the incenter of ΔABC.
(*Source: 2000 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).



Let *G* be the centroid and *I* be the incenter of ΔABC. Let line *AI* intersect side *BC* at *D*. Let *E* and *F* be the midpoints of *AB* and *AC* respectively. Let *O* be the circumcenter of ΔAEF. Let *M* be the midpoint of side *BC*.

We claim *I* is the circumcenter of ΔDEF. To see this, note *I* is on line *AD*. So

$$\frac{DB}{2EB}=\frac{DB}{AB}=\frac{DI}{AI}=\frac{DC}{AC}=\frac{DC}{2FC}=\lambda.$$

Also, $DB+DC=BC=(AB+AC)/2=EB+FC=2\lambda(DB+DC)$ implies $\lambda=1/2$. Then *DB=EB* and *DC=FC*. So lines *BI* and *CI* are the perpendicular bisectors of *DE* and *DF* respectively.

Now we show *I* is on the circumcircle of ΔAEF. To see this, we compute

$\angle EIF = 2\angle EDB$
$= 2(180^\circ-\angle BDE-\angle CDF)$
$= (180^\circ-2\angle BDE)+(180^\circ-2\angle CDF)$
$= \angle DBE+\angle DCF$
$= 180^\circ-\angle EAF.$

Finally, we show $OI\perp IG$. Since *IE=IF*, $OI\perp EF$. Since $EF\parallel BC$, we just need to show $IG\parallel BC$, which follows from $DI/AI=1/2=MG/AG$.

Problem 319. For a positive integer *n*, let *S* be the set of all integers *m* such

that $|m| < 2n$. Prove that whenever $2n+1$ elements are chosen from S, there exist three of them whose sum is 0.
(*Source: 1990 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery) and **Fai YUNG**.

For $n = 1$, $S = \{-1,0,1\}$. If 3 elements are chosen from S, then they are $-1,0,1$, which have zero sum.

Suppose case n is true. For the case $n+1$, S is the union of $S'=\{m: -2n+1 \le m \le 2n-1\}$ and $S''=\{-2n-1,-2n,2n,2n+1\}$. Let T be a $2n+3$ element subset of S.

Case 1: (T contains at most 2 elements of S''). Then T contains $2n+1$ elements of S'. By case n, T has 3 elements with zero sum.

Case 2: (T contains exactly 3 elements of S''.) There are 4 subcases:

Subcase 1: ($\pm 2n$ and $2n+1$ are in T.) If 0 is in T, then $\pm 2n$ and 0 are in T with zero sum. If -1 is in T, then $2n+1$, $-2n$, -1 are in T with zero sum.

Otherwise, the other $2n$ numbers of T are among $1, \pm 2, \pm 3, \ldots, \pm(2n-1)$, which (after removing n) can be divided into the $2n-2$ pairs $\{1, 2n-1\}$, $\{2, 2n-2\}$, …, $\{n-1, n+1\}$, $\{-2, -2n+1\}$, $\{-3, -2n+2\}$, …, $\{-n, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n$ or $-2n-1$, we can add the pair to $-2n$ or $2n+1$ to get three numbers in T with zero sum.

Subcase 2: ($2n$ and $\pm(2n+1)$ are in T.) If 0 is in T, then $\pm(2n+1)$ and 0 are in T with zero sum. If 1 is in T, then $-2n-1$, $2n$, 1 are in T with zero sum.

Otherwise, the other $2n$ numbers of T are among $-1, \pm 2, \pm 3, \ldots, \pm(2n-1)$, which (after removing $-n$) can be divided into the $2n-2$ pairs $\{2, 2n-1\}$, $\{3, 2n-2\}$, …, $\{n, n+1\}$, $\{-1, -2n+1\}$, $\{-2, -2n+2\}$, …, $\{-n+1, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n+1$ or $-2n$, we can add the pair to $-2n-1$ or $2n$ to get three numbers in T with zero sum.

Subcase 3: ($\pm 2n$ and $-2n-1$ are in T.) This can be handled as in subcase 1.

Subcase 4: ($-2n$ and $\pm(2n+1)$ are in T.) This can be handled as in subcase 2.

Case 3: (T contains S''.) If 0 is in T, then $-2n$, $2n$, 0 are in T with zero sum. If 1 is in T, then $-2n-1$, $2n$, 1 are in T with zero sum. If -1 is in T, then $2n+1$, $-2n$, -1 are in T with zero sum.

Otherwise, the other $2n-1$ numbers of T are among $\pm 2, \pm 3, \ldots, \pm(2n-1)$, which can be divided into the $2n-2$ pairs $\{2, 2n-1\}$, $\{3, 2n-2\}$, …, $\{n, n+1\}$, $\{-2, -2n+1\}$, $\{-3, -2n+2\}$, …, $\{-n, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n+1$ or $-2n-1$, we can add the pair to $-2n-1$ or $2n+1$ to get three numbers in T with zero sum.

This completes the induction and we are done.

Problem 320. For every positive integer $k > 1$, prove that there exists a positive integer m such that among the rightmost k digits of 2^m in base 10, at least half of them are 9's.
(*Source: 2005 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **G.R.A. 20 Problem Solving Group** (Roma, Italy).

We claim $m=2\times 5^{k-1}+k$ works. Let $f(k)=2\times 5^{k-1}$. We check by induction that

$$2^{f(k)} \equiv -1 \pmod{5^k}. \qquad (*)$$

First $f(2)=10$, $2^{10}=1024 \equiv -1 \pmod{5^2}$. Next, suppose case k is true. Then $2^{f(k)} = -1 + 5^k n$ for some integer n. We get

$$\begin{aligned} 2^{f(k+1)} &= (-1+5^k n)^5 \\ &= \sum_{j=0}^{5}\binom{5}{j}(-1)^{5-j}5^{kj}n^j \\ &\equiv -1 \pmod{5^{k+1}}, \end{aligned}$$

completing the induction.

By (*), we get $2^m \equiv -2^k \pmod{5^k}$. Also, clearly $2^m \equiv 0 \equiv -2^k \pmod{2^k}$. Hence,

$$2^m \equiv -2^k \equiv 10^k - 2^k \pmod{10^k}.$$

This implies the k rightmost digits in base 10 of 2^m and $10^k - 2^k$ are the same. For $k > 1$, $2^k < 10^{(k-1)/2}$. So

$$10^k - 1 \ge 10^k - 2^k > 10^k - 10^{(k-1)/2}.$$

The result follows from the fact that the k-digit number $10^k - 10^{(k-1)/2}$ in base 10 has at least half of its digits are 9's on the left.

Olympiad Corner

(continued from page 1)

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points A_1, B_1, A_2, B_2, A_3, B_3 as follows: For each $i = 1,2,3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

Problem 4. Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

Problem 5. Larry and Bob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90˚ left turn after every l kilometer driving from the start; Rob makes a 90˚ right turn after every r kilometer driving from the start, where l and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (l, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

Mathematical Excalibur

Volume 14, Number 2 *May - September, 2009*

Olympiad Corner

The following were the problems of the first day of the 2008 Chinese Girls' Math Olympiad.

Problem 1. (a) Determine if the set $\{1,2,\cdots,96\}$ can be partitioned into 32 sets of equal size and equal sum.

(b) Determine if the set $\{1,2,\cdots,99\}$ can be partitioned into 33 sets of equal size and equal sum.

Problem 2. Let $\varphi(x) = ax^3+bx^2+cx+d$ be a polynomial with real coefficients. Given that $\varphi(x)$ has three positive real roots and that $\varphi(x) < 0$, prove that $2b^3 + 9a^2d - 7abc \le 0$.

Problem 3. Determine the least real number a greater than 1 such that for any point P in the interior of square $ABCD$, the area ratio between some two of the triangles PAB, PBC, PCD, PDA lies in the interval $[1/a, a]$.

Problem 4. Equilateral triangles ABQ, BCR, CDS, DAP are erected outside the (convex) quadrilateral $ABCD$. Let X, Y, Z, W be the midpoints of the segments PQ, QR, RS, SP respectively. Determine the maximum value of

$$\frac{XZ+YW}{AC+BD}.$$

Acknowledgment: Thanks to Elina Chiu, Dept. of Math., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 3, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Remarks on IMO 2009

Leung Tat-Wing

2009 IMO Hong Kong Team Leader

The 50th International Mathematical Olympiad (IMO) was held in Bremen, Germany from 10th to 22nd July 2009. I arrived Bremen amid stormy and chilly (16°*C*) weather. Our other team members arrived three days later. The team eventually obtained 1 gold, 2 silver and 2 bronze medals, ranked (unofficially) 29 out of 104 countries/regions. This was the first time more than 100 countries participated. Our team, though not among the strongest teams, did reasonably well. But here I mainly want to give some remarks about this year's IMO, before I forget.

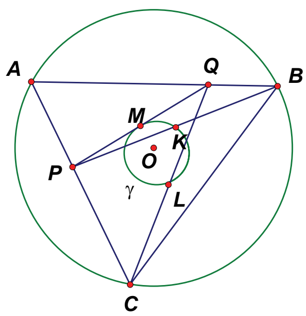
First, the problems of the contest:

Problem 1. Let n be a positive integer and let $a_1, a_2, \ldots, a_k$ $(k \ge 2)$ be distinct integers in the set $\{1,2,\ldots,n\}$ such that n divides $a_i(a_{i+1}-1)$ for $i=1,\ldots,k-1$. Prove that n does not divide $a_k(a_1-1)$.

This nice and easy number theory problem was the only number theory problem in the contest. Indeed it is not easy to find a sequence satisfying the required conditions, especially when k is close to n, or n is prime. Since adding the condition n divides $a_k(a_1-1)$ should be impossible, it was natural to prove the statement by contradiction. Clearly $2 \le k \le n$, and we have $a_1 \equiv a_1a_2 \pmod n$, $a_2 \equiv a_2a_3 \pmod n$, …, $a_{k-1} \equiv a_{k-1}a_k$ (mod n). The extra condition $a_k \equiv a_ka_1$ (mod n) would in fact "complete the circle". Now $a_1 \equiv a_1a_2 \pmod n$. Using the second condition, we get $a_1 \equiv a_1a_2 \equiv a_1a_2a_3 \pmod n$ and so on, until we get $a_1 \equiv a_1a_2\cdots a_k \pmod n$. However, in a circle every point is a starting point. So starting from a_2, using the second condition we have $a_2 \equiv a_2a_3 \pmod n$. By the third condition, we then have $a_2 \equiv a_2a_3a_4 \pmod n$. As now the circle is complete, we eventually have $a_2 \equiv a_2a_3\cdots a_ka_1 \pmod n$. Arguing in this manner we eventually have $a_1 \equiv a_2 \equiv \cdots \equiv a_k \pmod n$, which is of course a contradiction!

Problem 2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that PQ is tangent to the circle Γ. Prove that $OP=OQ$.

The nice geometry problem was supposed to be a medium problem, but it turned out it was easier than what the jury had thought. The trick was to understand the relations involved. A very nice solution provided by one of our members went as follows.



As $KM \| BQ$ (midpoint theorem), we have $\angle AQP = \angle QMK$. Since PQ is tangent to Γ, we have $\angle QMK = \angle MLK$ (angle of alternate segment). Therefore, $\angle AQP = \angle MLK$. By the same argument, we have $\angle APQ = \angle MKL$. Hence, $\Delta APQ \sim \Delta MKL$. Therefore,

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{2MK}{2ML} = \frac{BQ}{CP}.$$

This implies $AP \cdot PC = AQ \cdot QB$. But by considering the power of P with respect to the circle ABC, we have

$$\begin{aligned} AP \cdot PC &= (R+OP)(R-OP) \\ &= R^2 - OP^2, \end{aligned}$$

where R is the radius of the circumcircle of ΔABC.

Likewise,

$$AQ \cdot QB = (R+OQ)(R-OQ) = R^2 - OQ^2.$$

These force $OP^2 = OQ^2$, or $OP = OQ$, done!

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

This was one of the two hard problems (3 and 6). Fortunately, it turned out that it was still within reach.

One trouble is of course the notation. Of course, s_{s_1} stands for the s_1^{th} term of the s_i sequence and so on. Starting from an arithmetic progression (AP) with common difference d, then it is easy to check that both

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are APs with common difference d^2. The question is essentially proving the "converse". So the first step is to prove that the common differences of the two APs s_{s_i} and s_{s_i+1} are in fact the same, say s. It is not too hard to prove and is intuitively clear, for two lines of different slopes will eventually meet and cross each other, violating the condition of strictly increasing sequence. The next step is the show the difference between two consecutive terms of s_i is indeed $\sqrt{s}$, (thus s is a square). One can achieve this end by the method of descent, or max/min principle, etc.

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

This problem was also relatively easy. It is interesting to observe that an isosceles triangle can be the starting point of an IMO problem. With geometric software such as *Sketchpad*, one can easily see that $\angle CAB$ should be 60° or 90°. To prove the statement of the problem, one may either use synthetic method or coordinate method. One advantage of using the coordinate method is after showing the possible values of $\angle CAB$, one can go back to show these values do work by suitable substitutions. Some contestants lost marks either because they missed some values of $\angle CAB$ or forgot to check the two possible cases do work.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths a, $f(b)$ and $f(b+f(a)-1)$. (A triangle is *non-degenerate* if its vertices are not collinear.)

The Jury worried if the word <u>triangle</u> may be allowed to be degenerate in some places. But I supposed all our secondary school students would consider only non-degenerate triangles. This was a nice problem in functional inequality (triangle inequality). One proves the problem by establishing several basic properties of f. Indeed the first step is to prove $f(1)=1$, which is not entirely easy. Then one proceeds to show that f is injective and/or $f(f(x)) = x$, etc, and finally shows that the only possible function is the identity function $f(x) = x$ for all x.

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s=a_1+a_2+\cdots+a_n$. A grasshopper is to jump along the real axis, starting from the point O and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

It turned out that this problem was one of the most difficult problems in IMO history. Only three of the 564 contestants received full scores. (Perhaps it was second to problem 3 posed in IMO 2007, for which only 2 contestants received full scores.)

When I first read the solution provided by the Problem Committee, I felt I was reading a paper of analysis. Without reading the solution, of course I would say we could try to prove the problem by induction, as the cases of small n were easy. The trouble was how to establish the induction step. Later the Russians provided a solution by induction, by separating the problem into sub-cases min $M < a_n$ or min $M \geq a_n$, and then applying the principle-hole principle, etc judiciously to solve the problem. Terry Tao said (jokingly) that the six problems were easy. But in his blog, he admitted that he had spent sometime reading the problem and he even wrote an article about it (I have not seen the article.)

The two hard problems (3 and 6) were more combinatorial and/or algebraic in nature. I had a feeling that this year the Jury has been trying to avoid hard number theory problems, which were essentially corollaries of deep theorems (for example, IMO 2003 problem 6 by the Chebotarev density theorem or IMO 2008 problem 3 by a theorem of H. Iwaniec) or hard geometry problem using sophisticated geometric techniques (like IMO 2008 problem 6).

The Germans ran the program vigorously (obstinately). They had an organization (Bildung und Begabung) that looked after the entire event. They had also prepared a very detailed shortlist problem set and afterwards prepared very detailed marking schemes for each problem. The coordinators were very professional and they studied the problems well. Thus, there were not too many arguments about how many points should be awarded for each problem.

Three of the problems (namely 1, 2 and 4) were relatively easy, problems 3 and 5 were not too hard, so although problem 6 was hard, contestants still scored relatively high points. This explained why the cut-off scores were not low, 14 for bronze, 24 for silver and 32 for gold.

It might seem that we still didn't do the hard problems too well. But after I discussed with my team members, I found that they indeed had the potential and aptitude to do the hard problems. What may still be lacking are perhaps more sophisticated skills and/or stronger will to tackle such problems.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 3, 2009.***

Problem 326. Prove that $3^{4^5}+4^{5^6}$ is the product of two integers, each at least 10^{2009}.

Problem 327. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there is an even number of pieces on the black squares of the board.
(*Source: 1989 USSR Math Olympiad*)

Problem 328. *(Due to Tuan Le, Fairmont High School, Anaheim, Ca., USA)* Let $a,b,c>0$. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}$$
$$\geq \frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Problem 329. Let $C(n,k)$ denote the binomial coefficient with value $n!/(k!(n-k)!)$. Determine all positive integers n such that for all $k = 1, 2, \cdots, n-1$, we have $C(2n,2k)$ is divisible by $C(n,k)$.

Problem 330. In ΔABC, $AB = AC = 1$ and $\angle BAC = 90°$. Let D be the midpoint of side BC. Let E be a point inside segment CD and F be a point inside segment BD. Let M be the point of intersection of the circumcircles of ΔADE and ΔABF, other than A. Let N be the point of intersection of the circumcircle of ΔACE and line AF, other than A. Let P be the point of intersection of the circumcircle of ΔAMN and line AD, other than A. Determine the length of segment AP with proof.
(*Source: 2003 Chinese IMO team test*)

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Solutions

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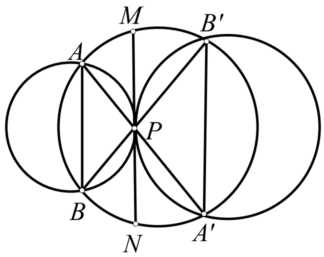
Problem 321. Let *AA', BB'* and *CC'* be three non-coplanar chords of a sphere and let them all pass through a common point *P* inside the sphere. There is a (unique) sphere S_1 passing through *A, B, C, P* and a (unique) sphere S_2 passing through *A', B', C', P*.

If S_1 and S_2 are externally tangent at *P*, then prove that *AA'=BB'=CC'*.

Solution. **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Jim Robert STUDMAN** (Hanford, Washington, USA).

Consider the intersection of the 3 spheres with the plane through *A, A', B, B'* and *P*.



Let *MN* be the common external tangent through *P* to the circle through *A, B, P* and the circle through *A', B'P* as shown above. We have ∠*ABP* = ∠*APM* = ∠*A'PN* = ∠*A'B'P* = ∠*A'B'B* = ∠*BAA'* = ∠*BAP*. Hence, *AP=BP*. Similarly, *A'P = B'P*. So *AA'* = *AP+A'P* = *BP+B'P* = *BB'*. Similarly, *BB'* = *CC'*.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6) and **LAM Cho Ho** (CUHK Math Year 1).

Problem 322. (*Due to Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam*) Let *a, b, c* be positive real numbers satisfying the condition $a+b+c = 3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2.$$

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and the proposer independently**.**

Observe that

$$\frac{a^2(b+1)}{a+b+ab}=a-\frac{ab}{a+b+ab}. \qquad (*)$$

Applying the AM-GM inequality twice, we have

$$\frac{ab}{a+b+ab}\leq\frac{ab}{3\sqrt[3]{a^2b^2}}=\frac{\sqrt[3]{ab}}{3}\leq\frac{a+b+1}{9}.$$

By (*), we have

$$\frac{a^2(b+1)}{a+b+ab}\geq a-\frac{a+b+1}{9}=\frac{8a-b-1}{9}.$$

Adding two other similar inequalities and using $a+b+c = 3$ on the right, we get the desired inequality.

Other commended solvers: **LAM Cho Ho** (CUHK Math Year 1), **Manh Dung NGUYEN** (Special High School for Gifted Students, HUS, Vietnam), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Stefan STOJCHEVSKI** (Yahya Kemal College, Skopje, Macedonia), **Jim Robert STUDMAN** (Hanford, Washington, USA) and **Dimitar TRENEVSKI** (Yahya Kemal College, Skopje, Macedonia).

Problem 323. Prove that there are infinitely many positive integers *n* such that 2^n+2 is divisible by *n*.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **LAM Cho Ho** (CUHK Math Year 1) and **WONG Ka Fai** (Wah Yan College Kowloon, Form 4).

We will prove the stronger statement that there are infinitely many positive even integers *n* such that 2^n+2 is divisible by *n* and also that 2^n+1 is divisible by $n-1$. Call such *n* a good number. Note $n = 2$ is good. Next, it suffices to prove that if *n* is good, then the larger integer $m = 2^n+2$ is also good.

Suppose *n* is good. Since *n* is even and $m = 2^n+2$ is twice an odd integer, so $m = nj$ for some odd integer *j*. Also, the odd integer $m-1 = 2^n+1 = (n-1)k$ for some odd integer *k*. Using the factorization $a^i+1 = (a+1)(a^{i-1}-a^{i-2}+\cdots+1)$ for positive odd integer *i*, we see that

$$\begin{aligned}2^m+2 &= 2(2^{(n-1)k}+1)\\ &= 2(2^{n-1}+1)\,(2^{(n-1)(k-1)}-\cdots+1)\end{aligned}$$

is divisible by $2(2^{n-1}+1) = m$ and

$$2^m+1 = 2^{nj}+1 = (2^n+1)(2^{n(j-1)}-\cdots+1)$$

is divisible by $2^n+1 = m-1$. Therefore, *m* is also good.

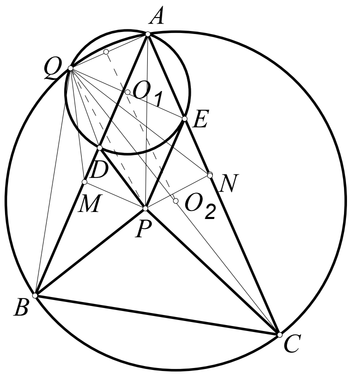
Problem 324. *ADPE* is a convex quadrilateral such that ∠ *ADP* = ∠ *AEP*. Extend side *AD* beyond *D* to a point *B* and extend side *AE* beyond *E* to a point *C* so that ∠ *DPB* = ∠ *EPC*. Let O_1 be the circumcenter of *ΔADE* and let O_2 be the circumcenter of *ΔABC*.

If the circumcircles of *ΔADE* and *ΔABC* are not tangent to each other, then prove that line O_1O_2 bisects line segment AP.

Solution. **Jim Robert STUDMAN** (Hanford, Washington, USA).

Let the circumcircle of ΔADE and the circumcircle of ΔABC intersect at A and Q.

Observe that line O_1O_2 bisects chord AQ and $O_1O_2 \perp AQ$. Hence, line O_1O_2 bisects line segment AP will follow if we can show that $O_1O_2 \parallel PQ$, or equivalently that $PQ \perp AQ$.



Let points M and N be the feet of perpendiculars from P to lines AB and AC respectively. Since $\angle ANP = 90° = \angle AMP$, points A, N, P, M lie on a circle Γ with AP as diameter. We claim that $\angle MQN = \angle MAN$. This would imply Q is also on circle Γ, and we would have $PQ \perp AQ$ as desired.

Since we are given $\angle ADP = \angle AEP$, we get $\angle BDP = \angle CEP$. This combines with the given fact $\angle DPB = \angle EPC$ imply ΔDPB and ΔEPC are similar, which yields $DB/EC = DP/EP = DM/EN$.

Since A,E,D,Q are concyclic, we have

$$\begin{aligned}\angle BDQ &= 180° - \angle ADQ \\ &= 180° - \angle AEQ = \angle CEQ.\end{aligned}$$

This and $\angle DBQ = \angle ABQ = \angle ACQ = \angle ECQ$ imply ΔDQB and ΔEQC are similar. So we have $QD/QE = DB/EC$. Combining with the equation at the end of the last paragraph, we get

$$QD/QE = DM/EN.$$

Using ΔDQB and ΔEQC are similar, we get $\angle MDQ = \angle BDQ = \angle CEQ = \angle NEQ$. These imply ΔMDQ and ΔNEQ are similar. Then $\angle MQD = \angle NQE$.

Finally, for the claim, we now have

$$\begin{aligned}\angle MQN &= \angle MQD + \angle DQN \\ &= \angle NQE + \angle DQN \\ &= \angle DQE \\ &= \angle DAE \\ &= \angle MAN.\end{aligned}$$

Comments: Some solvers used a bit of homothety to simplify the proof.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6), **LAM Cho Ho** (CUHK Math Year 1), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 7).

Problem 325. On a plane, n distinct lines are drawn. A point on the plane is called a *k-point* if and only if there are exactly k of the n lines passing through the point. Let $k_2, k_3, \ldots, k_n$ be the numbers of 2-points, 3-points, …, n-points on the plane, respectively.

Determine the number of regions the n lines divided the plane into in terms of n, $k_2, k_3, \ldots, k_n$.
(*Source: 1998 Jiangsu Province Math Competition*)

Solution. **LAM Cho Ho** (CUHK Math Year 1).

Take a circle of radius r so that all intersection points of the n lines are inside the circle and none of the n lines is tangent to the circle. Now each line intersects the circle at two points. These $2n$ points on the circle are the vertices of a convex $2n$-gon (call it M) as we go around the circle, say clockwise. Let the n lines partition the interior of M into P_3 triangles, P_4 quadrilaterals, $\cdots$, P_j j-gons, $\cdots$. These polygonal regions are all convex since the angles of these regions, which were formed by intersecting at least two lines, are all less than 180°. By convexity, no two sides of any polygonal region are parts of the same line. So we have $P_j = 0$ for $j>3n$.

Consider the sum of all the angles of these regions partitioning M. On one hand, it is $180°(P_3+2P_4+3P_5+\cdots)$ by counting region by region. On the other hand, it also equals $360°(k_2+k_3+\cdots+k_n)+(2n-2)180°$ by counting all the angles around each vertices of the regions. Cancelling 180°, we get

$$P_3+2P_4+3P_5+\cdots=2(k_2+k_3+\cdots+k_n)+(2n-2).$$

Next, consider the total number of all the edges of these regions partitioned M (with each of the edges inside M counted twice). On one hand, it is $3P_3+4P_4+5P_5+\cdots$ by counting region by region. On the other hand, it is also $(4k_2+6k_3+\cdots 2nk_n)+4n$ by counting the number of edges around the k-points and around the vertices of M. The $4n$ term is due to the $2n$ edges of M and each vertex of M (being not a k-point) issues exactly one edge into the interior of M. So we have

$$3P_3+4P_4+5P_5+\cdots=4k_2+6k_3+\cdots 2nk_n+4n.$$

Subtracting the last two displayed equations, we can obtain

$$P_3+P_4+P_5+\cdots=k_2+2k_3+(n-1)k_n+n+1.$$

Finally, the number of regions these n lines divided the plane into is the limit case r tends to infinity. Hence, it is exactly $k_2+2k_3+\cdots+(n-1)k_n+n+1$.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6) and **YUNG Fai**.

Remarks on IMO 2009

(continued from page 2)

As I found out from the stronger teams (Chinese, Japanese, Korean, or Thai, etc.), they were obviously more heavily or vigorously trained. For instance, a Thai boy/girl had to go through more like 10 tests to be selected as a team member.

Another thing I learned from the meeting was several countries were interested to host the event (South-East Asia countries and Asia-Minor countries). In fact, one country is going to host three international competitions of various subjects in a row for three years. Apparently they think hosting these events is good for gifted education.

The first IMO was held in Romania in 1959. Throughout these 51 years, only one year IMO was not held (1980). To commemorate the fiftieth anniversary of IMO in 2009, six notable mathematicians related to IMO (B. Bollabas, T. Gowers, L. Lovasz, S. Smirnov, T. Tao and J. C. Yoccoz) were invited to talk to the contestants. Of course, Yoccoz, Gowers and Tao were Fields medalists. The afternoon of celebration then became a series of (rather) heavy lectures (not bad). They described the effects of IMOs on them and other things. The effect of IMO on the contestants is to be seen later, of course!

Mathematical Excalibur

Volume 14, Number 3 *October-November, 2009*

Olympiad Corner

The 2009 Czech-Polish-Slovak Math Competition was held on June 21-24. The following were the problems.

Problem 1. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$(1+yf(x))(1-yf(x+y))=1$$

for all $x,y \in \mathbb{R}^+$.

Problem 2. Given positive integers a and k, the sequence a_1, a_2, a_3, … is defined by $a_1=a$ and $a_{n+1}=a_n+k\rho(a_n)$, where $\rho(m)$ stands for the product of the digits of m in its decimal representation (e.g. $\rho(413)=12$, $\rho(308)=0$). Prove that there exist positive integers a and k such that the sequence a_1, a_2, a_3, … contains exactly 2009 different numbers.

Problem 3. Given ΔABC, let k be the excircle at the side BC. Choose any line p parallel to BC intersecting line segments AB and AC at points D and E. Denote by ℓ the incircle of ΔADE. The tangents from D and E to the circle k not passing through A intersect at P. The tangents from B and C to the circle ℓ not passing through A intersect at Q. Prove that the line PQ passes through a point independent of p.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 1, 2009***.

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Probabilistic Method

Law Ka Ho

Roughly speaking, the probabilistic method helps us solve combinatorial problems via considerations related to probability.

We know that among any 6 people, there exist 3 who know each other or 3 who don't know each other (we assume if A knows B, then B knows A). When 6 is replaced by 5, this is no longer true, as can be seen by constructing a counter-example. When the numbers get large, constructing counterexamples becomes difficult. In this case the probabilistic method helps.

Example 1. Show that among 2^{100} people, there do not necessarily exist 200 people who know each other or 200 people who don't know each other.

Solution. Assign each pair of people to be knowing each other or not by flipping a fair coin. Among a set of 200 people, the probability that they know each other or they don't know each other is thus $2\times 2^{-C_2^{200}}=2^{-19899}$. As there are $C_{200}^{2^{100}}$ choices of 200 people, the probability that there exist 200 people who know each other or 200 people who don't know each other is at most

$$C_{200}^{2^{100}}\times 2^{-19899}<\frac{(2^{100})^{200}}{200!}\times 2^{-19899}$$
$$=\frac{2^{101}}{200!}<1$$

Hence the probability for the non-existence of 200 people who know each other or 200 people who don't know each other is greater than 0, which implies the result.

Here we see that the general rationale is to show that in a random construction of an example, the probability that it satisfies what we want is positive, which means that there exists such an example. Clearly, the

Example 2. In each cell of a 100×100 table, one of the integers 1, 2, …, 5000 is written. Moreover, each integer appears in the table exactly twice. Prove that one can choose 100 cells in the table satisfying the three conditions below:

(1) Exactly one cell is chosen in each row.

(2) Exactly one cell is chosen in each column.

(3) The numbers in the cells chosen are pairwise distinct.

Solution. Take a random permutation a_1, …, a_{100} of $\{1, …, 100\}$ and choose the a_i-th cell in the i-th row. Such choice satisfies (1) and (2). For j = 1, …, 5000, the probability of choosing both cells written j is

$$\begin{cases} 0 & \text{they are in the same row or column} \\ \dfrac{1}{100}\times\dfrac{1}{99} & \text{otherwise} \end{cases}$$

Hence the probability that such choice satisfies (3) is at least

$$1-5000\times\frac{1}{100}\times\frac{1}{99}>0\,.$$

Of course, one can easily transform the above two probabilistic solutions to merely using counting arguments (by counting the number of 'favorable outcomes' instead of computing the probabilities), which is essentially the same. But a probabilistic solution is usually neater and more natural.

Another common technique in the probabilistic method is to compute the average (or expected value) – the total is the average times the number of items, and there exists an item which is as good as the average. These are illustrated in the next two examples.

Example 3. (*APMO 1998*) Let F be the set of all n-tuples (A_1 , A_2 , …, A_n) where each A_i , i = 1, 2, …, n, is a subset of {1, 2, …, 1998}. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1, A_2, \ldots, A_n)} |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Solution. (*Due to Leung Wing Chung, 1998 Hong Kong IMO team member*) Note that the set {1, 2, …, 1998} has 2^{1998} subsets because we may choose to include or not to include each of the 1998 elements in a subset. Hence there are altogether 2^{1998n} terms in the summation.

Now we compute the average value of each term. For i = 1, 2, …, 1998, i is an element of $A_1 \cup A_2 \cup \cdots \cup A_n$ if and only if i is an element of at least one of $A_1, A_2, \ldots, A_n$. The probability for this to happen is $1-2^{-n}$. Hence the average value of each term in the summation is $1998(1-2^{-n})$, and so the answer is $2^{1998n} \cdot 1998(1-2^{-n})$.

Example 4. In a chess tournament there are 40 players. A total of 80 games have been played, and every two players compete at most once. For certain integer n, show that there exist n players, no two of whom have competed. (Of course, the larger the n, the stronger the result.)

Solution 4.1. If we use a traditional counting approach, we can prove the case $n=4$. Assume on the contrary that among any 4 players, at least one match is played. Then the number of games played is at least $C_4^{40} \div C_2^{38} = 260$, a contradiction. Note that this approach cannot prove the $n=5$ case since $C_5^{40} \div C_3^{38} = 78 < 80$.

Solution 4.2. We use a probabilistic approach to prove the $n=5$ case. Randomly choose some players such that each player has probability 0.25 to be chosen. Then discard all players who had lost in a match with another chosen player. In this way no two remaining players have played with each other.

What is the average number of players left? On average $40 \times 0.25 = 10$ players would be chosen. For each match played, the probability that both players are chosen is 0.25^2 , so on average there are $80 \times 0.25^2 = 5$ matches played among the chosen players. After discarding the losers, the average number of players left is at least 5 (in fact greater than 5 since the losers could repeat). That means there exists a choice in which we obtain at least 5 players who have not played against each other.

(Note: if we replace 0.25 by p, then the average number of players left would be $40p - 80p^2 = 5 - 80(p-0.25)^2$ and this explains the choice of the number 0.25.)

Solution 4.3. This time we use another probabilistic approach to prove the $n=8$ case. (!!) We assign a random ranking to the 40 players, and we pick those who have only played against players with lower ranking. Note that in this way no two of the chosen players have competed.

Suppose the i-th player has played d_i games. Since 80 games have been played, we have $d_1 + d_2 + \cdots + d_{40} = 80 \times 2$. Also, the i-th player is chosen if and only if he is assigned the highest ranking among himself and the players with whom he has competed, and the probability for this to happen is $1/(d_i+1)$. Hence the average number of players chosen is

$$\begin{aligned}\frac{1}{d_1+1} + \cdots + \frac{1}{d_{40}+1} &\geq \frac{40^2}{(d_1+1)+\cdots+(d_{40}+1)} \\ &= \frac{40^2}{160+40} = 8\end{aligned}$$

Here we made use of the Cauchy- Schwarz inequality. This means there exists 8 players, no two of whom have competed.

Remark. Solution 4.3 is the best possible result. Indeed, we may divide the 40 players into eight groups of 5 players each. If two players have competed if and only if they are from the same group, then the number of games played will be $8 \times C_2^5 = 80$ and it is clear that it is impossible to find 9 players, no two of whom have competed.

The above example shows that the probabilistic method can sometimes be more powerful than traditional methods. We conclude with the following example, which makes use of an apparently trivial property of probability, namely the probability of an event always lies between 0 and 1.

Example 5. In a public examination there are n subjects, each offered in Chinese and English. Candidates may sit for as many (or as few) subjects as they like, but each candidate may only choose one language version for each subject. For any two different subjects, there exists a candidate sitting for different language versions of the two subjects. If there are at most 10 candidates sitting for each subject, determine the maximum possible value of n.

Solution. The answer is 1024. The following example shows that n = 1024 is possible. Suppose there are 10 candidates (numbered 1 to 10), each sitting for all 1024 subjects (numbered 0 to 1023). For student i, the j-th subject is taken in Chinese if the i-th digit from the right is 0 in the binary representation of j, and the subject is taken in English otherwise. In this way it is easy to check that the given condition is satisfied. (The answer along with the example is not difficult to get if one begins by replacing 10 with smaller numbers and then observe the pattern.)

To show that 1024 is the maximum, we randomly assign each candidate to be 'Chinese' or 'English'. Let E_j be the event 'all candidates in the j-th subject are sitting for the language version which matches their assigned identity'. As there are at most 10 candidates in each subject, we have the probability

$$P(E_j) \geq 2^{-10} = \frac{1}{1024}.$$

Since 'for any two different subjects, there exists a candidate sitting for different language versions of the two subjects', no two E_j may occur simultaneously. It follows that

$$\begin{aligned}&P(\text{at least one } E_j \text{ happens}) \\ &= P(E_1) + P(E_2) + \cdots + P(E_n) \\ &\geq \frac{n}{1024}\end{aligned}$$

But since the probability of an event is at most 1, the above gives $1 \geq \frac{n}{1024}$, so we have $n \leq 1024$ as desired!

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 1, 2009.***

Problem 331. For every positive integer n, prove that

$$\sum_{k=0}^{n-1}(-1)^k\cos^n(k\pi/n)=\frac{n}{2^{n-1}}.$$

Problem 332. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let BD bisect OC perpendicularly. On diagonal AC, choose the point P such that $PC=OC$. Let line BP intersect line AD and the circumcircle of $ABCD$ at E and F respectively. Prove that PF is the geometric mean of EF and BF in length.

Problem 333. Find the largest positive integer n such that there exist n 4-element sets $A_1, A_2, \ldots, A_n$ such that every pair of them has exactly one common element and the union of these n sets has exactly n elements.

Problem 334. (*Due to FEI Zhenpeng, Northeast Yucai School, China*) Let $x,y \in (0,1)$ and x be the number whose n-th digit after the decimal point is the n^n-th digit after the decimal point of y for all $n=1,2,3,\ldots$. Show that if y is rational, then x is rational.

Problem 335. (*Due to Ozgur KIRCAK, Yahya Kemal College, Skopje, Macedonia*) Find all $a\in\mathbb{R}$ for which the functional equation f: $\mathbb{R}\to\mathbb{R}$

$$f(x-f(y))=a(f(x)-x)-f(y)$$

for all $x, y \in\mathbb{R}$ has a unique solution.

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Solutions

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Problem 326. Prove that $3^{4^5}+4^{5^6}$ is the product of two integers, each at least 10^{2009}.

Solution. **CHAN Ho Lam Franco** (GT (Ellen Yeung) College, Form 3), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Manh Dung NGUYEN** (Hanoi University of Technology, Vietnam), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (GT(Ellen Yeung) College) and **Pedro Henrique O. PANTOJA** (UFRN, Brazil).

Let $a=3^{256}$ and $b=4^{3906}$. Then

$$\begin{aligned}3^{4^5}+4^{5^6}&=a^4+4b^4\\&=(a^4+4a^2b^2+4b^4)-4a^2b^2\\&=(a^2+2b^2+2ab)(a^2+2b^2-2ab).\end{aligned}$$

Note that $a^2+2b^2+2ab > a^2+2b^2-2ab > 2b^2-2ab = 2b(b-a) > b > 2^{7800} > (10^3)^{780} > 10^{2009}$. The result follows.

Problem 327. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there is an even number of pieces on the black squares of the board.

(*Source: 1989 USSR Math Olympiad*)

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM) and **YUNG Fai**.

Without loss of generality, we may assume the square in row 1, column 1 is not black. Then, for all $i, j = 1,2,\ldots,8$, the square in row i, column j is black if and only if $i+j\equiv 1 \pmod 2$. Since the pieces are in different columns, the position of the piece contained in the i-th row is in column $p(i)$, where p is some permutation of $\{1,2,\ldots,8\}$. Therefore, the number of pieces on the black squares in mod 2 is congruent to

$$\sum_{i=1}^{8}(i+p(i))=\sum_{i=1}^{8}i+\sum_{i=1}^{8}p(i)=72,$$

which is even.

Other commended solvers: **Abby LEE** (SKH Lam Woo Memorial Secondary School) and **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 328. *(Due to Tuan Le, Fairmont High School, Anaheim, Ca., USA)* Let $a,b,c>0$. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}$$
$$\geq\frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Solution 1. **Manh Dung NGUYEN** (Hanoi University of Technology, Vietnam), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam),

Below we will use the cyclic notation

$$\sum_{cyc}f(a,b,c)=f(a,b,c)+f(b,c,a)+f(c,a,b).$$

By the Cauchy-Schwarz inequality, we have $(a^3+b^3)(a+b)\geq(a^2+b^2)^2$. Using this, the left side is

$$\sum_{cyc}\frac{\sqrt{a^3+b^3}}{a^2+b^2}\geq\sum_{cyc}\frac{1}{\sqrt{a+b}}$$
$$=\frac{\sum_{cyc}\sqrt{(a+b)(b+c)}}{\sqrt{(a+b)(b+c)(c+a)}}.$$

So it suffices to show

$$\sum_{cyc}\sqrt{(a+b)(b+c)}\geq\frac{6(ab+bc+ca)}{a+b+c}.\quad(*)$$

First we claim that

$$(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca)$$

and $(a+b+c)^2\geq 3(ab+bc+ca)$.

These follow from

$$9(a+b)(b+c)(c+a)-8(a+b+c)(ab+bc+ca)$$
$$=a(b-c)^2+b(c-a)^2+c(a-b)^2\geq 0$$

and

$$(a+b+c)^2-3(ab+bc+ca)$$
$$=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\geq 0.$$

By the AM-GM inequality,

$$\sum_{cyc}\sqrt{(a+b)(b+c)}\geq 3\sqrt[3]{(a+b)(b+c)(c+a)}.$$

To get (*), it remains to show

$$(a+b+c)\sqrt[3]{(a+b)(b+c)(c+a)}\geq 2(ab+bc+ca).$$

This follows by cubing both sides and using the two inequalities in the claim to get

$$(a+b+c)^3(a+b)(b+c)(c+a)$$
$$\geq\frac{8}{9}(a+b+c)^4(ab+bc+ca)$$
$$\geq 8(ab+bc+ca)^3.$$

Solution 2. **LEE Ching Cheong** (HKUST, Year 1).

Due to the homogeneity of the original inequality, without loss of generality we may assume $ab+bc+ca=1$. Then

$(a+b)(b+c) = 1+b^2$. The inequality (*) in solution 1 becomes

$$\sum_{cyc}\sqrt{1+b^2} \geq \frac{6}{a+b+c}.$$

Observe that

$$\sqrt{1+x^2} \geq \frac{1}{2}\left(x-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}} = \frac{x+\sqrt{3}}{2},$$

which can be checked by squaring both sides and simplified to $(\sqrt{3}x-1)^2 \geq 0$ (or alternatively, $f(x) = \sqrt{1+x^2}$ is a convex function on $\mathbb{R}$ and $y = (x+\sqrt{3})/2$ is the equation of the tangent line to the graph of $f(x)$ at $(1/\sqrt{3}, 2/\sqrt{3})$.)

Now $(a+b+c)^2 \geq 3(ab+bc+ca)$ can be expressed as

$$\sum_{cyc} b = a+b+c \geq \sqrt{3}.$$

Using these, inequality (*) follows as

$$\sum_{cyc}\sqrt{1+b^2} \geq \frac{\sum_{cyc} b + 3\sqrt{3}}{2}$$

$$\geq 2\sqrt{3} \geq \frac{6}{a+b+c}.$$

Other commended solvers: **Salem MALIKIĆ** (Student, University of Sarajevo, Bosnia and Herzegovina) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 329. Let $C(n,k)$ denote the binomial coefficient with value $n!/(k!(n-k)!)$. Determine all positive integers n such that for all $k = 1, 2, \cdots, n-1$, we have $C(2n,2k)$ is divisible by $C(n,k)$.

Solution. **HUNG Ka Kin Kenneth** (Diocesan Boys' School).

For $n < 6$, we can check that $n = 1, 2, 3$ and 5 are the only solutions. For $n \geq 6$, we will show there are no solutions. Observe that after simplification,

$$\frac{C(2n,2k)}{C(n,k)} = \frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{(2k-1)(2k-3)\cdots 1}.$$

Let n be an even integer with $n \geq 6$. Then $n-1 \geq 5$. So $n-1$ has a <u>*prime*</u> factor $p \geq 3$. Now $1 < (p+1)/2 \leq n/2 < n-1$. Let $k = (p+1)/2$. Then $p = 2k-1$, but p is not a factor of $2n-1, 2n-3, \ldots, 2n-2k+1$ since the closest consecutive multiples of p are $2n-2k-1 = 2(n-1)-p$ and $2n-2 = 2(n-1)$. Hence, $C(2n, 2k)/C(n, k)$ is not an integer. So such n cannot a solution for the problem.

For an odd integer $n \geq 7$, we divide into three cases.

<u>*Case 1*</u> : ($n-1 \neq 2^a$ for all a=1,2,3,…) Then $n-1$ has a <u>*prime*</u> factor $p \geq 3$. We repeat the argument above.

<u>*Case 2*</u> : ($n-2 \neq 3^b$ for all b=1,2,3,…) Then $n-2$ has a <u>*prime*</u> factor $p \geq 5$. Now $1 < (p+1)/2 \leq n/2 < n-1$. Let $k = (p+1)/2$. Then $p=2k-1$, but p is not a factor of $2n-1, 2n-3, \ldots, 2n-2k+1$ since again $2n-2k-3 = 2(n-2)-p$ and $2n-4 = 2(n-2)$ are multiples of p. Hence, $C(2n,2k)/C(n,k)$ is not an integer.

<u>*Case 3*</u> : ($n-1 = 2^a$ and $n-2 = 3^b$ for some positive integers a and b) Then $2^a - 3^b = 1$. Consider mod 3, we see a is even, say $a = 2c$. Then

$$3^b = 2^a - 1 = 2^{2c} - 1 = (2^c-1)(2^c+1).$$

Since 2^c+1 and 2^c-1 have a difference of 2 and they are powers of 3 by unique prime factorization, we must have $c = 1$. Then $a = 2$ and $n = 5$, which contradicts $n \geq 7$.

Other commended solvers: **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **O Kin Chit Alex** (GT(Ellen Yeung) College).

Problem 330. In ΔABC, $AB = AC = 1$ and $\angle BAC = 90°$. Let D be the midpoint of side BC. Let E be a point inside segment CD and F be a point inside segment BD. Let M be the point of intersection of the circumcircles of ΔADE and ΔABF, other than A. Let N be the point of intersection of the circumcircle of ΔACE and line AF, other than A. Let P be the point of intersection of the circumcircle of ΔAMN and line AD, other than A. Determine the length of segment AP with proof.
(*Source: 2003 Chinese IMO team test*)

Official Solution.

We will show A, B, P, C are concyclic. (Then, by symmetry, AP is a diameter of the circumcircle of ΔABC. We see $\angle ABP = 90°$, $AB = 1$ and $\angle BAP = 45°$, which imply $AP = \sqrt{2}$.)

Consider inversion with center at A and $r = 1$. Let X^* denote the image of point X. Let the intersection of lines XY and WZ be denoted by $XY \cap WZ$. We have $B^* = B$ and $C^* = C$. The line BC is sent to the circumcircle ω of ΔABC. The points F, D, E are sent to the intersection points F^*, D^*, E^* of lines AF, AD, AE with ω respectively.

The circumcircles of ΔADE and ΔABF are sent to lines D^*E^* and BF^*. So $M^* = D^*E^* \cap BF^*$. Also, the circumcircle of ΔACE and line AF are sent to lines CE^* and AF^*. Hence, $N^* = CE^* \cap AF^*$. Next, the circumcircle of ΔAMN and line AD are sent to lines M^*N^* and AD^*. So, $P^* = M^*N^* \cap AD^*$.

Now D^*, E^*, C, B, F^*, A are six points on ω. By Pascal's theorem, $M^* = D^*E^* \cap BF^*$, $N^* = E^*C \cap F^*A$ and $D = CB \cap AD^*$ are collinear. Since $P^* = M^*N^* \cap AD^*$, we get $D = P^*$. Then $P = D^*$ and A, B, P, C are all on ω.

Olympiad Corner

(continued from page 1)

Problem 4. Given a circle k and its chord AB which is not a diameter, let C be any point inside the longer arc AB of k. We denote by K and L the reflections of A and B with respect to the axes BC and AC. Prove that the distance of the midpoints of the line segments KL and AB is independent of the location of point C.

Problem 5. The n-tuple of positive integers $a_1,\ldots,a_n$ satisfies the following conditions:

(i) $1 \leq a_1 < a_2 < \cdots < a_n \leq 50$;

(ii) for any n-tuple of positive integers $b_1,\ldots, b_n$, there exist a positive integer m and an n-tuple of positive integers $c_1,\ldots, c_n$ such that

$$mb_i = c_i^{a_i} \text{ for } i = 1,\ldots,n.$$

Prove that $n \leq 16$ and find the number of different n-tuples $a_1,\ldots, a_n$ satisfying the given conditions for $n = 16$.

Problem 6. Given an integer $n \geq 16$, consider the set

$$G = \{(x,y): x,y \in \{1,2,\ldots,n\}\}$$

consisting of n^2 points in the plane. Let A be any subset of G containing at least $4n\sqrt{n}$ points. Prove that there are at least n^2 convex quadrangles with all their vertices in A such that their diagonals intersect in one common point.

Mathematical Excalibur

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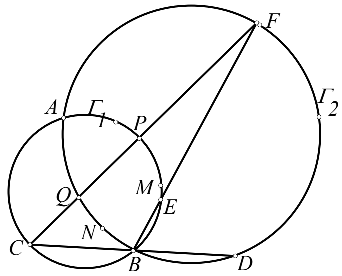
Olympiad Corner

The 2010 Chinese Mathematical Olympiad was held on January. Here are the problems.

Problem 1. As in the figure, two circles Γ_1, Γ_2 intersect at points A, B. A line through B intersects Γ_1, Γ_2 at C, D respectively. Another line through B intersects Γ_1, Γ_2 at E, F respectively. Line CF intersects Γ_1, Γ_2 at P, Q respectively. Let M, N be the midpoints of arcs PB, arc QB respectively. Prove that if $CD = EF$, then C, F, M, N are concyclic.



Problem 2. Let $k \geq 3$ be an integer. Sequence $\{a_n\}$ satisfies $a_k=2k$ and for all $n > k$, $a_n= a_{n-1} + 1$ if a_{n-1} and n are coprime and $a_n=2n$ if a_{n-1} and n are not coprime. Prove that the sequence $\{a_n-a_{n-1}\}$ contains infinitely many prime numbers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 17, 2010***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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A Refinement of Bertrand's Postulate

Neculai Stanciu

(Buzău, Romania)

In this article, we give an elementary demonstration of the famous Bertrand's postulate by using a theorem proved by the mathematician M. El Bachraoni in 2006.

Interesting is the distribution of prime numbers among the natural numbers and problems about their distributions have been stated in very simple ways, but they all turned out to be very difficult. The following *open problem* was stated by the Polish mathematician W. Sierpiński in 1958:

For all natural numbers $n > 1$ and $k \leq n$, there is at least one prime in the range $[kn,(k+1)n]$.

The case k=1 (known as Bertrand's postulate) was stated in 1845 by the French mathematician J. Bertrand and was proved by the Russian mathematician P. L. Chebysev. Simple proofs have been given by the Hungarian mathematician P. Erdos in 1932 and recently by the Romanian mathematician M. Tena [3]. The case k=2 was proved in 2006 by M. El Bachraoni (see [1]). His proof was relatively short and not too complicated. It is freely available on the internet [4].

Below we will present a refinement of Bertrand's postulate and it is perhaps the simplest demonstration of the postulate based on the following

Theorem 1. For any positive integer $n > 1$, there is a prime number between $2n$ and $3n$. (For the proof, see [1] or [4].)

The demonstration in [1] was typical of many theorems in number theory and was based on multiple inequalities valid for large values of n which can be calculated effectively. For the rest of the values of n, there are many basic improvisations, some perhaps difficult to follow.

Theorem 2. For $n \geq 1$, there is a prime number p such that $n < p < 3(n+1)/2$. (Since $3(n+1)/2<2n$ for $n > 3$, this is a refinement of the Bertrand's postulate.)

For the proof, the case n=1 follows from $1<p=2<3$. The case n=2 follows from $2<p=3<9/2$. For n even, say $n=2k$, by Theorem 1, we have a prime p such that $n=2k<p<3k<3(2k+1)/2=3(n+1)/2$. Similarly, for n odd, say $n=2k+1$, we have a prime p such that $n = 2k+1 < 2k+2=2(k+1) < p < 3(k+1)=3(n+1)/2$.

Concerning the distribution of prime numbers among the natural numbers, recently (in 2008) Rafael Jakimczuk has proved a formula (see [2] or [4]) for the n-th prime p_n, which provided a better error term than previous known approximate formulas for p_n. His formula is for $n\geq 4$,

$$p_n = n\log n + \log(n\log n)(n - Li(n\log n))$$
$$+\sum_{k=2}^{\infty}\frac{(-1)^k Q_{k-1}(\log(n\log n))}{k!n^{k-1}\log^{k-1} n}(n-Li(n\log n))^k$$
$$+O(h(n)),\text{ where}$$
$$Li(x)=\int_2^x \frac{dt}{\log t},\quad h(n)=\frac{n\log^2 n}{\exp(d\sqrt{\log n})}$$

and $Q_{k-1}(x)$ are polynomials.

References

[1] M. El Bachraoni, "*Primes in the Interval [2n,3n]*," Int. J. Contemp. Math. Sciences, vol. 1, 2006, no. 13, 617-621.

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[3] M. Tena, "*O demonstrație a postulatului lui Bertrand*," G. M.-B 10, 2008.

[4] http://www.m-hikari.com/ijcms.html

Max-Min Inequalities

Pedro Henrique O. Pantoja

(UFRN, NATAL, BRAZIL)

There are many inequalities. In this article, we would like to introduce the readers to some inequalities that involve maximum and minimum.

The first example was a problem from the Federation of Bosnia for Grade 1 in 2008.

Example 1 (*Bosnia-08*) For arbitrary real numbers x, y and z, prove the following inequality:

$$x^2+y^2+z^2-xy-yz-zx \geq \max\left\{\frac{3(x-y)^2}{4},\frac{3(y-z)^2}{4},\frac{3(z-x)^2}{4}\right\}.$$

Solution. Without loss of generality, suppose $x \geq y \geq z$. Then

$$\max\left\{\frac{3(x-y)^2}{4},\frac{3(y-z)^2}{4},\frac{3(z-x)^2}{4}\right\}=\frac{3}{4}(z-x)^2.$$

Let $a = x-y$, $b = y-z$ and $c = z-x$. Then $c = -(a+b)$. Hence, $(z-x)^2 = c^2 = (a+b)^2 = a^2+2ab+b^2$ and

$$\begin{aligned} x^2+y^2+z^2-xy-yz-zx &= \frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2] \\ &= \frac{1}{2}(a^2+b^2+a^2+2ab+b^2) \\ &= a^2+ab+b^2. \end{aligned}$$

So it suffices to show

$$a^2+ab+b^2 \geq \frac{3}{4}(a^2+2ab+b^2),$$

which is equivalent to $(a-b)^2 \geq 0$.

The next example was a problem on the 1998 Iranian Mathematical Olympiad.

Example 2. (Iran-98) Let a, b, c, d be positive real numbers such that $abcd=1$. Prove that

$$a^3+b^3+c^3+d^3 \geq \max\left\{a+b+c+d, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}.$$

Solution. It suffices to show

$$a^3+b^3+c^3+d^3 \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

and

$$a^3+b^3+c^3+d^3 \geq a+b+c+d.$$

For the first inequality, we observe that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{bcd+acd+abd+abc}{abcd} = bcd+acd+abd+abc.$$

Now, by the AM-GM inequality, we have $a^3+b^3+c^3 \geq 3abc$, $a^3+b^3+d^3 \geq 3abd$, $a^3+c^3+d^3 \geq 3acd$ and $b^3+c^3+d^3 \geq 3bcd$. Adding these four inequalities, we get the first inequality.

Next, let $S=a+b+c+d$. Then we have

$$S=a+b+c+d \geq 4(abcd)^{1/4}=4$$

by the AM-GM inequality and so $S^3 = S^2S \geq 16S$. The second inequality follows by applying the power mean inequality to obtain

$$\frac{a^3+b^3+c^3+d^3}{4} \geq \left(\frac{a+b+c+d}{4}\right)^3=\frac{S^3}{64} \geq \frac{S}{4}.$$

Example 3. Let a, b, c be positive real numbers. Prove that if $x = \max\{a,b,c\}$ and $y = \min\{a,b,c\}$, then

$$\frac{x}{y}+\frac{y}{x} \geq \frac{18abc}{(a+b+c)(a^2+b^2+c^2)}.$$

Solution. Suppose $a \geq b \geq c$. Then $x = a$ and $y = c$. Using the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{a}{c}+\frac{c}{a} &= \frac{a^2+c^2}{ac}=\frac{(a^2+c^2)b}{abc} \\ &\geq \frac{(2ac)b}{[(a+b+c)/3]^3}=\frac{54abc}{(a+b+c)^3} \\ &\geq \frac{54abc}{3(a^2+b^2+c^2)(a+b+c)}. \end{aligned}$$

The next example was problem 4 in the 2009 USA Mathematical Olympiad.

Example 4. (*USAMO-09*) For $n \geq 2$, let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \leq \left(n+\frac{1}{2}\right)^2.$$

Prove that

$$\max\{a_1,a_2,\ldots,a_n\} \leq 4\min\{a_1,a_2,\ldots,a_n\}.$$

Solution. Without loss of generality, we may assume

$$m=a_1 \leq a_2 \leq \cdots \leq a_n = M.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(n+\frac{1}{2}\right)^2 &\geq (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \\ &= (m+a_2+\cdots+M)\left(\frac{1}{M}+\frac{1}{a_2}+\cdots+\frac{1}{m}\right) \\ &\geq \left(\sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}}\right)^2. \end{aligned}$$

Taking square root of both sides,

$$n+\frac{1}{2} \geq \sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}}.$$

Simplifying, we get $2(m+M) \leq 5\sqrt{mM}$. Squaring both sides, we can get

$$4M^2-17mM+4m^2 \geq 0.$$

Factoring, we see

$$(4M-m)(M-4m) \geq 0.$$

Since $4M-m \geq 0$, we get $M-4m \geq 0$, which is the desired inequality.

The next example was problem 1 on the 2008 Greek National Math Olympiad.

Example 5. (*Greece-08*) For positive integers $a_1, a_2, \ldots, a_n$, prove that if $k=\max\{a_1,a_2,\ldots,a_n\}$ and $t=\min\{a_1,a_2,\ldots,a_n\}$, then

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \geq \prod_{i=1}^n a_i,$$

When does equality hold?

Solution. By the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^n a_i\right)^2 \leq \sum_{i=1}^n 1^2 \sum_{i=1}^n a_i^2 = n\sum_{i=1}^n a_i^2.$$

Hence,

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i} \geq \frac{\sum_{i=1}^n a_i}{n}.$$

Since each $a_i \geq 1$, the right side of the above inequality is at least one. Also, we have $kn/t \geq n$. So, applying the above inequality and the AM-GM inequality we have

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \geq \left(\frac{\sum_{i=1}^n a_i}{n}\right)^n \geq \prod_{i=1}^n a_i.$$

Equality holds if and only if all a_i's are equal.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 17, 2010.***

Problem 336. (*Due to Ozgur Kircak, Yahya Kemal College, Skopje, Macedonia*) Find all distinct pairs (x,y) of integers satisfying the equation

$$x^3 + 2009y = y^3 + 2009x.$$

Problem 337. In triangle ABC, $\angle ABC = \angle ACB = 40°$. P and Q are two points inside the triangle such that $\angle PAB = \angle QAC = 20°$ and $\angle PCB = \angle QCA = 10°$. Determine whether B, P, Q are collinear or not.

Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfies $a_0=1$, $b_0=0$ and for $n=0,1,2,\ldots$,

$$a_{n+1} = 7a_n + 6b_n - 3,$$
$$b_{n+1} = 8a_n + 7b_n - 4.$$

Prove that a_n is a perfect square for all $n=0,1,2,\ldots$

Problem 339. In triangle ABC, $\angle ACB = 90°$. For every n points inside the triangle, prove that there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$P_1P_2^2 + P_2P_3^2 + \cdots + P_{n-1}P_n^2 \le AB^2.$$

Problem 340. Let k be a given positive integer. Find the least positive integer N such that there exists a set of $2k+1$ distinct positive integers, the sum of all its elements is greater than N and the sum of any k elements is at most $N/2$.

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Solutions

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Problem 331. For every positive integer n, prove that

$$\sum_{k=0}^{n-1} (-1)^k \cos^n(k\pi/n) = \frac{n}{2^{n-1}}.$$

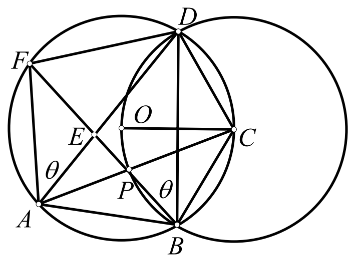
Solution. **Federico BUONERBA** (Università di Roma "Tor Vergata", Roma, Italy), **CHUNG Ping Ngai** (La Salle College, Form 6), **Ovidiu FURDUI** (Campia Turzii, Cluj, Romania), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Let $\omega = \cos(\pi/n) + i\sin(\pi/n)$. Then we have $\omega^n = -1$ and $(\omega^k + \omega^{-k})/2 = \cos(k\pi/n)$. So

$$\sum_{k=0}^{n-1} (-1)^k \cos^n(k\pi/n) = \sum_{k=0}^{n-1} \omega^{kn} \left(\frac{\omega^k + \omega^{-k}}{2}\right)^n$$
$$= \frac{1}{2^n}\sum_{k=0}^{n-1} \omega^{kn} \sum_{j=0}^{n} \binom{n}{j} \omega^{k(n-2j)}$$
$$= \frac{1}{2^n}\sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-1} (\omega^{2n-2j})^k$$
$$= \frac{1}{2^n}\left(\binom{n}{0}n + \binom{n}{n}n\right)$$
$$= \frac{n}{2^{n-1}}.$$

Problem 332. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let BD bisect OC perpendicularly. On diagonal AC, choose the point P such that $PC = OC$. Let line BP intersect line AD and the circumcircle of $ABCD$ at E and F respectively. Prove that PF is the geometric mean of EF and BF in length.

Solution. **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Abby LEE** (SKH Lam Woo Memorial Secondary School).



Since $PC=OC=BC$ and ΔBCP is similar to ΔAFP, we have $PF=AF$.

Next, $CB = CD = CP$ implies P is the incenter of ΔABD. Then BF bisects $\angle ABD$ yielding $\angle FAD = \angle ADF$, call it θ. (Alternatively, we have $\angle FAD = \angle PBD = ½\angle PCD$. Then

$$\angle AFD = 180° - \angle ACD$$
$$= 180° - \angle PCD$$
$$= 180° - 2\angle PBD$$
$$= 180° - 2\theta.$$

Hence, $\angle ADF = \theta$.) Also, we see $\angle AFE = \angle BFA$ and $\angle EAF = \theta = \angle ADF = \angle ABF$, which imply ΔAFE is similar to ΔBFA. So $AF/EF = BF/AF$. Then

$$PF = AF = \sqrt{EF \times BF}.$$

Comments: For those who are not aware of the incenter characterization used above, they may see Math Excalibur, vol. 11, no. 2 for details.

Other commended solvers: **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **Nicholas LEUNG** (St. Paul's School, London) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 333. Find the largest positive integer n such that there exist n 4-element sets $A_1, A_2, \ldots, A_n$ such that every pair of them has exactly one common element and the union of these n sets has exactly n elements.

Solution. **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Let the n elements be 1 to n. For $i=1$ to n, let s_i denote the number of sets in which i appeared. Then $s_1+s_2+\cdots+s_n = 4n$. On average, each i appeared in 4 sets.

Assume there is an element, say 1, appeared in more than 4 sets, say 1 is in $A_1, A_2, \ldots, A_5$. Then other than 1, the remaining $3\times5=15$ elements must all be distinct. Now 1 cannot be in all sets, otherwise there would be $3n+1>n$ elements in the union. So there is a set A_6 not containing 1. Its intersections with each of $A_1, A_2, \ldots, A_5$ must be different, yet A_6 only has 4 elements, contradiction. On the other hand, if there is an element appeared in less than 4 sets, then there would be another element appeared in more than 4 sets, contradiction. Hence, every i appeared in exactly 4 sets.

Suppose 1 appeared in A_1, A_2, A_3, A_4. Then we may assume that $A_1=\{1,2,3,4\}$, $A_2=\{1,5,6,7\}$, $A_3=\{1,8,9,10\}$ and $A_4=\{1,11,12,13\}$. Hence, $n \ge 13$. Assume $n \ge 14$. Then 14 would be in a set A_5. The other 3 elements of A_5 would come from A_1, A_2, A_3, say. Then A_4 and A_5 would have no common element, contradiction.

Hence, n can only be 13. Indeed, for the $n = 13$ case, we can take A_1, A_2, A_3, A_4, as above and $A_5=\{2,5,8,11\}$, $A_6=\{2,6,9,12\}$, $A_7=\{2,7,10,13\}$, $A_8=\{3,5,10,12\}$, $A_9=\{3,6,8,13\}$, $A_{10}=\{3,7,9,11\}$, $A_{11}=\{4,5,9,13\}$, $A_{12}=\{4, 6, 10,11\}$ and $A_{13}=\{4,7,8,12\}$.

Other commended solvers: **CHUNG**

Ping Ngai (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Problem 324. (*Due to FEI Zhenpeng, Northeast Yucai School, China*) Let $x,y \in (0,1)$ and x be the number whose n-th digit after the decimal point is the n^n-th digit after the decimal point of y for all $n = 1,2,3,\ldots$. Show that if y is rational, then x is rational.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6),

Since the decimal representation of y is eventually periodic, let L be the length of the period and let the decimal representation of y start to become periodic at the m-th digit. Let k be the least common multiple of $1,2,\ldots,L$. Let n be any integer at least L and $n^n \geq m$.

By the pigeonhole principle, there exist $i < j$ among $0,1,\ldots,L$ such that $n^i \equiv n^j \pmod L$. Then for all positive integer d, we have $n^i \equiv n^{i+d(j-i)} \pmod L$. Since k is a multiple of $j-i$ and $n \geq L > i$, so we have $n^n \equiv n^{n+k} \pmod L$. Since k is also a multiple of L, we have $(n+k)^{n+k} \equiv n^{n+k} \equiv n^n \pmod L$. Then the n-th and $(n+k)$-th digit of x are the same. So x is rational.

Other commended solvers: **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Problem 335. (*Due to Ozgur KIRCAK, Yahya Kemal College, Skopje, Macedonia*) Find all $a \in \mathbb{R}$ for which the functional equation $f: \mathbb{R} \to \mathbb{R}$

$$f(x - f(y)) = a(f(x) - x) - f(y)$$

for all $x, y \in \mathbb{R}$ has a unique solution.

Solution. **LE Trong Cuong** (Lam Son High School, Vietnam)

Let $g(x) = f(x) - x$. Then, in terms of g, the equation becomes

$$g(x-y-g(y)) = ag(x) - x.$$

Assume $f(y) = y + g(y)$ is not constant. Let r, s be distinct elements in the range of $f(y) = y + g(y)$. For every real x,

$$g(x-r) = ag(x) - x = g(x-s).$$

This implies $g(x)$ is periodic with period $T = |r-s| > 0$. Then

$$\begin{aligned} ag(x) - x &= g(x-y-g(y)) \\ &= g(x+T-y-g(y)) \\ &= ag(x+T) - (x+T) \\ &= ag(x) - x - T. \end{aligned}$$

This implies $T=0$, contradiction. Thus, f is constant, i.e. there exists a real number c so that for all real y, $f(y)=c$. Then the original equation yields $c = a(c-x) - c$ for all real x, which forces $a=0$ and $c=0$.

Other commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Olympiad Corner

(continued from page 1)

Problem 3. Let a,b,c be complex numbers such that for every complex number z with $|z| \leq 1$, we have $|az^2+bz+c| \leq 1$. Find the maximum of $|bc|$.

Problem 4. Let m,n be integers greater than 1. Let $a_1 < a_2 < \cdots < a_m$ be integers. Prove that there exists a subset T of the set of all integers such that the number of elements of T, denoted by $|T|$, satisfies

$$|T| \leq 1 + \frac{a_m - a_1}{2n+1}$$

and for every $i \in \{1,2,\cdots,m\}$, there exist $t \in T$ and $s \in [-n,n]$ such that $a_i = t+s$.

Problem 5. For $n \geq 3$, we place a number of cards at points $A_1, A_2, \cdots, A_n$ and O. We can perform the following operations:

(1) if the number of cards at some point A_i is not less than 3, then we can remove 3 cards from A_i and transfer 1 card to each of the points A_{i-1}, A_{i+1} and O (here $A_0 = A_n$, $A_{n+1} = A_1$); or

(2) if the number of cards at O is not less than n, then we can remove n cards from O and transfer 1 card to each $A_1, A_2, \cdots, A_n$.

Prove that if the sum of all the cards placed at these $n+1$ points is not less than n^2+3n+1, then we can always perform finitely many operations so that the number of cards at each of the points is not less than $n+1$.

Problem 6. Let a_1, a_2, a_3, b_1, b_2, b_3 be distinct positive integers satisfying

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n \mid (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$

for all positive integer n. Prove that there exists a positive integer k such that $b_i = ka_i$ for $i=1,2,3$.

Max-Min Inequalities

(continued from page 2)

The inequality in the next example was very hard. It was proposed by Reid Barton and appeared among the 2003 IMO shortlisted problems.

Example 6. Let n be a positive integer and let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ be two sequences of positive real numbers. Let $(z_1, z_2, \ldots, z_{2n})$ be a sequence of positive real numbers such that for all $1 \leq i, j \leq n$, $z_{i+j}^2 \geq x_i y_j$. Let $M = \max\{z_1, z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M+z_2+\cdots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+\cdots+x_n}{n}\right)\left(\frac{y_1+\cdots+y_n}{n}\right).$$

Solution. (*Due to Reid Barton and Thomas Mildorf*) Let

$$X = \max\{x_1, x_2, \ldots, x_n\}$$

and

$$Y = \min\{x_1, x_2, \ldots, x_n\}.$$

By replacing x_i by $x_i' = x_i/X$, y_i by $y_i' = y_i/Y$ and z_i by $z_i' = z_i/(XY)^{1/2}$, we may assume $X=Y=1$. It suffices to prove

$$M + z_2 + \cdots + z_{2n} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n. \quad (*)$$

Then

$$\frac{M+z_2+\cdots+z_{2n}}{2n} \geq \frac{1}{2}\left(\frac{x_1+\cdots+x_n}{n} + \frac{y_1+\cdots+y_n}{n}\right),$$

which implies the desired inequality by applying the AM-GM inequality to the right side.

To prove (*), we will *claim* that for any $r \geq 0$, the number of terms greater than r on the left side is at least the number of such terms on the right side. Then the k-th largest term on the left side is greater than the k-th largest term on the right side for each k, proving (*).

For $r \geq 1$, there are no terms greater than 1 on the right side. For $r < 1$, let $A = \{i: x_i > r\}$, $B = \{j: y_j > r\}$, $A+B = \{i+j: i \in A, j \in B\}$ and $C = \{k: k > 1, z_k > r\}$. Let $|A|$, $|B|$, $|A+B|$, $|C|$ denote the number of elements in A, B, $A+B$, C respectively.

Since $X=Y=1$, so $|A|$, $|B|$ are at least 1. Now $x_i > r$, $y_j > r$ imply $z_{i+j} > r$. So $A+B$ is a subset of C. If A is consisted of $i_1 < \cdots < i_a$ and B is consisted of $j_1 < \cdots < j_b$, then $A+B$ contains

$$i_1+j_1 < i_1+j_2 < \cdots < i_1+j_b < i_2+j_b < \cdots < i_a+j_b.$$

Hence, $|C| \geq |A+B| \geq |A|+|B|-1 \geq 1$. So $z_k > r$ for some k. Then $M > r$. So the left side of (*) has $|C|+1 \geq |A|+|B|$ terms greater than r, which finishes the proof of the claim.

Mathematical Excalibur

Volume 14, Number 5 *March - April, 2010*

Olympiad Corner

Here are the Asia Pacific Math Olympiad problems on March 2010.

Problem 1. Let ABC be a triangle with $\angle BAC \neq 90°$. Let O be the circumcenter of triangle ABC and let Γ be the circumcircle of triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Problem 2. For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Problem 3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 21, 2010***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Ramsey Numbers and Generalizations

Law Ka Ho

The following problem is classical: among any 6 people, there exist 3 who know each other or 3 who don't know each other (we assume if A knows B, then B knows A). When 6 is replaced by 5, this is no longer true, as can be seen by constructing a counterexample. We write $R(3,3)=6$ and this is called a *Ramsey number*. In general, $R(m,n)$ denotes the smallest positive integer k such that, among any k people, there exist m who know each other or n who don't know each other.

How do we know that $R(m,n)$ exists for all m, n? A key result is the following.

Theorem 1. For any m, $n > 1$, we have $R(m,n) \le R(m-1,n)+R(m,n-1)$.

Proof. Take $R(m-1,n)+R(m,n-1)$ people. We need to show that there exist m people who know each other or n people who don't know each other. If a person X knows $R(m-1,n)$ others, then among the people X knows, there exist either $m-1$ who know each other (so that together with m, there are m people who know each other) or n people who don't know each other, so we are done. Similarly, if X doesn't know $R(m,n-1)$ others, we are also done. But one of these two cases must occur because the total number of 'others' is $R(m-1,n)+R(m,n-1)-1$.

Using Theorem 1, one can easily show (by induction on $m+n$) that $R(m,n) \le C_{m-1}^{m+n-2}$. This establishes an upper bound on $R(m,n)$. To establish a lower bound, we need a counter-example. While construction of counter-examples is in general very difficult, the probabilistic method (see Vol. 14, No. 3) may be able to help us in getting a non-constructive proof. Yet to get the exact value of a Ramsey number, the lower and upper bounds must match, which is extremely difficult. For m, $n > 3$, fewer than 10 values of $R(m,n)$ are known:

$R(3,4)=9$, $R(3,5)=14$, $R(3,6)=18$
$R(3,7)=23$, $R(3,8)=28$, $R(3,9)=36$
$R(4,4)=18$, $R(4,5)=25$

Even $R(5,5)$ is unknown at present. The best lower and upper bounds obtained so far are respectively 43 and 49. Paul Erdös once made the following remark.

> *Suppose an evil alien would tell mankind "Either you tell me [the value of R(5,5)] or I will exterminate the human race." … It would be best in this case to try to compute it, both by mathematics and with a computer.*
>
> *If he would ask [for the value of R(6,6)], the best thing would be to destroy him before he destroys us, because we couldn't [determine R(6,6)].*

Problems related to the Ramsey numbers occur often in mathematical competitions.

Example 2. (CWMO 2005) There are n new students. Among any three of them there exist two who know each other, and among any four of them there exist two who do not know each other. Find the greatest possible value of n.

Solution. The answer is 8. First, n can be 8 if the 8 students are numbered 1 to 8 and student i knows student j if and only if $|i-j| \not\equiv 1,4 \pmod 8$. Next, suppose $n=9$ is possible. Then no student may know 6 others, for among the 6 either 3 don't know each other or 3 know each other (so together with the original student there exist 4 who know each other). Similarly, it cannot happen that a student doesn't know 4 others. Hence each student knows exactly 5 others. But this is impossible, because if we sum the number of others whom each student know, we get $9\times 5=45$, which is odd, yet each pair of students who know each other is counted twice.

Remark. The answer to the above problem is $R(3,4)-1$, as can be seen by comparing with the definition of $R(3,4)$.

The Ramsey number can be generalised in many different directions. One is to increase the number of statuses from 2 (know or don't know) to more than 2, as the following example shows.

Example 3. (IMO 1964) Seventeen people correspond by mail with one another — each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

Solution. Suppose the three topics are *A*, *B* and *C*. Pick any person; he writes to 16 others. By the pigeonhole principle, he writes to 6 others on the same topic, say *A*. If any two of the 6 people write to each other on *A*, then we are done. If not, then these 6 people write to each other on *B* or *C*. Since $R(3,3)=6$, either 3 of them write to each other on *B*, or 3 of them write to each other on *C*. In any case there exist 3 people who write to each other about the same topic.

Remark. The above problem proves $R(3,3,3)\le 17$, where $R(m,n,p)$ is defined analogously as $R(m,n)$ except that there are now three possible statuses instead of two. It can be shown that $R(3,3,3)=17$ by constructing a counterexample when there are only 16 people.

Another direction of generalization is to generalise '*m* people who know each other' or '*n* people who don't know each other' to other structures. (Technically, the *graph Ramsey number* $R(G,H)$ is the smallest positive integer *k* such that when every two of *k* points are joined together by a red or blue edge, there must exist a red copy of *G* or a blue copy of *H*. Hence $R(m,n)=R(K_m,K_n)$, where K_m denotes the complete graph on *m* vertices, i.e. *m* points among which every two are joined by an edge).

Example 4. *N* people attend a meeting, and some of them shake hands with each other. Suppose that each person shakes hands with at most 100 other people, and among any 50 people there exist at least two who have shaken hands with each other. Find the greatest possible value of *N*.

Solution. The answer is 4949. We first show that *N* = 4949 is possible: suppose there are 49 groups of 101 people each, and two people shake hands if and only if they are in the same group. It is easy to check that the requirements of the question are satisfied. Now suppose *N* = 4950 and each person shakes hands with at most 100 others. We will show that there exist 50 people who have not shaken hands with each other, thus contradicting the given condition. To do this, pick a first person P_1 and cross out all those who have shaken hands with him. Then pick P_2 from the rest and again cross out those who have shaken hands with him, and so on. In this way, at most 100 people are crossed out each time. After P_{49} is chosen, at least $4950-49-49\times 100=1$ person remains, so we will be able to choose P_{50}. Because of the 'crossing out' algorithm, we see that no two of P_1, P_2, …, P_{50} have shaken hands with each other.

Remark. By identifying each person with a point and joining two points by a red line if two people have shaken hands and a blue line otherwise, we see that the above problem proves $R(K_{1,100},K_{50})=4950$. Here $K_{1,100}$ is the graph on 101 points by joining 1 point to the other 100 points.

The *Van der Waerden number* $W(r,k)$ is the smallest positive integer *N* such that if each of 1, 2, …, *N* is assigned one of *r* colours, then there exist a monochromatic *k*-term arithmetic progression. The following example shows that we have $W(2,3)\le 325$.

Example 5. If each of the integers 1, 2, …, 325 is assigned red or blue colour, there exist three integers *p*, *q*, *r* which are assigned the same colour and which form an arithmetic progression.

Solution. Divide the 325 integers into 65 groups G_1 = {1, 2, 3, 4, 5}, G_2 = {6, 7, 8, 9, 10}, …, G_{65} = {321, 322, 323, 324, 325}. There are $2^5 = 32$ possible colour patterns for each group. Hence there exist three groups G_a and G_b, $1\le a<b\le 33$, whose colour patterns are the same. We note that $2b-a\le 65$ and that *a*, *b*, $2b-a$ form an arithmetic progression. Now two of the first three numbers of G_a are of the same colour, say, the first and third are red (it can be seen that the proof goes exactly the same way if it is the first and second, or second and third). If the fifth is also red, then we are done. Otherwise, the first and third numbers of both G_a and G_b (recall that they have identical colour patterns) are red while the fifth is blue. If the fifth number of G_{2b-a} is red, then it together with the first number of G_a and the third number of G_b form a red arithmetic progression; if it is blue, then it together with the fifth numbers of G_a and G_b form a blue arithmetic progression.

It can be shown via a two-dimensional inductive argument that $W(r,k)$ exists for all *r*, *k*. We see that the existence of Ramsey numbers and van der Waerden numbers are very similar: both say that the desired structure exists in a sufficiently large population.

An analogy to this (though not mathematically rigorous) is that when there are sufficiently many stars in the sky, one can form from them whatever picture one wishes. (This is one of the lines in the movie *A Beautiful Mind*!)

Yet another generalization of the van der Waerden Theorem (which says that $W(r,k)$ exists for all *r*, *k*) is the *Hales-Jewett Theorem*. The exact statement of the theorem is rather technical, but we can look at an informal version here. We are familiar with the two-person tic-tac-toe game played on a 3×3 square in two dimensions. We also have the two-person tic-tac-toe game played on a $4\times 4\times 4$ cube in three dimensions (try it out at http://www.mathdb.org/fun/games/tie_toe/e_tie_toe.htm!). Both games can end in a draw. However, it is easy to see that a two-person tic-tac-toe game played on a 2×2 square in two dimensions cannot end in a draw. The Hales-Jewett Theorem says that for any *n* and *k*, the *k*-person tic-tac-toe game played on an $n\times n\times\cdots\times n$ (*D* factors of *n*, where *D* is the dimension) hypercube cannot end in a draw when *D* is large enough! (For instance, we have just seen that when *n* = 2 and *k* = 2, then *D* = 2 is large enough, while when *n* = 3 and *k* = 2, then *D* = 2 is not large enough.) In case *k* = 2 (i.e. a two-person game) and when *D* is large enough so that a draw is impossible, it can be shown (via a so-called strategy stealing argument) that the first player has a winning strategy!

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***May 21, 2010.***

Problem 341. Show that there exists an infinite set S of points in the 3-dimensional space such that every plane contains at least one, but not infinitely many points of S.

Problem 342. Let $f(x)=a_nx^n+\cdots+a_1x+p$ be a polynomial with coefficients in the integers and degree $n>1$, where p is a prime number and

$$|a_n|+|a_{n-1}|+\cdots+|a_1|<p.$$

Then prove that $f(x)$ is not the product of two polynomials with coefficients in the integers and degrees less than n.

Problem 343. Determine all ordered pairs (a,b) of positive integers such that $a\neq b$, $b^2+a=p^m$ (where p is a prime number, m is a positive integer) and a^2+b is divisible by b^2+a.

Problem 344. $ABCD$ is a cyclic quadrilateral. Let M, N be midpoints of diagonals AC, BD respectively. Lines BA, CD intersect at E and lines AD, BC intersect at F. Prove that

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\frac{2MN}{EF}.$$

Problem 345. Let $a_1, a_2, a_3, \cdots$ be a sequence of integers such that there are infinitely many positive terms and also infinitely many negative terms. For every positive integer n, the remainders of $a_1, a_2, \cdots, a_n$ upon divisions by n are all distinct. Prove that every integer appears exactly one time in the sequence.

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Solutions

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Problem 336. (*Due to Ozgur Kircak, Yahya Kemal College, Skopje, Macedonia*) Find all distinct pairs (x,y) of integers satisfying the equation

$$x^3+2009y=y^3+2009x.$$

Solution. **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Pedro Henrique O. PANTOJA** (UFRN, Natal, Brazil), **PUN Ying Anna** (HKU), **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College), **Simon YAU Chi-Keung** and **Fai YUNG**.

All pairs (x,x) satisfy the equation. If (x,y) satisfies the equation and $x\neq y$, then

$$x^2+xy+y^2=\frac{x^3-y^3}{x-y}=2009\equiv 2(\mathrm{mod}\,3).$$

However, $x^2+xy+y^2\equiv x^2-2xy+y^2=(x-y)^2\equiv 0$ or 1 (mod 3). So there are no solutions with $x\neq y$.

Problem 337. In triangle ABC, $\angle ABC = \angle ACB=40°$. P and Q are two points inside the triangle such that $\angle PAB = \angle QAC = 20°$ and $\angle PCB = \angle QCA = 10°$. Determine whether B, P, Q are collinear or not.

***Solution* 1.** **CHUNG Ping Ngai** (La Salle College, Form 6) and **HUNG Ka Kin Kenneth** (Diocesan Boys' School).

Let $\angle PBA=a$, $\angle PBC=b$, $\angle QBA=a'$ and $\angle QBC=b'$. By the trigonometric form of Ceva's theorem, we have

$$1=\frac{\sin\angle PBA}{\sin\angle PBC}\frac{\sin\angle PCB}{\sin\angle PCA}\frac{\sin\angle PAC}{\sin\angle PAB}=\frac{\sin a\sin 10^\circ\sin 80^\circ}{\sin b\sin 30^\circ\sin 20^\circ},$$

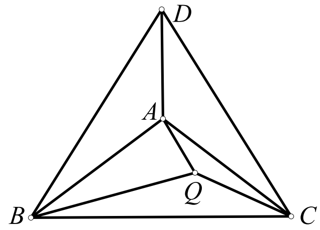
$$1=\frac{\sin\angle QBA}{\sin\angle QBC}\frac{\sin\angle QCB}{\sin\angle QCA}\frac{\sin\angle QAC}{\sin\angle QAB}=\frac{\sin a'\sin 30^\circ\sin 20^\circ}{\sin b'\sin 10^\circ\sin 80^\circ}.$$

As $\sin10°\sin80° = \sin10°\cos10° =½\sin20° = \sin30°\sin20°$, we obtain $\sin a= \sin b$ and $\sin a' = \sin b'$. Since $0<a,b,a',b'<90°$ and $a+b=40°=a'+b'$, we get $a=b=a'=b'=20°$, i.e. $\angle PBA=\angle PBC=\angle QBA=\angle QBC$. Therefore, B, P, Q are collinear.

***Solution* 2.** **LEE Kai Seng**.

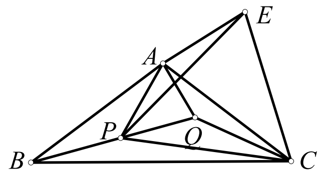
We will show B,P,Q collinear by proving lines BQ and BP bisect $\angle ABC$.

Draw an equilateral triangle BDC with D on the same side of BC as A. Since $\angle ABC = \angle ACB=40°$, $AB=AC$. Then both D and A are equal distance from B and C. So DA bisects $\angle BDC$. We have



$\angle QCD = 60°-\angle BCQ = 30° = \angle ADC.$

Also, $\angle DCA = \angle QCD - \angle QCA = 20° = \angle QAC$, which implies $QA\|CD$. Then $AQCD$ is an isosceles trapezoid, so $AD = QC$. This with $BD=BC$ and $\angle BDA = 30° = \angle QCB$ imply $\Delta BDA \cong \Delta QCB$. Then $BA=BQ$. Since $\angle BAQ = \angle BAC - \angle QAC = 100°-20° = 80°$, we get $\angle ABQ = 20° = ½\angle ABC$. So BQ bisects $\angle ABC$.



Extend BA to a point E so that $BE=BC$. Then $\angle BCE = ½(180°-\angle ABC) = 70°$. Next, we will show ΔEPC is equilateral.

We have $\angle PCE=\angle BCE-\angle PCB=60°$, $\angle ACE=\angle BCE-\angle BCA=30°=½\angle PCE$. So CA bisects $\angle PCE$. Next, $\angle CAE = 180°-\angle BAC = 80° = \angle BAC-\angle BAP = \angle CAP$. Then $\Delta CAE \cong \Delta CAP$. So $CE = CP$ and ΔEPC is equilateral. Then B, P are equal distance from E and C. Hence BP bisects $\angle ABC$.

Other commended solvers: **CHAN Chun Wai** (St. Paul's College), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **PUN Ying Anna** (HKU).

Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_0=1$, $b_0=0$ and for $n=0,1,2,\ldots$,

$$a_{n+1}=7a_n+6b_n-3,$$
$$b_{n+1}=8a_n+7b_n-4.$$

Prove that a_n is a perfect square for all $n=0,1,2,\ldots$

***Solution* 1.** **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (G.T. (Ellen Yeung) College), **Ercole SUPPA** (Teramo, Italy) and **YEUNG Chun Wing** (St. Paul's College).

Solving for b_n in the first equation and putting it into the second equation, we have

$a_{n+2}=14a_{n+1}-a_n-6$ for $n=0,1,2,\ldots$ (*)

with $a_0=1$ and $a_1=4$. Let $d_n=a_n-½$. Then (*) becomes $d_{n+2}=14d_{n+1}-d_n$. Since the roots of $x^2-14x+1=0$ are $7\pm 4\sqrt{3}$, we get d_n is of the form $\alpha(7-4\sqrt{3})^n+\beta(7+4\sqrt{3})^n$. Using $d_0=½$ and $d_1=3½$, we get $\alpha=¼$ and $\beta=¼$. So

$$a_n=d_n+\frac{1}{2}=\frac{2+(7-4\sqrt{3})^n+(7+4\sqrt{3})^n}{4}.$$

Now, consider the sequence $\{c_n\}$ of positive integers, defined by $c_0=1$, $c_1=2$ and

$c_{n+2}=4c_{n+1}-c_n$ for $n=0,1,2,\ldots.$ (**)

Since the roots of $x^2-4x+1=0$ are $2\pm\sqrt{3}$, as above we get

$$c_n=\frac{(2-\sqrt{3})^n+(2+\sqrt{3})^n}{2}.$$

Squaring c_n, we see $a_n=c_n^2$.

***Solution* 2.** **William CHAN** and **Invisible MAK** (Carmel Alison Lam Foundation Secondary School).

The equations imply

$a_{n+2}=14a_{n+1}-a_n-6$ for $n=0,1,2,\ldots$ (*)

We will prove $a_na_{n+2}=(a_{n+1}+3)^2$ by math induction. The case $n=0$ is $1\times 49=(4+3)^2$. Suppose $a_{n-1}a_{n+1}=(a_n+3)^2$. Then

$$\begin{aligned}
&a_na_{n+2}-(a_{n+1}+3)^2\\
&=a_n(14a_{n+1}-a_n-6)-(a_{n+1}+3)^2\\
&=14a_na_{n+1}-a_n^2-6a_n-a_{n+1}^2-6a_{n+1}-9\\
&=(14a_n-a_{n+1}-6)a_{n+1}-(a_n+3)^2\\
&=a_{n-1}a_{n+1}-(a_n+3)^2\\
&=0.
\end{aligned}$$

This completes the induction.

Next, we will show all a_n's are perfect squares. Now $a_0=1^2$ and $a_1=2^2$. Suppose $a_{n-1}=r^2$ and $a_n=s^2$, we get $a_{n+1}=(a_n+3)^2/r^2$ and $a_{n+2}=(a_{n+1}+3)^2/s^2$. Since the square root of a positive integer is an integer or an irrational number, a_{n+1} and a_{n+2} are perfect squares. By mathematical induction, the result follows.

Other commended solvers: **PUN Ying Anna** (HKU), **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College).

Problem 339. In triangle $ABC, \angle ACB=90°$. For every n points inside the triangle, prove that there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$P_1P_2^2+P_2P_3^2+\cdots+P_{n-1}P_n^2\le AB^2.$$

Solution. **Federico BUONERBA** (Università di Roma "Tor Vergata", Roma, Italy), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **PUN Ying Anna** (HKU).

We will prove the following more general result:

Let ABC be a triangle with $\angle ACB=90°$. For every n points inside or on the sides of the triangle, there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$AP_1^2+P_1P_2^2+\cdots+P_{n-1}P_n^2+P_nB^2\le AB^2.$$

We prove this by induction on n. For the case $n=1$, since $\angle AP_1B\ge 90°$, the cosine law gives $AP_1^2+P_1B^2\le AB^2$.

Next we assume all cases less than n are true. For the case n, we can divide the original right triangle into two right triangles by taking the altitude from C to H on the hypotenuse AB. We can assume that the two smaller right triangles AHC and BHC contain $m>0$ and $n-m>0$ points respectively (otherwise, one of these two smaller triangles contains all the points and we keep dividing in the same way the smaller right triangle which contains all the points). Since $m<n$ and $n-m<n$, by the induction hypothesis, there exist a labeling of points in triangle AHC as $P_1, P_2, \ldots, P_m$ such that

$$AP_1^2+P_1P_2^2+\cdots+P_{m-1}P_m^2+P_mC^2\le AC^2$$

and a labeling of points in triangle BHC as $P_{m+1}, P_{m+2}, \ldots, P_m$ such that

$$CP_{m+1}^2+P_{m+1}P_{m+2}^2+\cdots+P_nB^2\le CB^2.$$

Since $\angle P_mCP_{m+1}\le 90°$, the cosine law gives $P_mP_{m+1}^2\le P_mC^2+CP_{m+1}^2$. Then

$$\begin{aligned}
&AP_1^2+P_1P_2^2+\cdots+P_{n-1}P_n^2+P_nB^2\\
&\le AC^2+CB^2=AB^2.
\end{aligned}$$

Problem 340. Let k be a given positive integer. Find the least positive integer N such that there exists a set of $2k+1$ distinct positive integers, the sum of all its elements is greater than N and the sum of any k elements is at most $N/2$.

Solution. **CHAN Chun Wai** (St. Paul's College), **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **PUN Ying Anna** (HKU).

Let $a_1, a_2, \ldots, a_{2k+1}$ be such a set of $2k+1$ of positive integers arranged in increasing order. We have

$$\sum_{i=1}^{2k+1}a_i\ge N+1\ge 2\sum_{i=k+2}^{2k+1}a_i+1.$$

Then

$$\begin{aligned}
a_{k+1}&\ge\sum_{i=k+2}^{2k+1}a_i-\sum_{i=1}^{k}a_i+1\\
&=\sum_{i=1}^{k}(a_{i+k+1}-a_i)+1\\
&\ge\sum_{i=1}^{k}(k+1)+1\\
&=k^2+k+1.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{N}{2}&\ge\sum_{i=k+2}^{2k+1}a_i=\sum_{i=k+2}^{2k+1}(a_i-a_{k+1})+\sum_{i=k+2}^{2k+1}a_{k+1}\\
&\ge\sum_{i=k+2}^{2k+1}(i-k-1)+k(k^2+k+1)\\
&=\frac{2k^3+3k^2+3k}{2}.
\end{aligned}$$

Now all inequalities above become equality if we take $a_i=k^2+i$ for $i=1, 2, \ldots, 2k+1$. So the least positive value of N is $2k^3+3k^2+3k$.

Olympiad Corner

(continued from page 1)

Problem 4. Let ABC be an acute triangle satisfying the condition $AB>BC$ and $AC>BC$. Denote by O and H the circumcenter and orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Problem 5. Find all functions f from the set $\boldsymbol{R}$ of real numbers into $\boldsymbol{R}$ which satisfy for all $x, y, z\in\boldsymbol{R}$ the identity

$$\begin{aligned}
&f(f(x)+f(y)+f(z))\\
&=f(f(x)-f(y))+f(2xy+f(z))+2f(xz-yz).
\end{aligned}$$

Mathematical Excalibur

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Primitive Roots Modulo Primes

Kin Y. Li

The well-known Fermat's little theorem asserts that if p is a prime number and x is an integer not divisible by p, then

$$x^{p-1}\equiv 1 \pmod p.$$

For positive integer $n>1$ and integer x, if there exists a least positive integer d such that $x^d\equiv 1 \pmod n$, then we say d is the <u>order</u> of $x \pmod n$. We denote this by $ord_n(x) = d$. It is natural to ask for a prime p, if there exists x such that $ord_p(x) = p-1$. Such x is called a <u>primitive root (mod p)</u>. Indeed, we have the following

<u>Theorem.</u> For every prime number p, there exists a primitive root (mod p). (We will comment on the proof at the end of the article.)

As a consequence, if x is a primitive root (mod p), then $1, x, x^2, \ldots, x^{p-2} \pmod p$ are distinct and they form a permutation of $1, 2, \ldots, p-1 \pmod p$. This is useful in solving some problems in math competitions. The following are some examples. (Below, we will use the common notation $a|b$ to denote a is a divisor of b.)

<u>Example 1.</u> (*2009 Hungary-Israel Math Competition*) Let $p \ge 2$ be a prime number. Determine all positive integers k such that $S_k = 1^k + 2^k + \cdots + (p-1)^k$ is divisible by p.

<u>Solution.</u> Let x be a primitive root (mod p). Then

$$S_k \equiv 1+x^k+\cdots+x^{(p-2)k} \pmod p.$$

If $p-1|k$, then $S_k\equiv 1+\cdots+1=p-1 \pmod p$. If $p-1\nmid k$, then since $x^k\not\equiv 1 \pmod p$ and $x^{(p-1)k}\equiv 1 \pmod p$, we have

$$S_k \equiv \frac{x^{(p-1)k}-1}{x^k-1}\equiv 0 \pmod p.$$

Therefore, all the k's that satisfy the requirement are precisely those integers that are not divisible by $p-1$.

<u>Example 2.</u> Prove that if p is a prime number, then $(p-1)!\equiv -1 \pmod p$. This is <u>*Wilson's theorem*</u>.

<u>Solution.</u> The case $p = 2$ is easy. For $p > 2$, let x be a primitive root (mod p). Then

$$(p-1)! \equiv x^1x^2\cdots x^{p-1} = x^{(p-1)p/2} \pmod p.$$

By the property of x, $w=x^{(p-1)/2}$ satisfies $w\not\equiv 1 \pmod p$ and $w^2\equiv 1 \pmod p$. So $w \equiv -1 \pmod p$. Then

$$(p-1)! \equiv x^{(p-1)p/2} = w^p = -1 \pmod p.$$

<u>Example 3.</u> (1993 *Chinese IMO Team Selection Test*) For every prime number $p \ge 3$, define

$$F(p)=\sum_{k=1}^{(p-1)/2} k^{120},\quad f(p)=\frac{1}{2}-\left\{\frac{F(p)}{p}\right\},$$

where $\{x\}=x-[x]$ is the fractional part of x. Find the value of $f(p)$.

<u>Solution.</u> Let x be a primitive root (mod p). If $p-1\nmid 120$, then $x^{120}\not\equiv 1 \pmod p$ and $x^{120(p-1)}\equiv 1 \pmod p$. So

$$F(p)\equiv\frac{1}{2}\sum_{i=1}^{p-1} x^{120i}$$

$$=\frac{x^{120}(x^{120(p-1)}-1)}{2(x^{120}-1)}\equiv 0 \pmod p.$$

Then $f(p) = 1/2$.

If $p-1|\ 120$, then $p\in\{3, 5, 7, 11, 13, 31, 41, 61\}$ and $x^{120}\equiv 1 \pmod p$. So

$$F(p)\equiv\frac{1}{2}\sum_{i=1}^{p-1} x^{120i}=\frac{p-1}{2} \pmod p.$$

Then

$$f(p)=\frac{1}{2}-\frac{p-1}{2p}=\frac{1}{2p}.$$

<u>Example 4.</u> If a and b are nonnegative integers such that $2^a \equiv 2^b \pmod{101}$, then prove that $a \equiv b \pmod{100}$.

Olympiad Corner

Below are the First Round problems of the 26th Iranian Math Olympiad.

Problem 1. In how many ways can one choose $n-3$ diagonals of a regular n-gon, so that no two have an intersection strictly inside the n-gon, and no three form a triangle?

Problem 2. Let ABC be a triangle. Let I_a be the center of its A-excircle. Assume that the A-excircle touches AB and AC in B' and C', respectively. Let I_aB and I_aC intersect $B'C'$ in P and Q, respectively. Let M be the intersection of CP and BQ. Prove that the distance between M and the line BC is equal to the inradius of ΔABC.

Problem 3. Let a, b, c and d be real numbers, and at least one of c or d is not zero. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\frac{ax+b}{cx+d}.$$

Assume that $f(x) \ne x$ for every $x\in\mathbb{R}$. Prove that there exists at least one p such that $f^{1387}(p) = p$, then for every x, for which $f^{1387}(x)$ is defined, we have $f^{1387}(x) = x$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***July 10, 2010***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
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Solution. We first check 2 is a primitive root of (mod 101). If d is the least positive integer such that $2^d \equiv 1$ (mod 101), then dividing 100 by d, we get $100 = qd + r$ for some integers q, r, where $0 \le r < d$. By Fermat's little theorem,

$$1 \equiv 2^{100} = (2^d)^q 2^r \equiv 2^r \pmod{101},$$

which implies the remainder $r = 0$. So $d|100$.

Assume $d < 100$. Then $d|50$ or $d|20$, which implies 2^{20} or $2^{50} \equiv 1$ (mod 101). But $2^{10} = 1024 \equiv 14$(mod 101) implies $2^{20} \equiv 14^2 \equiv -6$ (mod 101) and $2^{50} \equiv 14(-6)^2 \equiv -1$ (mod 101). So $d = 100$.

Finally, $2^a \equiv 2^b$ (mod 101) implies $2^{|a-b|} \equiv 1$ (mod 101). Then as above, dividing $|a-b|$ by 100, we will see the remainder is 0. Therefore, $a \equiv b$ (mod 100).

Comments: The division argument in the solution above shows if $ord_n(x) = d$, then $x^k \equiv 1$ (mod n) if and only if $d \mid k$. This is useful.

Example 5. (1994 *Putnam Exam*) For any integer a, set

$$n_a = 101a - 100\times 2^a.$$

Show that for $0 \le a,b,c,d \le 99$,

$$n_a + n_b \equiv n_c + n_d \pmod{10100}$$

implies $\{a,b\} = \{c,d\}$.

Solution. Since 100 and 101 are relatively prime, $n_a+n_b \equiv n_c+n_d$ (mod 10100) is equivalent to

$$n_a + n_b \equiv n_c + n_d \pmod{100}$$

and

$$n_a + n_b \equiv n_c + n_d \pmod{101}.$$

As $n_a \equiv a$ (mod 100) and $n_a \equiv 2^a$ (mod 101). These can be simplified to

$$a+b \equiv c+d \pmod{100} \quad (*)$$

and

$$2^a+2^b \equiv 2^c+2^d \pmod{101}.$$

Using $2^{100} \equiv 1$(mod 101) and (*), we get

$$2^a 2^b = 2^{a+b} \equiv 2^{c+d} = 2^c 2^d \pmod{101}.$$

Since $2^b \equiv 2^c+2^d-2^a$ (mod 101), we get $2^a(2^c+2^d-2^a) \equiv 2^c 2^d$ (mod 101). This can be rearranged as

$$(2^a-2^c)(2^a-2^d) \equiv 0 \pmod{101}.$$

Then $2^a \equiv 2^c$ (mod 101) or $2^a \equiv 2^d$ (mod 101). By the last example, we get $a \equiv c$ or d (mod 100). Finally, using $a+b \equiv c+d$ (mod 100), we get $\{a,b\} = \{c,d\}$.

Example 6. Find all two digit numbers n (i.e. $n = 10a + b$, where $a, b \in \{0,1,\ldots,9\}$ and $a \neq 0$) such that for all integers k, we have $n \mid k^a - k^b$.

Solution. Clearly, $n = 11, 22, \ldots, 99$ work. Suppose n is such an integer with $a \neq b$. Let p be a prime divisor of n. Let x be a primitive root (mod p). Then $p \mid x^a - x^b$, which implies $x^{|a-b|} \equiv 1$(mod p). By the comment at the end of example 4, we have $p-1 \mid |a-b| \le 9$. Hence, $p = 2, 3, 5$ or 7.

If $p = 7 \mid n$, then $6 \mid |a-b|$ implies $n = 28$. Now $k^2 \equiv k^8$ (mod 4) and (mod 7) hold by property of (mod 4) and Fermat's little theorem respectively. So $n = 28$ works.

Similarly the $p = 5$ case will lead to $n = 15$ or 40. Checking shows $n = 15$ works. The $p = 3$ case will lead to $n = 24$ or 48. Checking shows $n = 48$ works. The $p = 2$ case will lead to $n = 16$, 32 or 64, but checking shows none of them works. Therefore, the only answers are 11, 22, …, 99, 28, 15, 48.

Example 7. Let p be an odd prime number. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $m,n \in \mathbb{Z}$,

(i) if $m \equiv n$ (mod p), then $f(m) = f(n)$ and
(ii) $f(mn) = f(m)f(n)$.

Solution. For such functions, taking $m = n = 0$, we have $f(0) = f(0)^2$, so $f(0) = 0$ or 1. If $f(0) = 1$, then taking $m = 0$, we have $1 = f(0) = f(0)\,f(n) = f(n)$ for all $n \in \mathbb{Z}$, which is clearly a solution.

If $f(0) = 0$, then $n \equiv 0$ (mod p) implies $f(n) = 0$. For $n \not\equiv 0$ (mod p), let x be a primitive root (mod p). Then $n \equiv x^k$(mod p) for some $k \in \{1,2,\ldots,p-1\}$. So $f(n) = f(x^k) = f(x)^k$. By Fermat's little theorem, $x^p \equiv x$ (mod p). This implies $f(x)^p = f(x)$. So $f(x) = 0$, 1 or -1. If $f(x) = 0$, then $f(n) = 0$ for all $n \in \mathbb{Z}$. If $f(x) = 1$, then $f(n) = 1$ for all $n \not\equiv 0$ (mod p). If $f(x) = -1$, then for n congruent to a nonzero square number (mod p), $f(n) = 1$, otherwise $f(n) = -1$.

After seeing how primitive roots can solve problem, it is time to examine the proof of the theorem more closely. We will divide the proofs into a few observations.

For a polynomial $f(x)$ of degree n with coefficients in (mod p), the congruence

$$f(x) \equiv 0 \pmod{p}$$

has at most n solutions (mod p). This can be proved by doing induction on n and imitating the proof for real coefficient polynomials having at most n roots.

If $d|p-1$, then $x^d-1 \equiv 0$(mod p) has exactly d solutions (mod p). To see this, let $n = (p-1)/d$, then

$$x^{p-1}-1 = (x^d-1)(x^{(n-1)d}+x^{(n-2)d}+\cdots+1).$$

Since $x^{p-1}-1 \equiv 0$ (mod p) has $p-1$ solutions by Fermat's little theorem, so if $x^d-1 \equiv 0$ (mod p) has less than d solutions, then

$$(x^d-1)(x^{(n-1)d}+x^{(n-2)d}+\cdots+1) \equiv 0 \pmod{p}$$

would have less than $d + (n-1)d = p-1$ solutions, which is a contradiction.

Suppose the prime factorization of $p-1$ is $p_1^{e_1} \cdots p_k^{e_k}$, where p_i's are distinct primes and $e_i \ge 1$. For $i = 1, 2, \ldots, k$, let $m_i = p_i^{e_i}$. Using the observation in the last paragraph, we see there exist $m_i - m_i/p_i > 1$ solutions x_i of equation $x^{m_i} - 1 \equiv 0$ (mod p), which are not solutions of $x^{m_i/p_i} - 1 \equiv 0$ (mod p). It follows that the least positive integer d such that $x_i^d - 1 \equiv 0$ (mod p) is $m_i = p_i^{e_i}$. That means x_i has order $m_i = p_i^{e_i}$ in (mod p).

Let r be the order of $x_i x_j$ in (mod p). By the comment at the end of example 4, we have $r \mid p_i^{e_i} p_j^{e_j}$. Now

$$x_j^{rd} \equiv (x_i^d)^r x_j^{rd} = (x_i x_j)^{rd} \equiv 1 (\text{mod} p),$$

which by the comment again, we get $p_j^{e_j} \mid rd$. Since $p_j^{e_j}$ and $d = p_i^{e_i}$ are relatively prime, we get $p_j^{e_j} \mid r$. Interchanging the roles of p_i and p_j, we also get $p_i^{e_j} \mid r$. So $p_i^{e_i} p_j^{e_j} \mid r$. Then $r = p_i^{e_i} p_j^{e_j}$. So $x = x_1 x_2 \cdots x_k$ will have order $p_1^{e_1} \cdots p_k^{e_k} = p-1$, which implies x is a primitive root (mod p).

For $n > 1$, Euler's theorem asserts that if x and n are relatively prime integers, then $x^{\varphi(n)} \equiv 1$(mod n), where $\varphi(n)$ is the number of positive integers among $1,2,\ldots,n$ that are relatively prime to n. Similarly, we can define x to be a primitive root (mod n) if and only if the least positive integer d satisfying $x^d \equiv 1$(mod n) is $\varphi(n)$. For the inquisitive mind who wants to know for which n, there exists primitive roots (mod n), the answers are $n = 2, 4, p^k$ and $2p^k$, where p is an odd prime. This is much harder to prove. The important thing is for such a primitive root x (mod n), the numbers x^i (mod n) for $i = 1$ to $\varphi(n)$ is a permutation of the $\varphi(n)$ numbers among $1,2,\ldots,n$ that are relatively prime to n.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***July 10, 2010.***

Problem 346. Let k be a positive integer. Divide $3k$ pebbles into five piles (with possibly unequal number of pebbles). Operate on the five piles by selecting three of them and removing one pebble from each of the three piles. If it is possible to remove all pebbles after k operations, then we say it is a *harmonious ending*.

Determine a necessary and sufficient condition for a harmonious ending to exist in terms of the number k and the distribution of pebbles in the five piles.

(*Source*: 2008 *Zhejiang Province High School Math Competition*)

Problem 347. $P(x)$ is a polynomial of degree n such that for all $w\in\{1, 2, 2^2, \ldots, 2^n\}$, we have $P(w) = 1/w$.

Determine $P(0)$ with proof.

Problem 348. In $\triangle ABC$, we have $\angle BAC = 90°$ and $AB < AC$. Let D be the foot of the perpendicular from A to side BC. Let I_1 and I_2 be the incenters of $\triangle ABD$ and $\triangle ACD$ respectively. The circumcircle of $\triangle AI_1I_2$ (with center O) intersects sides AB and AC at E and F respectively. Let M be the intersection of lines EF and BC.

Prove that I_1 or I_2 is the incenter of the $\triangle ODM$, while the other one is an excenter of $\triangle ODM$.

(*Source:* 2008 *Jiangxi Province Math Competition*)

Problem 349. Let $a_1, a_2, \ldots, a_n$ be rational numbers such that for every positive integer m,

$$a_1^m + a_2^m + \cdots + a_n^m$$

is an integer. Prove that $a_1, a_2, \ldots, a_n$ are integers.

Problem 350. Prove that there exists a positive constant c such that for all positive integer n and all real numbers $a_1, a_2, \ldots, a_n$, if

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

then

$$\max_{x\in[0,2]}|P(x)| \le c^n \max_{x\in[0,1]}|P(x)|.$$

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Solutions

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Problem 341. Show that there exists an infinite set S of points in the 3-dimensional space such that every plane contains at least one, but not infinitely many points of S.

Solution. **Emanuele NATALE and Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Consider the curve $\sigma : \mathbb{R}\to\mathbb{R}^3$ defined by $\sigma(x) = (x, x^3, x^5)$. Let S be the graph of σ. If $ax+by+cz=d$ is the equation of a plane in $\mathbb{R}^3$, then the intersection of the plane and the curve is determined by the equation

$$ax + bx^3 + cx^5 = d,$$

which has at least one and at most five solutions.

Other commended solvers: **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Problem 342. Let $f(x)=a_nx^n+\cdots+a_1x+p$ be a polynomial with coefficients in the integers and degree $n>1$, where p is a prime number and

$$|a_n|+|a_{n-1}|+\cdots+|a_1| < p.$$

Then prove that $f(x)$ is not the product of two polynomials with coefficients in the integers and degrees less than n.

Solution. **The 6B Mathematics Group** (Carmel Alison Lam Foundation Secondary School), **CHUNG Ping Ngai** (La Salle College, Form 6), **LEE Kai Seng** (HKUST), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Pedro Henrique O. PANTOJA** (University of Lisbon, Portugal).

Let w be a root of $f(x)$ in $\mathbb{C}$. Assume $|w|\le 1$. Using $a_nw^n+\cdots+a_1w+p=0$ and the triangle inequality, we have

$$p = \left|\sum_{i=1}^n a_i w^i\right| \le \sum_{i=1}^n |a_i|\,|w|^i \le \sum_{i=1}^n |a_i|,$$

which contradicts the given inequality. So all roots of $f(x)$ have absolute values greater than 1.

Assume $f(x)$ is the product of two integral coefficient polynomials $g(x)$ and $h(x)$ with degrees less than n. Let b and c be the nonzero coefficients of the highest degree terms of $g(x)$ and $h(x)$ respectively. Then $|b|$ and $|c| \ge 1$. By Vieta's theorem, $|g(0)/b|$ and $|h(0)/c|$ are the products of the absolute values of their roots respectively. Since their roots are also roots of $f(x)$, we have $|g(0)/b| > 1$ and $|h(0)/c| > 1$. Now $p = |f(0)| = |g(0)h(0)|$, but $g(0)$, $h(0)$ are integers and $|g(0)| > |b| \ge 1$ and $|h(0)| > |c| \ge 1$, which contradicts p is prime.

Problem 343. Determine all ordered pairs (a,b) of positive integers such that $a\ne b$, $b^2+a=p^m$ (where p is a prime number, m is a positive integer) and a^2+b is divisible by b^2+a.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

For such (a,b),

$$\frac{a^2+b}{a+b^2} = a - b^2 + \frac{b^4+b}{a+b^2}$$

implies $p^m = a + b^2 \mid b^4 + b = b(b^3+1)$. From $a \ne b$, we get $b < 1+b < a+b^2$. As $\gcd(b, b^3+1) = 1$, so p^m divides $b^3+1 = (b+1)(b^2-b+1)$.

Next, by the Euclidean algorithm, we have $\gcd(b+1,b^2-b+1) = \gcd(b+1,3) \mid 3$.

Assume we have $\gcd(b+1,b^2-b+1)=1$. Then $b^2+a=p^m$ divides only one of $b+1$ or b^2-b+1. However, both $b+1, b^2-b+1< b^2+a=p^m$. Hence, $b+1$ and b^2-b+1 must be divisible by p. Then the assumption is false and

$$p = \gcd(b+1,b^2-b+1) = 3. \qquad (*)$$

If $m = 1$, then $b^2+a = 3$ has no solution. If $m = 2$, then $b^2+a = 9$ yields $(a,b) = (5,2)$.

For $m \ge 3$, by (*), one of $b+1$ or b^2-b+1 is divisible by 3, while the other one is divisible by 3^{m-1}. Since

$$b+1 < \sqrt{b^2+a}+1 = 3^{m/2}+1 < 3^{m-1},$$

so $3^{m-1} \mid b^2-b+1$. Since $m \ge 3$, we have $b^2 - b + 1 \equiv 0 \pmod 9$. Checking $b \equiv -4, -3, -2, -1, 0, 1, 2, 3, 4 \pmod 9$ shows there cannot be any solution.

Problem 344. *ABCD* is a cyclic quadrilateral. Let *M, N* be midpoints of diagonals *AC, BD* respectively. Lines *BA, CD* intersect at *E* and lines *AD, BC* intersect at *F*. Prove that

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\frac{2MN}{EF}.$$

***Solution* 1. LEE Kai Seng** (HKUST).

Without loss of generality, let the circumcircle of *ABCD* be the unit circle in the complex plane. We have

$M=(A+C)/2$ and $N=(B+D)/2$.

The equations of lines *AB* and *CD* are

$$Z+AB\overline{Z}=A+B$$

and

$$Z+CD\overline{Z}=C+D$$

respectively. Solving for *Z*, we get

$$E=Z=\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{\overline{AB}-\overline{CD}}.$$

Similarly,

$$F=\frac{\overline{A}-\overline{B}-\overline{C}+\overline{D}}{\overline{AD}-\overline{BC}}.$$

In terms of *A, B, C, D*, we have

$2MN=|A+C-B-D|$,

$$EF=\left|\overline{E}-\overline{F}\right|$$

$$=\left|\frac{A+B-C-D}{AB-CD}-\frac{A-B-C+D}{AD-BC}\right|$$

$$=\left|\frac{(B-D)(C-A)(A+C-B-D)}{(AB-CD)(AD-BC)}\right|.$$

The left and right hand sides of the equation become

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\left|\frac{|B-D|^2-|A-C|^2}{(A-C)(B-D)}\right|,$$

$$\frac{2MN}{EF}=\left|\frac{(AB-CD)(AD-BC)}{(B-D)(C-A)}\right|.$$

It suffices to show the numerators of the right sides are equal. We have

$$\left||B-D|^2-|A-C|^2\right|$$

$$=\left|(B-D)(\overline{B}-\overline{D})-(A-C)(\overline{A}-\overline{C})\right|$$

$$=\left|A\overline{C}+C\overline{A}-B\overline{D}-D\overline{B}\right|$$

and

$$\left|(AB-CD)(AD-BC)\right|$$

$$=\left|(AB-CD)(\overline{AD}-\overline{BC})\right|$$

$$=\left|B\overline{D}-C\overline{A}-A\overline{C}+D\overline{B}\right|.$$

Comments: For complex method of solving geometry problems, please see <u>Math Excalibur</u>, *vol*. 9, *no*. 1.

***Solution* 2. CHUNG Ping Ngai** (La Salle College, Form 6).

Without loss of generality, let $AC > BD$. Since $\angle EAC=\angle EDB$ and $\angle AEC=\angle DEB$, we get $\Delta AEC\sim\Delta DEB$. Then

$$\frac{AE}{DE}=\frac{AC}{DB}=\frac{AM}{DN}=\frac{MC}{DB}$$

and $\angle ECA=\angle EBD$. So $\Delta AEM\sim\Delta DEN$ and $\Delta CEM\sim\Delta BEN$. Similarly, we have $\Delta AFC\sim\Delta BFD$, $\Delta AFM\sim\Delta BFN$ and $\Delta CFM\sim\Delta DFN$. Then

$$\frac{EN}{EM}=\frac{DE}{AE}=\frac{BD}{AC}=\frac{FB}{FA}=\frac{FN}{FM}.\quad (*)$$

Define *Q* so that *QENF* is a parallelogram. Let $P=MQ\cap EF$. Then

$\angle EQF=\angle FNE=180°-\angle ENB-\angle FND$
$=180°-\angle EMC-\angle FMC=180°-\angle EMF.$

Hence, *M, E, Q, F* are concyclic. Then $\angle MEQ=180°-\angle MFQ$.

By (1), $EN\times FM=EM\times FN$. Then

$[EMQ]=½\, EM\times FN\sin\angle MEQ$
$=½\, EN\times FM\sin\angle MFQ=[FMQ]$,

where $[XYZ]$ denotes the area of ΔXYZ. Then $EP=FP$, which implies *M, N, P, Q* are collinear. Due to *M, E, Q, F* concyclic, so $\Delta PEM\sim\Delta PQF$ and $\Delta PEQ\sim\Delta PMF$. Then

$$\frac{EM}{EN}=\frac{EM}{QF}=\frac{PM}{PF},\ \frac{FN}{FM}=\frac{QE}{FM}=\frac{QP}{PF}=\frac{NP}{PF}.$$

Using these relations, we have

$$\frac{AC}{BD}-\frac{BD}{AC}=\frac{EM}{EN}-\frac{FN}{FM}$$

$$=\frac{MP}{PF}-\frac{NP}{PF}=\frac{MN}{EF/2},$$

which is the desired equation.

Problem 345. Let a_1, a_2, a_3, ⋯ be a sequence of integers such that there are infinitely many positive terms and also infinitely many negative terms. For every positive integer *n*, the remainders of a_1, a_2, ⋯, a_n upon divisions by *n* are all distinct. Prove that every integer appears exactly one time in the sequence.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE and Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Assume there are $i>j$ such that $a_i=a_j$. Then for $n>i$, $a_i\equiv a_j \pmod n$, which is a contradiction. So any number appears at most once.

Next, for every positive integer *n*, let $S_n=\{a_1, a_2, \ldots, a_n\}$, max $S_n=a_v$ and min $S_n=a_w$. If $k=a_v-a_w\ge n$, then $k\ge n\ge v, w$ and $a_v\equiv a_w \pmod k$, contradicting the given fact. So

$$\max S_n-\min S_n=a_v-a_w\le n-1.$$

Now $S_n\subseteq[\min S_n, \max S_n]$ and both contain *n* integers. So the *n* numbers in S_n are the *n* consecutive integers from min S_n to max S_n.

Now for every integer *m*, since there are infinitely many positive terms and also infinitely many negative terms, there exists a_p and a_q such that $a_p<m<a_q$. Let $r>\max\{p,q\}$, then *m* is in S_r. Therefore, every integer appears exactly one time in the sequence.

Comment: An example of such a sequence is 0, 1, −1, 2, −2, 3, −3, ….

Olympiad Corner

(continued from page 1)

Problem 4. Let $a\in\mathbb{N}$ be such that for every $n\in\mathbb{N}$, $4(a^n+1)$ is a perfect cube. Show that $a=1$.

Problem 5. We want to choose some phone numbers for a new city. The phone numbers should consist of exactly ten digits, and 0 is not allowed as a digit in them. To make sure that different phone numbers are not confused with each other, we want every two phone numbers to either be different in at least two places or have digits separated by at least 2 units, in at least one of the ten places.

What is the maximum number of phone numbers that can be chosen, satisfying the constraints? In how many ways can one choose this amount of phone numbers?

Problem 6. Let *ABC* be a triangle and *H* be the foot of the altitude drawn from *A*. Let *T, T'* be the feet of the perpendicular lines drawn from *H* onto *AB, AC*, respectively. Let *O* be the circumcenter of ΔABC, and assume that $AC=2OT$. Prove that $AB=2OT'$.

// Mathematical Excalibur

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Olympiad Corner

Below are the problems used in the selection of the Indian team for IMO-2010.

Problem 1. Is there a positive integer n, which is a multiple of 103, such that $2^{2n+1} \equiv 2 \pmod n$?

Problem 2. Let a, b, c be integers such that b is even. Suppose the equation $x^3+ax^2+bx+c=0$ has roots α, β, γ such that $\alpha^2 = \beta+\gamma$. Prove that α is an integer and $\beta \neq \gamma$.

Problem 3. Let ABC be a triangle in which $BC < AC$. Let M be the midpoint of AB; AP be the altitude from A on to BC; and BQ be the altitude from B on to AC. Suppose QP produced meet AB (extended) in T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

Problem 4. Let $ABCD$ be a cyclic quadrilateral and let E be the point of intersection of its diagonals AC and BD. Suppose AD and BC meet in F. Let the midpoints of AB and CD be G and H respectively. If Γ is the circumcircle of triangle EGH, prove that FE is tangent to Γ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 20, 2010***.

For individual subscription for the next five issues for the 10-11 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Lagrange Interpolation Formula

Kin Y. Li

Let n be a positive integer. If we are given two collections of n+1 real (or complex) numbers $w_0, w_1, \ldots, w_n$ and $c_0, c_1, \ldots, c_n$ with the w_k's distinct, then there *exists* a *unique* polynomial $P(x)$ of degree at most n satisfying $P(w_k) = c_k$ for $k = 0,1,\ldots,n$. The uniqueness is clear since if $Q(x)$ is also such a polynomial, then $P(x)-Q(x)$ would be a polynomial of degree at most n and have roots at the n+1 numbers $w_0, w_1, \ldots, w_n$, which leads to $P(x)-Q(x)$ be the zero polynomial.

Now, to exhibit such a polynomial, we define $f_0(x)=(x-w_1)(x-w_2)\cdots(x-w_n)$ and similarly for i from 1 to n, define

$f_i(x)=(x-w_0)\cdots(x-w_{i-1})(x-w_{i+1})\cdots(x-w_n)$.

Observe that $f_i(w_k) = 0$ if and only if $i \neq k$. Using this, we see

$$P(x)=\sum_{i=0}^{n} c_i \frac{f_i(x)}{f_i(w_i)}$$

satisfies $P(w_k) = c_k$ for $k = 0,1,\ldots,n$. This is the famous *Lagrange interpolation formula*.

Below we will present some examples of using this formula to solve math problems.

Example 1. (*Romanian Proposal to* 1981 *IMO*) Let P be a polynomial of degree n satisfying for $k = 0,1,\ldots,n$,

$$P(k)=\binom{n+1}{k}^{-1}.$$

Determine $P(n+1)$.

Solution. For $k = 0,1,\ldots,n$, let $w_k=k$ and

$$c_k=\binom{n+1}{k}^{-1}=\frac{k!(n+1-k)!}{(n+1)!}.$$

Define $f_0, f_1, \ldots, f_n$ as above. We get

$$f_k(k) = (-1)^{n-k}k!(n-k)!$$

and

$$f_k(n+1)=\frac{(n+1)!}{(n+1-k)}.$$

By the Lagrange interpolation formula,

$$P(n+1)=\sum_{k=0}^{n} c_k \frac{f_k(n+1)}{f_k(k)}=\sum_{k=0}^{n}(-1)^{n-k},$$

which is 0 if n is odd and 1 if n is even.

Example 2. (*Vietnamese Proposal to* 1977 *IMO*) Suppose $x_0, x_1, \ldots, x_n$ are integers and $x_0 > x_1 > \cdots > x_n$. Prove that one of the numbers $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$ is at least $n!/2^n$, where $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ is a polynomial with real coefficients.

Solution. Define $f_0, f_1, \ldots, f_n$ using $x_0, x_1, \ldots, x_n$. By the Lagrange interpolation formula, we have

$$P(x)=\sum_{i=0}^{n} P(x_i)\frac{f_i(x)}{f_i(x_i)},$$

since both sides are polynomials of degrees at most n and are equal at $x_0, x_1, \ldots, x_n$. Comparing coefficients of x^n, we get

$$1=\sum_{i=0}^{n}\frac{P(x_i)}{f_i(x_i)}.$$

Since $x_0, x_1, \ldots, x_n$ are strictly decreasing integers, we have

$$|f_i(x_i)|=\prod_{j=0}^{i-1}|x_j-x_i|\prod_{j=i+1}^{n}|x_j-x_i|$$
$$\geq i!(n-i)!=\frac{1}{n!}\binom{n}{i}.$$

Let the maximum of $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$ be $|P(x_k)|$. By the triangle inequality, we have

$$1\leq\sum_{i=0}^{n}\frac{|P(x_i)|}{|f_i(x_i)|}\leq\frac{|P(x_k)|}{n!}\sum_{i=0}^{n}\binom{n}{i}=\frac{2^n|P(x_k)|}{n!}.$$

Then $|P(x_k)| \geq n!/2^n$.

Example 3. Let P be a point on the plane of ΔABC. Prove that

$$\frac{PA}{BC}+\frac{PB}{CA}+\frac{PC}{AB}\geq\sqrt{3}.$$

Solution. We may take the plane of ΔABC to be the complex plane and let *P, A, B, C* be corresponded to the complex numbers w, w_1, w_2, w_3 respectively. Then $PA=|w-w_1|$, $BC=|w_2-w_3|$, etc.

Now the only polynomial $P(x)$ of degree at most 2 that equals 1 at w_1, w_2, w_3 is the constant polynomial $P(x)\equiv 1$. So, expressing $P(x)$ by the Lagrange interpolation formula, we have

$$\frac{(x-w_1)(x-w_2)}{(w_3-w_1)(w_3-w_2)}+\frac{(x-w_2)(x-w_3)}{(w_1-w_2)(w_1-w_3)}$$

$$+\frac{(x-w_3)(x-w_1)}{(w_2-w_3)(w_2-w_1)}\equiv 1.$$

Next, setting $x = w$ and applying the triangle inequality, we get

$$\frac{PA}{BC}\frac{PB}{CA}+\frac{PB}{CA}\frac{PC}{AB}+\frac{PC}{AB}\frac{PA}{BC}\ge 1.\quad (*)$$

The inequality $(r+s+t)^2\ge 3(rs+st+tw)$, after subtracting the two sides, reduces to $[(r-s)^2+(s-t)^2+(t-r)^2]/2\ge 0$, which is true. Setting $r= PA/BC$, $s=PB/CA$ and $t=PC/AB$, we get

$$\left(\frac{PA}{BC}+\frac{PB}{CA}+\frac{PC}{AB}\right)^2\ge 3\left(\frac{PA}{BC}\frac{PB}{CA}+\frac{PB}{CA}\frac{PC}{AB}+\frac{PC}{AB}\frac{PA}{BC}\right).$$

Taking square roots of both sides and applying (*), we get the desired inequality.

Example 4. (2002 *USAMO*) Prove that any *monic* polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Solution. Suppose $F(x)$ is a monic real polynomial. Choose real $y_1, y_2, \ldots ,y_n$ such that for odd i, $y_i<\min\{0,2F(i)\}$ and for even i, $y_i>\max\{0,2F(i)\}$.

By the Lagrange interpolation formula, there is a polynomial of degree less than n such that $P(i)=y_i$ for i=1,2,…,n. Let

$$G(x) = P(x)+(x-1)(x-2)\cdots(x-n)$$

and

$$H(x) = 2F(x)-G(x).$$

Then $G(x)$ and $H(x)$ are monic real polynomials of degree n and their average is $F(x)$.

As $y_1, y_3, y_5, \ldots<0$ and $y_2, y_4, y_6, \cdots>0$, $G(i)=y_i$ and $G(i+1)=y_{i+1}$ have opposite signs (hence $G(x)$ has a root in $[i,i+1]$) for i=1,2,…,$n-1$. So $G(x)$ has at least $n-1$ real roots. The other root must also be real since non-real roots come in conjugate pair. Therefore, all roots of $G(x)$ are real.

Similarly, for odd i, $G(i) = y_i < 2F(i)$ implies $H(i)=2F(i)-G(i) > 0$ and for even i, $G(i) = y_i > 2F(i)$ implies $H(i) = 2F(i)-G(i) < 0$. These imply $H(x)$ has n real roots by reasoning similar to $G(x)$.

Example 5. Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ be real numbers such that $b_i-a_j\neq 0$ for i,j=1,2,3,4. Suppose there is a unique set of numbers X_1, X_2, X_3, X_4 such that

$$\frac{X_1}{b_1-a_1}+\frac{X_2}{b_1-a_2}+\frac{X_3}{b_1-a_3}+\frac{X_4}{b_1-a_4}=1,$$

$$\frac{X_1}{b_2-a_1}+\frac{X_2}{b_2-a_2}+\frac{X_3}{b_2-a_3}+\frac{X_4}{b_2-a_4}=1,$$

$$\frac{X_1}{b_3-a_1}+\frac{X_2}{b_3-a_2}+\frac{X_3}{b_3-a_3}+\frac{X_4}{b_3-a_4}=1,$$

$$\frac{X_1}{b_4-a_1}+\frac{X_2}{b_4-a_2}+\frac{X_3}{b_4-a_3}+\frac{X_4}{b_4-a_4}=1.$$

Determine $X_1+X_2+X_3+X_4$ in terms of the a_i's and b_i's.

Solution. Let

$$P(x)=\prod_{i=1}^{4}(x-a_i)-\prod_{i=1}^{4}(x-b_i).$$

Then the coefficient of x^3 in $P(x)$ is

$$\sum_{i=1}^{4}b_i-\sum_{i=1}^{4}a_i.$$

Define f_1, f_2, f_3, f_4 using a_1, a_2, a_3, a_4 as above to get the Lagrange interpolation formula

$$P(x)=\sum_{i=i}^{4}P(a_i)\frac{f_i(x)}{f_i(a_i)}.$$

Since the coefficient of x^3 in $f_i(x)$ is 1, the coefficient of x^3 in $P(x)$ is also

$$\sum_{i=1}^{4}\frac{P(a_i)}{f_i(a_i)}.$$

Next, observe that $P(b_j)/f_i(b_j) = b_j - a_i$, which are the denominators of the four given equations! For j = 1,2,3,4, setting $x = b_j$ in the interpolation formula and dividing both sides by $P(b_j)$, we get

$$1=\sum_{i=i}^{4}\frac{P(a_i)}{P(b_j)}\frac{f_i(b_j)}{f_i(a_i)}=\sum_{i=1}^{4}\frac{P(a_i)/f_i(a_i)}{b_j-a_i}.$$

Comparing with the given equations, by uniqueness, we get $X_i=P(a_i)/f_i(a_i)$ for i = 1,2,3,4. So

$$\sum_{i=1}^{4}X_i=\sum_{i=1}^{4}\frac{P(a_i)}{f_i(a_i)}=\sum_{i=1}^{4}b_i-\sum_{i=}^{4}a_i.$$

Comment: This example is inspired by problem 15 of the 1984 American Invitational Mathematics Examination.

Example 6. (*Italian Proposal to* 1997 *IMO*) Let p be a prime number and let $P(x)$ be a polynomial of degree d with integer coefficients such that:

(i) $P(0) = 0$, $P(1) = 1$;
(ii) for every positive integer n, the remainder of the division of $P(n)$ by p is either 0 or 1.

Prove that $d\ge p-1$.

Solution. By (i) and (ii), we see

$P(0)+P(1)+\cdots+P(p-1)\equiv k \pmod p$ (#)

for some $k\in\{1,2,\ldots, p-1\}$.

Assume $d\le p-2$. Then $P(x)$ will be uniquely determined by the values $P(0)$, $P(1)$, …, $P(p-2)$. Define $f_0, f_1, \ldots, f_{p-2}$ using 0, 1, …, $p-2$ as above to get the Lagrange interpolation formula

$$P(x)=\sum_{k=0}^{p-2}P(k)\frac{f_k(x)}{f_k(k)}.$$

As in example (1), we have

$$f_k(k) = (-1)^{p-2-k}k!(p-2-k)!,$$

$$f_k(p-1)=\frac{(p-1)!}{p-1-k}$$

and so

$$P(p-1)=\sum_{k=0}^{p-2}P(k)(-1)^{p-k}\binom{p-1}{k}.$$

Next, we claim that

$$\binom{p-1}{k}\equiv(-1)^k \pmod p\quad for\quad 0\le k\le p-2.$$

This is true for $k = 0$. Now for $0 < i < p$,

$$\binom{p}{i}=\frac{p!}{i!(p-i)!}\equiv 0 \pmod p$$

because p divides $p!$, but not $i!(p-i)!$. If the claim is true for k, then

$$\binom{p-1}{k+1}=\binom{p}{k+1}-\binom{p-1}{k}\equiv(-1)^{k+1}\pmod p$$

and the induction step follows. Finally the claim yields

$$P(p-1)\equiv(-1)^p\sum_{k=0}^{p-2}P(k)\pmod p.$$

So $P(0)+P(1)+\cdots+P(p-1)\equiv 0 \pmod p$, a contradiction to (#) above.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 20, 2010.***

Problem 351. Let S be a unit sphere with center O. Can there be three arcs on S such that each is a 300° arc on some circle with O as center and no two of the arcs intersect?

Problem 352. (*Proposed by Pedro Henrique O. PANTOJA*, *University of Lisbon, Portugal*) Let a, b, c be real numbers that are at least 1. Prove that

$$\frac{a^2bc}{\sqrt{bc}+1}+\frac{b^2ca}{\sqrt{ca}+1}+\frac{c^2ab}{\sqrt{ab}+1}\ge\frac{3}{2}.$$

Problem 353. Determine all pairs (x, y) of integers such that $x^5-y^2=4$.

Problem 354. For 20 boxers, find the least number n such that there exists a schedule of n matches between pairs of them so that for every three boxers, two of them will face each other in one of the matches.

Problem 355. In a plane, there are two *similar* convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$ Prove that if lines BB_1, CC_1 and DD_1 concur, then $ABCD$ is cyclic. Is the converse also true?

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Solutions

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Problem 346. Let k be a positive integer. Divide $3k$ pebbles into five piles (with possibly unequal number of pebbles). Operate on the five piles by selecting three of them and removing one pebble from each of the three piles. If it is possible to remove all pebbles after k operations, then we say it is a *harmonious ending*.

Determine a necessary and sufficient condition for a harmonious ending to exist in terms of the number k and the distribution of pebbles in the five piles.

(*Source*: 2008 *Zhejiang Province High School Math Competition*)

Solution. **CHOW Tseung Man** (True Light Girl's College), **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1).

The necessary and sufficient condition is every pile has at most k pebbles in the beginning.

The necessity is clear. If there is a pile with more than k pebbles in the beginning, then in each of the k operations, we can only remove at most 1 pebble from that pile, hence we cannot empty the pile after k operations.

For the sufficiency, we will prove by induction. In the case k=1, three pebbles are distributed with each pebble to a different pile. So we can finish in one operation. Suppose the cases less than k are true. For case k, since $3k$ pebbles are distributed. So at most 3 piles have k pebbles. In the first operation, we remove one pebble from each of the three piles with the maximum numbers of pebbles. This will take us to a case less than k. We are done by the inductive assumption.

Problem 347. $P(x)$ is a polynomial of degree n such that for all $w\in\{1, 2, 2^2, \ldots, 2^n\}$, we have $P(w) = 1/w$.

Determine $P(0)$ with proof.

***Solution* 1.** **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy). **William CHAN Wai-lam** (Carmel Alison Lam Foundation Secondary School) and **Thien Nguyen** (Nguyen Van Thien Luong High School, Dong Nai Province, Vietnam).

Let $Q(x) = xP(x)-1 = a(x-1)(x-2)\cdots(x-2^n)$. For $x\neq 1, 2, 2^2, \ldots, 2^n$,

$$\frac{Q'(x)}{Q(x)}=\frac{1}{x-1}+\frac{1}{x-2}+\cdots+\frac{1}{x-2^n}.$$

Since $Q(0)= -1$ and $Q'(x)=P(x)+xP'(x)$,

$$P(0)=Q'(0)=-\frac{Q'(0)}{Q(0)}=\sum_{k=0}^{n}\frac{1}{2^k}=2-\frac{1}{2^n}.$$

Solution 2. **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1), **Abby LEE** (SKH Lam Woo Memorial Secondary School, Form 5) and **WONG Kam Wing** (HKUST, Physics, Year 2).

Let $Q(x) = xP(x)-1 = a(x-1)(x-2)\cdots(x-2^n)$. Now $Q(0) = -1 = a(-1)^{n+1}2^s$, where $s = 1+2+\cdots+n$. So $a = (-1)^n\,2^{-s}$. Then $P(0)$ is the coefficient of x in $Q(x)$, which is

$$a(-1)^n(2^s+2^{s-1}+\cdots+2^{s-n})=\sum_{k=0}^{n}\frac{1}{2^k}=2-\frac{1}{2^n}.$$

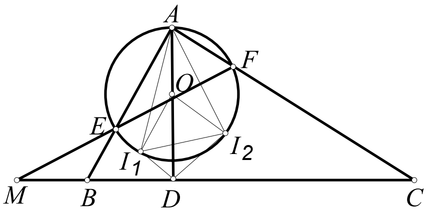
Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil),

Problem 348. In $\triangle ABC$, we have $\angle BAC = 90°$ and $AB < AC$. Let D be the foot of the perpendicular from A to side BC. Let I_1 and I_2 be the incenters of $\triangle ABD$ and $\triangle ACD$ respectively. The circumcircle of $\triangle AI_1I_2$ (with center O) intersects sides AB and AC at E and F respectively. Let M be the intersection of lines EF and BC.

Prove that I_1 or I_2 is the incenter of the $\triangle ODM$, while the other one is an excenter of $\triangle ODM$.

(*Source:* 2008 *Jiangxi Province Math Competition*)

Solution. **CHOW Tseung Man** (True Light Girl's College).



We *claim* EF intersects AD at O. Since $\angle EAF$=90°, EF is a diameter through O. Next we will show O is on AD.

Since AI_1, AI_2 bisect $\angle BAD$, $\angle CAD$ respectively, we get $\angle I_1AI_2$=45°. Then $\angle I_1OI_2$=90°. Since OI_1=OI_2, $\angle OI_1I_2$=45°. Also, DI_1, DI_2 bisect $\angle BDA$, $\angle CDA$ respectively implies $\angle I_1DI_2$=90°. Then D, I_1, O, I_2 are concyclic. So

$$\angle ODI_2=\angle OI_1I_2=45^\circ=\angle ADI_2.$$

Then O is on AD and the claim is true.

Since $\angle EOI_1 = 2\angle EAI_1 = 2\angle DAI_1 = \angle DOI_1$ and I_1 is on the angle bisector of $\angle ODM$, we see I_1 is the incenter of $\triangle ODM$. Similarly, replacing E by F and I_1 by I_2 in the last sentence, we see I_2 is an excenter of $\triangle ODM$.

Other commended solvers: **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1) and **Abby LEE** (SKH Lam Woo Memorial Secondary School, Form 5).

Problem 349. Let $a_1, a_2, \ldots, a_n$ be rational numbers such that for every positive integer m,

$$a_1^m+a_2^m+\cdots+a_n^m$$

is an integer. Prove that $a_1, a_2, \ldots, a_n$ are integers.

Solution. **CHUNG Ping Ngai** (MIT Year 1) and **HUNG Ka Kin Kenneth** (CalTech Year 1)**.**

We may first remove all the integers among $a_1, a_2, \ldots, a_n$ since their m-th powers are integers, so the rest of $a_1, a_2, \ldots, a_n$ will still have the same property. Hence, without loss of generality, we may assume all $a_1, a_2, \ldots, a_n$ are rational numbers and not integers. First write every a_i in simplest term. Let Q be their least common denominator and for all $1 \le i \le n$, let $a_i = k_i/Q$. Take a prime factor p of Q. Then p is not a prime factor of one of the k_i's. So one of the remainders r_i when k_i is divided by p is nonzero! Since $k_i \equiv r_i (\text{mod } p)$, so for every positive integer m,

$$\sum_{i=1}^{n} r_i^m \equiv \sum_{i=1}^{n} k_i^m = \left(\sum_{i=1}^{n} a_i^m\right) Q^m \equiv 0 (\text{mod} p^m).$$

This implies $p^m \le \sum_{i=1}^{n} r_i^m$. Since $r_i < p$,

$$1 \le \lim_{m\to\infty} \frac{1}{p^m} \sum_{i=1}^{n} r_i^m = \lim_{m\to\infty} \sum_{i=1}^{n} \left(\frac{r_i}{p}\right)^m = 0,$$

which is a contradiction.

Comments: In the above solution, it does not need all positive integers m, just an infinite sequence of positive integers m with the given property will be sufficient.

Problem 350. Prove that there exists a positive constant c such that for all positive integer n and all real numbers $a_1, a_2, \ldots, a_n$, if

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

then

$$\max_{x\in[0,2]} |P(x)| \le c^n \max_{x\in[0,1]} |P(x)|.$$

(*Ed.*-Both solutions below show the conclusion holds for any polynomial!)

Solution 1. **LEE Kai Seng.**

Let S be the maximum of $|P(x)|$ for all $x \in [0,1]$. For $i=0,1,2,\ldots,n$, let $b_i = i/n$ and

$$f_i(x) = (x-b_0)\cdots(x-b_{i-1})(x-b_{i+1})\cdots(x-b_n).$$

By the Lagrange interpolation formula, for all real x,

$$P(x) = \sum_{i=0}^{n} P(b_i) \frac{f_i(x)}{f_i(b_i)}.$$

For every $w \in [0,2]$, $|w - b_k| \le |2 - b_k|$ for all $k = 0,1,2,\ldots,n$. So

$$|f_i(w)| \le |f_i(2)| = \prod_{i=0}^{n} \left(2 - \frac{i}{n}\right)$$

$$= \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n^n}$$

$$= \frac{(2n)!}{n!n^n}.$$

Also, $|P(b_i)| \le S$ and

$$|f_i(b_i)| = \frac{i!(n-i)!}{n^n}.$$

By the triangle inequality,

$$|P(w)| \le \sum_{i=0}^{n} |P(b_i)| \frac{|f_i(w)|}{|f_i(b_i)|} \le S \sum_{i=0}^{n} \binom{2n}{i}\binom{2n-i}{n}.$$

Finally,

$$\sum_{i=0}^{n} \binom{2n}{i}\binom{2n-i}{n} \le \sum_{i=0}^{n} \binom{2n}{i}\binom{2n}{n} = 2^{2n}\binom{2n}{n} \le 2^{4n}.$$

Then

$$\max_{w\in[0,2]} |P(w)| \le 2^{4n} S = 16^n \max_{x\in[0,1]} |P(x)|.$$

Solution 2. **G.R.A.20 Problem Solving Group** (Roma, Italy).

For a bounded closed interval I and polynomial $f(x)$, let $\|f\|_I$ denote the maximum of $|f(x)|$ for all x in I. The Chebyschev polynomial of order n is defined by $T_0(x) = 1$, $T_1(x) = x$ and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \text{ for } n \ge 2.$$

(*Ed.*-By induction, we can obtain

$$T_n(x) = 2^n x^n + c_{n-1} x^{n-1} + \cdots + c_0$$

and $T_n(\cos\theta) = \cos n\theta$. So $T_n(\cos(\pi k/n)) = (-1)^k$, which implies all n roots of $T_n(x)$ are in $(-1,1)$ as it changes sign n times.)

It is known that for any polynomial $Q(x)$ with degree at most $n>0$ and all $t \notin [-1,1]$,

$$|Q(t)| \le \|Q\|_{[-1,1]} |T_n(t)|. \qquad (!)$$

To see this, we may assume $\|Q\|_{[-1,1]} = 1$ by dividing $Q(x)$ by such maximum. Assume $x_0 \notin [-1,1]$ and $|Q(x_0)| > |T_n(x_0)|$. Let

$$a = T(x_0)/Q(x_0) \text{ and } R(x) = aQ(x) - T_n(x).$$

For $k = 0, 1, 2, \cdots, n$, since $T_n(\cos(\pi k/n)) = (-1)^k$ and $|a|<1$, we see $R(\cos(\pi k/n))$ is positive or negative depending on whether k is odd or even. (In particular, $R(x) \not\equiv 0$.) By continuity, $R(x)$ has $n+1$ distinct roots on $[-1,1] \cup \{x_0\}$, which contradicts the degree of $R(x)$ is at most n.

Next, for the problem, we claim that for every $t \in [1,2]$, we have $|P(t)| \le 6^n \|P\|_{[0,1]}$.

(*Ed.*-Observe that the change of variable $t = (s+1)/2$ is a bijection between $s \in [-1,1]$ and $t \in [0,1]$. It is also a bijection between $s \in [1,3]$ and $t \in [1,2]$.) By letting $Q(s) = P((s+1)/2)$, the claim is equivalent to proving that for every $s \in [1,3]$, we have $|Q(s)| \le 6^n \|Q\|_{[-1,1]}$. By (!) above, it suffices to show that $|T_n(s)| \le 6^n$ for every $s \in [1,3]$.

Clearly, $|T_0(s)| = 1 = 6^0$. For $n=1$ and $s \in [1,3]$, $|T_1(s)| = s \le 3 < 6$. Next, since the largest root of T_n is less than 1, we see all $T_n(s) > 0$ for all $s \in [1,3]$. Suppose cases $n-2$ and $n-1$ are true. Then for all $s \in [1,3]$, we have $2sT_{n-1}(s)$, $T_{n-2}(s) > 0$ and so

$$\begin{aligned} |T_n(s)| &= |2sT_{n-1}(s) - T_{n-2}(s)| \\ &\le \max(2sT_{n-1}(s), T_{n-2}(s)) \\ &\le \max(6 \cdot 6^{n-1}, 6^{n-2}) = 6^n. \end{aligned}$$

This finishes everything.

Olympiad Corner

(continued from page 1)

Problem 5. Let $A = (a_{jk})$ be a 10×10 array of positive real numbers such that the sum of the numbers in each row as well as in each column is 1. Show that there exist $j<k$ and $l<m$ such that

$$a_{jl}a_{km} + a_{jm}a_{kl} \ge \frac{1}{50}.$$

Problem 6. Let ABC be a triangle. Let AD, BE, CF be cevians such that $\angle BAD = \angle CBE = \angle ACF$. Suppose these cevians concur at a point Ω. (Such a point exists for each triangle and it is called a Brocard point.) Prove that

$$\frac{A\Omega^2}{BC^2} + \frac{B\Omega^2}{CA^2} + \frac{C\Omega^2}{AB^2} \ge 1.$$

(*Ed.*-A cevian is a line segment which joins a vertex of a triangle to a point on the opposite side or its extension.)

Problem 7. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y) + xy = f(x)f(y)$$

for all reals x,y.

Problem 8. Prove that there are infinitely many positive integers m for which there exist consecutive odd positive integers p_m, q_m $(=p_m+2)$ such that the pairs (p_m, q_m) are all distinct and

$$p_m^2 + p_m q_m + q_m^2, \quad p_m^2 + mp_m q_m + q_m^2$$

are both perfect squares.

Mathematical Excalibur

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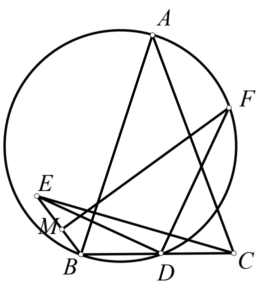
Olympiad Corner

Below are the problems of the 2010 Chinese Girls' Math Olympiad, which was held on August 10-11, 2010.

Problem 1. Let n be an integer greater than two, and let A_1, A_2, …, A_{2n} be pairwise disjoint nonempty subsets of $\{1,2,\ldots,n\}$. Determine the maximum value of $\sum_{i=1}^{2n}\frac{|A_i \cap A_{i+1}|}{|A_i|\cdot|A_{i+1}|}$. (Here we set $A_{2n+1}=A_1$. For a set X, let $|X|$ denote the number of elements in X.)

Problem 2. In $\triangle ABC$, $AB=AC$. Point D is the midpoint of side BC. Point E lies outside $\triangle ABC$ such that $CE\perp AB$ and $BE=BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc AD of the circumcircle of $\triangle ABD$ such that $MF\perp BE$. Prove that $ED\perp FD$.



(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 14, 2011***.

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IMO Shortlisted Problems

Kin Y. Li

Every year, before the IMO begins, a problem selection committee collects problem proposals from many nations. Then it prepares a short list of problems for the leaders to consider when the leaders meet at the IMO site. The following were some of the interesting shortlisted problems in past years that were not chosen. Perhaps some of the ideas may reappear in later proposals in coming years.

Example 1. (*1985 IMO Proposal by Israel*) For which integer $n \ge 3$ does there exist a regular n-gon in the plane such that all its vertices have integer coordinates in a rectangular coordinate system?

Solution. Let A_i have coordinates (x_i,y_i), where x_i, y_i are integers for $i=1,2,\cdots,n$. In the case $n = 3$, if $A_1A_2A_3$ is equilateral, then on one hand, its area is

$$\frac{\sqrt{3}}{4}A_1A_2^2=\frac{\sqrt{3}}{4}\left((x_1-x_2)^2+(y_1-y_2)^2\right),$$

which is irrational. On the other hand, its area is also

$$\frac{\left|\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\right|}{2}=\pm\frac{1}{2}\begin{vmatrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{vmatrix},$$

which is rational. Hence, the case $n = 3$ leads to contradiction. The case $n = 4$ is true by taking (0,0),(0,1),(1,1) and (1,0). The case $n = 6$ is false since $A_1A_3A_5$ would be equilateral.

For the other cases, suppose $A_1A_2\cdots A_n$ is such a regular n-gon with minimal side length. For $i=1,2,\cdots,n$, define point B_i so that $A_iA_{i+1}A_{i+2}B_i$ is a parallelogram (where $A_{n+1}=A_1$ and $A_{n+2}=A_2$). Since $A_{i+1}A_{i+2}$ is parallel to A_iA_{i+3} (where $A_{n+3}=A_3$) and $A_{i+1}A_{i+2} < A_iA_{i+3}$, we see B_i is between A_i and A_{i+3} on the segment A_iA_{i+3}. In particular, B_i is inside $A_1A_2\cdots A_n$.

Next the coordinates of B_i are $(x_{i+2}-x_{i+1}+x_i,\ y_{i+2}-y_{i+1}+y_i)$, both of which are integers. Using A_iA_{i+3} is parallel to $A_{i+1}A_{i+2}$, by subtracting coordinates, we can see $B_i \ne B_{i+1}$ and B_iB_{i+1} is parallel to $A_{i+1}A_{i+2}$. By symmetry, $B_1B_2\cdots B_n$ is a regular n-gon inside $A_1A_2\cdots A_n$. Hence, the side length of $B_1B_2\cdots B_n$ is less than the side length of $A_1A_2\cdots A_n$. This contradicts the side length of $A_1A_2\cdots A_n$ is supposed to be minimal. Therefore, n=4 is the only possible case.

Example 2. (*1987 IMO Proposal by Yugoslavia*) Prove that for every natural number k ($k \ge 2$) there exists an irrational number r such that for every natural number m,

$$[r^m] \equiv -1 \pmod{k}.$$

(Here $[x]$ denotes the greatest integer less than or equal to x.)

(*Comment:* The congruence equation is equivalent to $[r^m]+1$ is divisible by k. Since $[r^m]\le r^m < [r^m]+1$, we want to add a small amount $\delta \in(0,1]$ to r^m to make it an integer divisble by k. If we can get $\delta = s^m$ for some $s\in(0,1)$, then some algebra may lead to a solution.)

Solution. If I have a quadratic equation

$$f(x) = x^2-akx+bk = 0$$

with a, b integers and irrational roots r and s such that $s\in(0,1)$, then $r+s=ak\equiv0 \pmod{k}$ and $rs=bk\equiv0 \pmod{k}$. Using

$$r^{m+1}+s^{m+1}=(r+s)(r^m+s^m)-rs(r^{m-1}+s^{m-1}),$$

by induction on m, we see r^m+s^m is also an integer as cases m=0,1 are clear. So

$$[r^m]+1= r^m+s^m \equiv (r+s)^m \equiv 0 \pmod{k}.$$

Finally, to get such a quadratic, we compute the discriminant $\Delta= a^2k^2-4bk$. By taking $a = 2$ and $b = 1$, we have

$$(2k-2)^2 < \Delta= 4k^2-4k < (2k-1)^2.$$

This leads to roots r, s irrational and

$$\frac{1}{2} < s = \frac{2k-\sqrt{\Delta}}{2} < 1.$$

In the next example, we will need to compute the exponent e of a prime number p such that p^e is the largest power of p dividing $n!$. The formula is

$$e=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots.$$

Basically, since $n!=1\times2\times\cdots\times n$, we first factor out p from numbers between 1 to n that are divisible by p (this gives $[n/p]$ factors of p), then we factor out another p from numbers between 1 to n that are divisible by p^2 (this give $[n/p^2]$ more factors of p) and so on.

Example 3. (*1983 and 1991 IMO Proposal by USSR*) Let a_n be the last nonzero digit (from left to right) in the decimal representation of $n!$. Prove that the sequence $a_1, a_2, a_3, \ldots$ is not periodic after a finite number of terms (equivalently $0.a_1a_2a_3\ldots$ is irrational).

Solution. Assume beginning with the term a_M, the sequence becomes periodic with period t. Then for $m \ge M$, we have $a_{m+t} = a_m$. To get a contradiction, we will do it in steps.

Step 1. For every positive integer k, $(10^k)! = (10^k-1)!\times10^k$ implies

$$a_{10^k}=a_{10^k-1}.$$

Step 2. We can get integers $k > m \ge M$ such that 10^k-10^m is a multiple of t as follow. We factor t into the form 2^r5^sw, where w is an integer relatively prime to 10. By Euler's theorem, $10^{\varphi(w)}-1$ is a number divisible by w. Choose $m = \max\{M,r,s\}$ and $k = m + \varphi(w)$. Then $10^k-10^m = 2^m5^m(10^{\varphi(w)}-1)$ is a multiple of t, say $10^k-10^m=ct$ for some integer c.

Step 3. Let $n = 10^k - 1 + ct$. By periodicity, we have

$$a_n=a_{10^k-1}=a_{10^k}=a_{n+1}.$$

Let $a_n=d$, that is the last nonzero digit of $n!$ is d. Since $(n+1)!=(n+1)\times n!$ and the last nonzero digit of $n+1= 2\times10^k-10^m$ is 9, we see $a_{n+1}=a_n$ implies the units digit of $9d$ is d. Checking d=1 to 9, we see only $d = 5$ is possible. So $n!$ ends in $50\ldots0$.

Step 4. By step 3, we see the prime factorization of $n!$ is of the form 2^r5^sw with w relatively prime to 10 and $s \ge r+1 > r$. However,

$$r=\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]+\left[\frac{n}{2^3}\right]+\cdots$$
$$>\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\cdots=s.$$

This is a contradiction and we are done.

Example 4. (*2001 IMO Proposal by Great Britain*) Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that

$$AP\cdot AG+BP\cdot BG+CP\cdot CG$$

is minimum, and express this minimum value in terms of the side lengths of ABC.

Solution. (*Due to the late Professor Murray Klamkin*) Use a vector system with the origin taken to be the centroid of ABC. Denoting the vector from the origin to the point X by $\boldsymbol{X}$, we have

$$\begin{aligned}&AP\cdot AG+BP\cdot BG+CP\cdot CG\\&=|\boldsymbol{A}-\boldsymbol{P}||\boldsymbol{A}|+|\boldsymbol{B}-\boldsymbol{P}||\boldsymbol{B}|+|\boldsymbol{C}-\boldsymbol{P}||\boldsymbol{C}|\\&\ge|(\boldsymbol{A}-\boldsymbol{P})\cdot\boldsymbol{A}|+|(\boldsymbol{B}-\boldsymbol{P})\cdot\boldsymbol{B}|+|(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{C}|\\&=|\boldsymbol{A}|^2+|\boldsymbol{B}|^2+|\boldsymbol{C}|^2\quad(\text{since }\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{0})\\&=(BC^2+CA^2+AB^2)/3.\end{aligned}$$

Equality holds if and only if

$$|\boldsymbol{A}-\boldsymbol{P}||\boldsymbol{A}|=|(\boldsymbol{A}-\boldsymbol{P})\cdot\boldsymbol{A}|,$$
$$|\boldsymbol{B}-\boldsymbol{P}||\boldsymbol{B}|=|(\boldsymbol{B}-\boldsymbol{P})\cdot\boldsymbol{B}|$$
and $$|\boldsymbol{C}-\boldsymbol{P}||\boldsymbol{C}|=|(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{C}|,$$

which is equivalent to P is on the lines GA, GB and GC, i.e. $P=G$.

The next example is a proof of a theorem of Fermat. It is (the contrapositive of) an infinite descent argument that Fermat might have used.

Example 5. (*1978 IMO Proposal by France*) Prove that for any positive integers x, y, z with $xy-z^2=1$ one can find nonnegative integers a, b, c, d such that $x=a^2+b^2$, $y=c^2+d^2$ and $z=ac+bd$. Set $z = (2n)!$ to deduce that for any prime number $p=4n+1$, p can be represented as the sum of squares of two integers.

Solution. We will prove the first statement by induction on z. If z=1, then $(x,y) = (1,2)$ or $(2,1)$ and we take $(a,b,c,d) = (0,1,1,1)$ or $(1,1,0,1)$ respectively.

Next for integer $w > 1$, suppose cases $z = 1$ to $w-1$ are true. Let positive integers u,v,w satisfy $uv-w^2=1$ with $w>1$. Note $u=v$ leads to w=0, which is absurd. Also $u=w$ leads to w=1, again absurd. Due to symmetry in u, v, we may assume $u < v$. Let $x=u$, $y=u+v-2w$ and $z=w-u$. Since

$$uv = w^2+1 > w^2 = uv-1 > u^2-1,$$

so $y \ge 2(uv)^{1/2}-2w > 0$ and $z = w-u > 0$. Next we can check $xy-z^2 = uv-w^2 = 1$. By inductive hypothesis, we have

$$x = a^2+b^2,\ y = c^2+d^2,\ z = ac+bd.$$

So $u=x=a^2+b^2$, $w= x+z =a^2+b^2+ ac+bd = a(a+c) + b(b+d)$ and $v = y-u+2w = (a+c)^2+(b+d)^2$. This completes the proof of the first statement.

For the second statement, we have

$$\begin{aligned}z^2&=(2n)!(2n)(2n-1)\cdots1\\&=(2n)!(p-(2n+1))\cdots(p-4n)\\&\equiv(-1)^{2n}(4n)!=(p-1)!\equiv-1\pmod{p},\end{aligned}$$

where the last congruence is by Wilson's theorem. This implies z^2+1 is a multiple of p, i.e. $z^2+1=py$ for some positive integer y. By the first statement, we see $p =a^2+b^2$ for some positive integers a and b.

Example 6. (*1997 IMO Proposal by Russia*) An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

Solution. Let a be the first term and d be the common difference. We will prove by induction on d. If d=1, then the terms are consecutive integers, hence the result is true. Next, suppose d>1. Let $r = \gcd(a,d)$ and $h=d/r$, then $\gcd(a/r,h)$=1. We have two cases.

Case 1: $\gcd(r,h) = 1$. Then $\gcd(a,h)$=1. Since there exist x^2 and y^3 in the progression, so x^2 and $y^3 \equiv a \pmod{d}$. Since h divides d, x^2 and $y^3 \equiv a \pmod{h}$. From $\gcd(a,h)$=1, we get $\gcd(y,h)$=1. Then there exists an integer t such that $ty \equiv x \pmod{h}$. So

$$t^6a^2 \equiv t^6y^6 \equiv x^6 \equiv a^3 \pmod{h}.$$

Since $\gcd(a,h)$=1, we may cancel a^2 to get $t^6 \equiv a \pmod{h}$.

Since $\gcd(r,h)$=1, there exists an integer k such that $kh \equiv -t \pmod{r}$. Then we have $(t+kh)^6 \equiv 0 \equiv a \pmod{r}$ and also $(t+kh)^6 \equiv a \pmod{h}$. Since $\gcd(r,h)$=1 and $rh=d$, we get $(t+kh)^6 \equiv a \pmod{d}$. Hence, $(t+kh)^6$ is in the progression.

Case 2: $\gcd(r,h) > 1$. Let p be a prime dividing $\gcd(r,h)$. Then p divides r, which divides a and d. Let p^m be the greatest power of p dividing a and p^n be the greatest power of p dividing d. Since $d = rh$, p divides h and $\gcd(a,d) = r$, we see $n > m \ge 1$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 14, 2011.***

Problem 356. A and B alternately color points on an initially colorless plane as follow. A plays first. When A takes his turn, he will choose a point not yet colored and paint it red. When B takes his turn, he will choose 2010 points not yet colored and paint them blue. When the plane contains three red points that are the vertices of an equilateral triangle, then A wins. Following the rules of the game, can B stop A from winning?

Problem 357. Prove that for every positive integer n, there do not exist four integers a, b, c, d such that $ad=bc$ and $n^2 < a < b < c < d < (n+1)^2$.

Problem 358. $ABCD$ is a cyclic quadrilateral with AC intersects BD at P. Let E, F, G, H be the feet of perpendiculars from P to sides AB, BC, CD, DA respectively. Prove that lines EH, BD, FG are concurrent or are parallel.

Problem 359. (*Due to Michel BATAILLE*) Determine (with proof) all real numbers x,y,z such that $x+y+z \ge 3$ and

$$x^3+y^3+z^3+x^4+y^4+z^4 \le 2(x^2+y^2+z^2).$$

Problem 360. (*Due to Terence ZHU, Affiliated High School of Southern China Normal University*) Let n be a positive integer. We call a set S of at least n distinct positive integers a <u>*n-divisible*</u> set if among every n elements of S, there always exist two of them, one is divisible by the other.

Determine the least integer m (in terms of n) such that every n-divisible set S with m elements contains n integers, one of them is divisible by all the remaining $n-1$ integers.

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Solutions

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Problem 351. Let S be a unit sphere with center O. Can there be three arcs on S such that each is a 300° arc on some circle with O as center and no two of the arcs intersect?

Solution. **Andy LOO** (St. Paul's Co-ed College).

The answer is no. Assume there exist three such arcs l_1, l_2 and l_3. For k=1,2,3, let C_k be the unit circle with center O that l_k is on. Since l_k is a 300° arc on C_k, <u>every point P on C_k is on l_k or its reflection point with respect to O is on l_k</u>. Let P_{ij} and P_{ji} be the intersection points of C_i and C_j. (Since P_{ij} and P_{ji} are reflection points with respect to O, if P_{ij} does not lie on both l_i and l_j, then P_{ji} will be on l_i and l_j, contradiction.) So we may let P_{ij} be the point on l_i and not on l_j and P_{ji} be the point on l_j and not on l_i.

Now P_{21} and P_{31} are on C_1 and outside of l_1, so $\angle P_{21}OP_{31} < 60°$. Hence the length of arcs $P_{21}P_{31}$ and $P_{12}P_{13}$ are equal and are less than $\pi/3$ (and similarly for $\angle P_{32}OP_{12}$, $\angle P_{13}OP_{23}$ and their arcs). Denote the distance (i.e. the length of shortest path) between P and Q on S by $d(P,Q)$. We have

$$\begin{aligned}\pi &= d(P_{12},P_{21})\\ &\le d(P_{12},P_{32})+d(P_{32},P_{31})+d(P_{31},P_{21})\\ &< \pi/3+\pi/3+\pi/3=\pi,\end{aligned}$$

which is absurd.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Problem 352. (*Proposed by Pedro Henrique O. PANTOJA, University of Lisbon, Portugal*) Let a, b, c be real numbers that are at least 1. Prove that

$$\frac{a^2bc}{\sqrt{bc}+1}+\frac{b^2ca}{\sqrt{ca}+1}+\frac{c^2ab}{\sqrt{ab}+1}\ge\frac{3}{2}.$$

Solution. **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

From $a^2\sqrt{bc}\ge\sqrt{bc}\ge 1$, we get

$$\sum_{cyclic}\frac{a^2bc}{\sqrt{bc}+1}\ge\sum_{cyclic}\frac{a^2bc}{2\sqrt{bc}}=\sum_{cyclic}\frac{a^2\sqrt{bc}}{2}\ge\frac{3}{2}.$$

Moreover, we will prove the stronger fact: if $a, b, c > 0$ and $abc \ge 1$, then the inequality still holds. From $k = abc \ge 1$, we get

$$\frac{a^2bc}{\sqrt{bc}+1}=\frac{ka^{3/2}}{\sqrt{k}+a^{1/2}}\ge\frac{a^{3/2}}{1+a^{1/2}},\quad (*)$$

where the inequality can be checked by cross-multiplication. For $x > 0$, define

$$f(x)=\frac{x^{3/2}}{1+x^{1/2}}-\frac{5}{8}\ln x.$$

Its derivative is

$$f'(x)=\frac{(\sqrt{x}-1)(8x^{3/2}+20x+15x^{1/2}+5)}{8x(\sqrt{x}+1)^2}.$$

This shows $f(1)=1/2$ is the minimum value of f, since $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Then by (*),

$$\sum_{cyclic}\frac{a^2bc}{\sqrt{bc}+1}\ge\sum_{cyclic}\frac{a^{3/2}}{1+a^{1/2}}\ge\frac{3}{2}+\frac{5}{8}\ln abc\ge\frac{3}{2}.$$

Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil), **CHAN Chiu Yuen Oscar** (Wah Yan College Hong Kong), **Ozgur KIRCAK** (Jahja Kemal College, Skopje, Macedonia), **LAM Lai Him** (HKUST Math UG Year 2), **Andy LOO** (St. Paul's Co-ed College), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Salem MALIKIĆ** (Student, University of Sarajevo, Bosnia and Herzegovina), **NG Chau Lok** (HKUST Math UG Year 1), **Thien NGUYEN** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (GT(Ellen Yeung) College), **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Karatapanis SAVVAS** (3rd Senior High School of Rhoades, Greece), **TRAN Trong Hoang Tuan John** (Bac Lieu Specialized Secondary School, Vietnam), **WONG Chi Man** (CUHK Info Engg Grad), **WONG Sze Nga** (Diocesan Girls' School), **WONG Tat Yuen Simon** and **POON Lok Wing** (Carmel Divine Grace Foundation Secondary School) and **Simon YAU.**

Problem 353. Determine all pairs (x, y) of integers such that $x^5-y^2=4$.

Solution. **Ozgur KIRCAK** (Jahja Kemal College, Skopje, Macedonia), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy), **Anderson TORRES** (São Paulo, Brazil) and **Ghaleo TSOI Kwok-Wing** (University of Warwick, Year 1).

Let x, y take on values −5 to 5. We see $x^5 \equiv 0$, 1 or 10 (mod 11), but $y^2+4 \equiv 2$, 4, 5, 7, 8 or 9 (mod 11). Therefore, there can be no solution.

Other commended solvers: **Andy LOO** (St. Paul's Co-ed College).

Problem 354. For 20 boxers, find the least number n such that there exists a schedule of n matches between pairs of them so that for every three boxers, two of them will face each other in one of the matches.

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College) and **Andy LOO** (St. Paul's Co-ed College).

Among the boxers, let A be a boxer that will be in the least number of matches, say m matches. For the $19-m$ boxers that do not have a match with A, each pair of them with A form a triple. Since A doesn't play them, every one of these $(19-m)(18-m)/2$ pairs must play each other in a match by the required condition.

For the m boxers that have a match with A, each of them (by the minimal condition on A) has at least m matches. Since each of these matches may be counted at most twice, we get at least $(m+1)m/2$ more matches. So

$$n \geq \frac{(19-m)(18-m)}{2} + \frac{(m+1)m}{2}$$
$$= (m-9)^2 + 90 \geq 90.$$

Finally, $n = 90$ is possible by dividing the 20 boxers into two groups of 10 boxers and in each group, every pair is scheduled a match. This gives a total of 90 matches.

Other commended solvers: **WONG Sze Nga** (Diocesan Girls' School).

Problem 355. In a plane, there are two *similar* convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$ Prove that if lines BB_1, CC_1 and DD_1 concur, then $ABCD$ is cyclic. Is the converse also true?

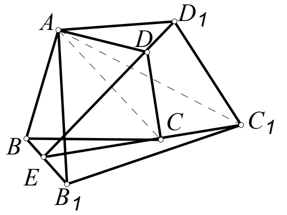
Solution. **CHAN Chiu Yuen Oscar** (Wah Yan College Hong Kong) and **LEE Shing Chi** (SKH Lam Woo Memorial Secondary School).

Since $ABCD$ and $AB_1C_1D_1$ are similar, we have

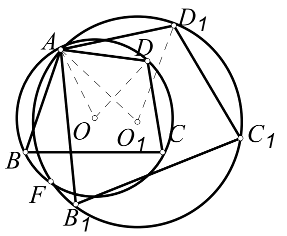
$$\frac{AB}{AB_1} = \frac{AC}{AC_1} = \frac{AD}{AD_1}. \qquad (1)$$

Also, $\triangle ABC$ and $\triangle AB_1C_1$ are similar. Then $\angle BAC = \angle B_1AC_1$. Subtracting $\angle B_1AC$ from both sides, we get $\angle BAB_1 = \angle CAC_1$. Similarly, $\angle CAC_1 = \angle DAD_1$. Along with (1), these give us $\triangle BAB_1$, $\triangle CAC_1$ and $\triangle DAD_1$ are similar. So

$$\angle AB_1B = \angle AC_1C = \angle AD_1D. \qquad (2)$$



Now if lines BB_1, CC_1 and DD_1 concur at E, then (2) can be restated as $\angle AB_1E = \angle AC_1E = \angle AD_1E$. These imply A, B_1, C_1, D_1, E are concyclic. So $AB_1C_1D_1$ is cyclic. Then by similarity, $ABCD$ is cyclic.



For the converse, suppose $ABCD$ is cyclic, then $AB_1C_1D_1$ is cyclic by similarity. Let the two circumcircles intersect at A and F. Let O be the circumcenter of $ABCD$ and O_1 be the circumcenter of $AB_1C_1D_1$. It follows $\triangle AOD$ and $\triangle AO_1D_1$ are similar. Hence $\angle AOD = \angle AO_1D_1$. From this we get

$$\angle AFD = \frac{1}{2}\angle AOD = \frac{1}{2}\angle AOD_1 = \angle AFD_1.$$

This implies line DD_1 passes through F. Similarly, lines BB_1 and CC_1 pass through F. Therefore, lines BB_1, CC_1 and DD_1 concur.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that for every given positive integer n, there exists a prime p and an integer m such that

(a) $p \equiv 5 \pmod 6$;
(b) $p \nmid n$;
(c) $n \equiv m^3 \pmod p$.

Problem 4. Let $x_1, x_2, \ldots, x_n$ (with $n \geq 2$) be real numbers such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

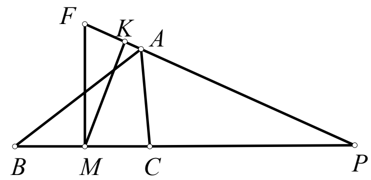
Prove that

$$\sum_{k=1}^{n}\left(1 - \frac{k}{\sum_{i=1}^{n} i x_i^2}\right)^2 \frac{x_k^2}{k} \leq \left(\frac{n-1}{n+1}\right)^2 \sum_{k=1}^{n} \frac{x_k^2}{k}.$$

Determine when equality holds.

Problem 5. Let $f(x)$ and $g(x)$ be strictly increasing linear functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x)$ is an integer if and only if $g(x)$ is an integer. Prove that for any real number x, $f(x) - g(x)$ is an integer.

Problem 6. In acute $\triangle ABC$, $AB > AC$. Let M be the midpoint of side BC. The exterior angle bisector of $\angle BAC$ meets ray BC at P. Points K and F lie on line PA such that $MF \perp BC$ and $MK \perp PA$. Prove that $BC^2 = 4PF \cdot AK$.



Problem 7. Let n be an integer greater than or equal to 3. For a permutation $p = (x_1, x_2, \ldots, x_n)$ of $(1,2,\ldots,n)$, we say x_j *lies between* x_i *and* x_k if $i < j < k$. (For example, in the permutation $(1,3,2,4)$, 3 lies between 1 and 4, and 4 does not lie between 1 and 2.) Set $S = \{p_1, p_2, \ldots, p_m\}$ consists of (distinct) permutations p_i of $(1,2,\ldots,n)$. Suppose that among every three distinct numbers in $\{1,2,\ldots,n\}$, one of these numbers does not lie between the other two numbers in every permutation $p_i \in S$. Determine the maximum value of m.

Problem 8. Determine the least odd number $a > 5$ satisfying the following conditions: There are positive integers m_1, m_2, n_1, n_2 such that $a = m_1^2 + n_1^2$, $a^2 = m_2^2 + n_2^2$ and $m_1 - n_1 = m_2 - n_2$.

IMO Shortlisted Problems

(continued from page 2)

Then p^m divides a and d, hence all terms a, $a+d$, $a+2d$, $\cdots$ of the progression. In particular, p^m divides x^2 and y^3. Hence, m is a multiple of 6.

Consider the arithmetic progression obtained by dividing all terms of a, $a+d$, $a+2d$, $\cdots$ by p^6. All terms are positive integers, the common difference is $d/p^6 < d$ and also contains $(x/p^3)^2$ and $(y/p^2)^3$. By induction hypothesis, this progression contains a sixth power j^6. Then $(pj)^6$ is a sixth power in a, $a+d$, $a+2d$, $\cdots$ and we are done.

Mathematical Excalibur

Volume 15, Number 4 January 2011

Olympiad Corner

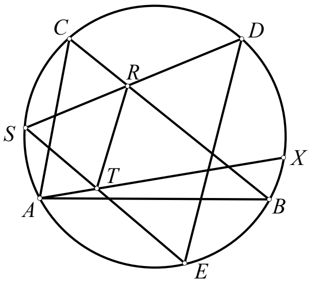
Below are the problems of the 2011 Chinese Math Olympiad, which was held on January 2011.

Problem 1. Let $a_1,a_2,\ldots,a_n$ $(n\geq 3)$ be real numbers. Prove that

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \leq \left[\frac{n}{2}\right](M-m)^2,$$

where $a_{n+1}=a_1$, $M=\max_{1\leq i\leq n} a_i$, $m=\min_{1\leq i\leq n} a_i$, $[x]$ denotes the greatest integer not exceeding x.

Problem 2. In the figure, D is the midpoint of the arc BC on the circumcircle Γ of triangle ABC. Point X is on arc BD. E is the midpoint of arc AX. S is a point on arc AC. Lines SD and BC intersect at point R. Lines SE and AX intersect at point T. Prove that if $RT \parallel DE$, then the incenter of triangle ABC is on line RT.



(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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Klamkin's Inequality

Kin Y. Li

In 1971 Professor Murray Klamkin established the following

Theorem. For any real numbers x,y,z, integer n and angles α,β,γ of any triangle, we have

$$x^2+y^2+z^2 \geq (-1)^{n+1}2(yz\cos n\alpha + zx\cos n\beta + xy\cos n\gamma).$$

Equality holds if and only if

$$\frac{x}{\sin n\alpha}=\frac{y}{\sin n\beta}=\frac{z}{\sin n\gamma}.$$

The proof follows immediately from expanding

$$\left(x+(-1)^n(y\cos n\gamma + z\cos n\beta)\right)^2 + (y\sin n\gamma - z\sin n\beta)^2 \geq 0.$$

There are many nice inequalities that we can obtain from this inequality. The following are some examples (see references [1] and [2] for more).

Example 1. For angles α,β,γ of any triangle, if n is an odd integer, then

$$\cos n\alpha + \cos n\beta + \cos n\gamma \leq 3/2.$$

If n is an even integer, then

$$\cos n\alpha + \cos n\beta + \cos n\gamma \geq -3/2.$$

(This is just the case $x=y=z=1$.)

Example 2. For angles α,β,γ of any triangle,

$$\sqrt{3}\cos\alpha + 2\cos\beta + 2\sqrt{3}\cos\gamma \leq 4.$$

(This is just the case $n=1$, $x=\sin 90^\circ$, $y=\sin 60^\circ$, $z=\sin 30^\circ$.)

There are many symmetric inequalities in α,β,γ, which can be proved by standard identities or methods. However, if we encounter *asymmetric* inequality like the one in example 2, it may be puzzling in coming up with a proof.

Example 3. Let a,b,c be sides of a triangle with area Δ. If r,s,t are any real numbers, then prove that

$$\left(\frac{ar+bs+ct}{4\Delta}\right)^2 \geq \frac{st}{bc}+\frac{tr}{ca}+\frac{rs}{ab}.$$

Solution. Let α,β,γ be the angles of the triangle. We first observe that

$$4\Delta^2 = a^2b^2\sin^2\gamma = b^2c^2\sin^2\alpha = c^2a^2\sin^2\beta$$

and $\cos 2\theta = 1-2\sin^2\theta$. So we can try to set $n=2$, $x=ar$, $y=bs$, $z=ct$. Indeed, after applying Klamkin's inequality, we get the result.

Example 4. Let a,b,c be sides of a triangle with area Δ. Prove that

$$\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2 \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}.$$

Comment: It may seem that we can use example 3 by setting $r=a$, $s=b$, $t=c$, but unfortunately

$$\frac{st}{bc}+\frac{tr}{ca}+\frac{rs}{ab}=3\geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}$$

holds only when $a=b=c$ by the AM-GM inequality.

Solution. To solve this one, we bring in the circumradius R of the triangle. We recall that $2\Delta = bc\sin\alpha$ and by extended sine law, $2R=a/(\sin\alpha)$. So $4\Delta R=abc$. Now we set $r=bcx$, $s=cay$ and $t=abz$. Then the inequality in example 3 becomes

$$(x+y+z)^2R^2 \geq yza^2+zxb^2+xyc^2. \quad (*)$$

Next, we set $yz=1/b^2$, $zx=1/c^2$, $xy=1/a^2$, from which we can solve for x,y,z to get

$$x=\frac{b}{ac}=\frac{b^2}{4\Delta R},\quad y=\frac{c}{4\Delta R},\quad z=\frac{a}{4\Delta R}.$$

Then (*) becomes

$$\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2 \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}.$$

Example 5. (*1998 Korean Math Olympiad*) Postive real numbers a,b,c satisfy $a+b+c=abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\le\frac{3}{2}$$

and determine when equality holds.

Solution. Let $a = \tan u$, $b = \tan v$ and $c = \tan w$, where $u,v,w > 0$. As $a+b+c=abc$,

$\tan u+\tan v+\tan w = \tan u \tan v \tan w$,

which can be written as

$$-\tan u=\frac{\tan v+\tan w}{1-\tan v\tan w}=\tan(v+w).$$

This implies $u+v+w=n\pi$ for some odd positive integer n. Let $\alpha = u/n$, $\beta = v/n$ and $\gamma = w/n$. Taking $x = y = z = 1$ in Klamkin's inequality (as in example 1), we have

$$\cos n\alpha+\cos n\beta+\cos n\gamma\le 3/2,$$

which is the desired inequality. Equality holds if and only if $a = b = c =\sqrt{3}$.

For the next two examples, we will introduce the following

Fact: Three positive real numbers x,y,z satisfy the equation

$$x^2+y^2+z^2+xyz = 4 \qquad (**)$$

if and only if there exists an acute triangle with angles α,β,γ such that

$$x = 2\cos \alpha,\ y = 2\cos \beta,\ z = 2\cos \gamma.$$

Proof. If $x,y,z > 0$ and $x^2+y^2+z^2+xyz = 4$, then x^2, y^2, $z^2 < 4$. So $0 < x, y, z < 2$. Hence, there are positive $\alpha,\beta,\gamma < \pi/2$ such that

$x = 2\cos \alpha$, $y = 2\cos \beta$ and $z = 2\cos \gamma$.

Substituting these into (**) and simplifying, we get $\cos \gamma = -\cos (\alpha+\beta)$, which implies $\alpha+\beta+ \gamma = \pi$. We can get the converse by using trigonometric identities.

Example 6. (*1995 IMO Shortlisted Problem*) Let a,b,c be positive real numbers. Determine all positive real numbers x,y,z satisfying the system of equations

$$x+y+z = a+b+c,$$

$$4xyz-(a^2x+b^2y+c^2z) = abc.$$

Solution. We can rewrite the second equation as

$$\left(\frac{a}{\sqrt{yz}}\right)^2+\left(\frac{b}{\sqrt{zx}}\right)^2+\left(\frac{c}{\sqrt{xy}}\right)^2+\frac{abc}{xyz}=4.$$

By the fact, there exists an acute triangle with angles α,β,γ such that

$$\frac{a}{\sqrt{yz}}=2\cos\alpha,\frac{b}{\sqrt{zx}}=2\cos\beta,\frac{c}{\sqrt{xy}}=2\cos\gamma.$$

Then the first equation becomes

$$x+y+z=2(\sqrt{yz}\cos\alpha+\sqrt{zx}\cos\beta+\sqrt{xy}\cos\gamma).$$

This is the equality case of Klamkin's inequality. So

$$\frac{\sqrt{x}}{\sin\alpha}=\frac{\sqrt{y}}{\sin\beta}=\frac{\sqrt{z}}{\sin\gamma}.$$

As $\gamma+\beta = \pi-\alpha$, so $\sin(\gamma+\beta)/\sin \alpha=1$. Then

$$\frac{b}{2x}+\frac{c}{2x}=\frac{\sqrt{z}}{\sqrt{x}}\cos\beta+\frac{\sqrt{y}}{\sqrt{x}}\cos\gamma$$

$$=\frac{\sin\gamma\cos\beta+\sin\beta\cos\gamma}{\sin\alpha}=1.$$

So $x = (b+c)/2$. Similarly, $y = (c+a)/2$ and $z = (a+b)/2$.

Example 7. (*2007 IMO Chinese Team Training Test*) Positive real numbers u,v,w satisfy the equation $u+v+w+\sqrt{uvw}=4$.

Prove that

$$\sqrt{\frac{vw}{u}}+\sqrt{\frac{uw}{v}}+\sqrt{\frac{uv}{w}}\ge u+v+w.$$

Solution. By the fact, there exists an acute triangle with angles α,β,γ such that

$$\sqrt{u}=2\cos\alpha,\sqrt{v}=2\cos\beta,\sqrt{w}=2\cos\gamma.$$

The desired inequality becomes

$$\frac{2\cos\beta\cos\gamma}{\cos\alpha}+\frac{2\cos\gamma\cos\alpha}{\cos\beta}+\frac{2\cos\alpha\cos\beta}{\cos\gamma}$$

$$\ge 4(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

Comparing with Klamkin's inequality, all we have to do is to take $n = 1$ and

$$x=\sqrt{\frac{2\cos\beta\cos\gamma}{\cos\alpha}},\qquad y=\sqrt{\frac{2\cos\gamma\cos\alpha}{\cos\beta}}$$

$$z=\sqrt{\frac{2\cos\alpha\cos\beta}{\cos\gamma}}.$$

Example 8. (*1988 IMO Shortlisted Problem*) Let n be an integer greater than 1. For $i=1,2,\ldots,n$, $\alpha_i> 0$, $\beta_i> 0$ and

$$\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=\pi.$$

Prove that $\displaystyle\sum_{i=1}^n\frac{\cos\beta_i}{\sin\alpha_i}\le\sum_{i=1}^n\cot\alpha_i.$

Solution. For $n = 2$, we have equality

$$\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}=\frac{\cos\beta_1}{\sin\alpha_1}-\frac{\cos\beta_1}{\sin\alpha_1}$$

$$=0=\cot\alpha_1+\cot\alpha_2.$$

For $n = 3$, α_1, α_2, α_3 are angles of a triangle, say with opposite sides a,b,c. Let Δ be the area of the triangle. Now $2\Delta = bc\sin \alpha_1 = ca\sin \alpha_2 = ab\sin \alpha_3$. Combining with the cosine law, we get

$$\cot\alpha_1=\frac{\cos\alpha_1}{\sin\alpha_1}=\frac{b^2+c^2-a^2}{4\Delta}$$

and similarly for $\cot \alpha_2$ and $\cot \alpha_3$. By Klamkin's inequality,

$$\sum_{i=1}^n\frac{4\Delta\cos\beta_i}{\sin\alpha_i}=2(bc\cos\beta_1+ca\cos\beta_2+ab\cos\beta_3)$$

$$\le a^2+b^2+c^2=\sum_{i=1}^3 4\Delta\cot\alpha_i.$$

Cancelling 4Δ, we will finish the case $n = 3$. For the case $n > 3$, suppose the case $n-1$ is true. We have

$$\sum_{i=1}^n\frac{\cos\beta_i}{\sin\alpha_i}=\left[\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}-\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$+\left[\sum_{i=3}^n\frac{\cos\beta_i}{\sin\alpha_i}+\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$=\left[\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}+\frac{\cos(\pi-(\beta_1+\beta_2))}{\sin(\pi-(\alpha_1+\alpha_2))}\right]$$

$$+\left[\sum_{i=3}^n\frac{\cos\beta_i}{\sin\alpha_i}+\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$\le\left[\cot\alpha_1+\cot\alpha_2+\cot(\pi-(\alpha_1+\alpha_2))\right]$$

$$+\left[\sum_{i=3}^n\cot\alpha_i+\cot(\alpha_1+\alpha_2)\right]$$

$$=\sum_{i=1}^n\cot\alpha_i.$$

This finishes the induction.

References

[1] M.S.Klamkin, "*Asymetric Triangle Inequalities*," Publ.Elektrotehn. Fak. Ser. Mat. Fiz. Univ. Beograd, No. 357-380 (1971) pp. 33-44.

[2] Zhu Hua-Wei, From Mathematical Competitions to Competition Mathematics, Science Press, 2009 (in Chinese).

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 28, 2011.***

Problem 361. Among all real numbers a and b satisfying the property that the equation $x^4+ax^3+bx^2+ax+1=0$ has a real root, determine the minimum possible value of a^2+b^2 with proof.

Problem 362. Determine all positive rational numbers x,y,z such that

$$x+y+z,\ \ xyz,\ \ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

are integers.

Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB=AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP = \angle ACB$.

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

Problem 365. For nonnegative real numbers a,b,c satisfying $ab+bc+ca=1$, prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c}\geq 2.$$

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Solutions

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Problem 356. A and B alternately color points on an initially colorless plane as follow. A plays first. When A takes his turn, he will choose a point not yet colored and paint it red. When B takes his turn, he will choose 2010 points not yet colored and paint them blue. When the plane contains three red points that are the vertices of an equilateral triangle, then A wins. Following the rules of the game, can B stop A from winning?

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Anna PUN Ying** (HKU Math) and **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School).

The answer is negative. In the first $2n$ moves, A can color n red points <u>*on a line*</u>, while B can color $2010n$ blue points. For each pair of the n red points A colored, there are two points (on the perpendicular bisector of the pair) that can be chosen as vertices for making equilateral triangles with the pair. When $n > 2011$, we have

$$2\binom{n}{2}=n(n-1)>2010n.$$

Then B cannot stop A from winning.

Other commended solvers: **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Andy LOO** (St. Paul's Co-ed College),**Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy) and **Lorenzo PASCALI** (Università di Roma "La Sapienza", Roma, Italy), **WONG Sze Nga** (Diocesan Girls' School).

Problem 357. Prove that for every positive integer n, there do not exist four integers a, b, c, d such that $ad=bc$ and $n^2 < a < b < c < d < (n+1)^2$.

Solution. **U. BATZORIG** (National University of Mongolia) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

We first prove a useful

<u>*Fact (Four Number Theorem)*</u>: Let a,b,c,d be positive integers with $ad=bc$, then there exists positive integers p,q,r,s such that $a=pq,\ b=qr,\ c=ps,\ d=rs.$

To see this, let $p=\gcd(a,c)$, then $p|a$ and $p|c$. So $q=a/p$ and $s=c/p$ are positive integers. Now $p=\gcd(a,c)$ implies $\gcd(q,s)=1$. From $ad=bc$, we get $qd=sb$. Then $s|d$. So $r=d/s$ is a positive integer and $a=pq,\ b=qr,\ c=ps,\ d=rs.$

For the problem, assume a,b,c,d exist as required. Applying the fact, since $d > b > a$, we get $s>q$ and $r>p$. Then $s\geq q+1$, $r\geq p+1$ and we get

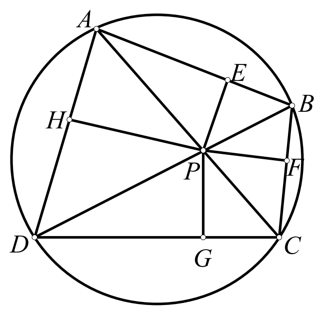
$$d=rs\geq(p+1)(q+1)\geq(\sqrt{pq}+1)^2$$
$$=(\sqrt{a}+1)^2>(n+1)^2,$$

a contradiction.

Other commended solvers: **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Anna PUN Ying** (HKU Math), **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **WONG Sze Nga** (Diocesan Girls' School).

Problem 358. $ABCD$ is a cyclic quadrilateral with AC intersects BD at P. Let E, F, G, H be the feet of perpendiculars from P to sides AB, BC, CD, DA respectively. Prove that lines EH, BD, FG are concurrent or are parallel.

Solution. **U. BATZORIG** (National University of Mongolia), **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Abby LEE Shing Chi** (SKH Lam Woo Memorial Secondary School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Anna PUN Ying** (HKU Math), **Anderson TORRES** (São Paulo, Brazil) and **WONG Sze Nga** (Diocesan Girls' School).



Since $ABCD$ is cyclic, $\angle BAC = \angle CDB$ and $\angle ABD = \angle DCA$, which imply ΔAPB and ΔDPC are similar. As E and G are feet of perpendiculars from P to these triangles (and similarity implies the corresponding segments of triangles are proportional), we get $AE/EB=DG/GC$. Similarly, we get $AH/HD=BF/FC$.

If $EH \parallel BD$, then $AE/EB = AH/HD$, which is equivalent to $DG/GC=BF/FC$, and hence $FG \parallel BD$.

Otherwise, lines EH and BD intersect at some point I. By Menelaus theorem and its converse, we have

$$\frac{\overrightarrow{AE}}{\overrightarrow{EB}}\cdot\frac{\overrightarrow{BI}}{\overrightarrow{ID}}\cdot\frac{\overrightarrow{DH}}{\overrightarrow{HA}}=-1,$$

which is equivalent to

$$\frac{\overrightarrow{BI}}{\overrightarrow{ID}}\cdot\frac{\overrightarrow{DG}}{\overrightarrow{GC}}\cdot\frac{\overrightarrow{CF}}{\overrightarrow{FB}}=-1,$$

and lines BD and FG also intersect at I.

Other commended solvers: **Lorenzo PASCALI** (Università di Roma "La Sapienza", Roma, Italy).

Problem 359. (*Due to Michel BATAILLE*) Determine (with proof) all real numbers x,y,z such that $x+y+z \ge 3$ and

$$x^3+y^3+z^3+x^4+y^4+z^4 \le 2(x^2+y^2+z^2).$$

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy) and **Terence ZHU** (Affilated High School of South China Normal University).

Let x,y,z be real numbers satisfying the conditions. For all real w, $w^2+3w+3 \ge (w+3/2)^2$ implies $(w^2+3w+3)(w-1)^2 \ge 0$. Expanding, we get (*) $w^4+w^3-2w^2 \ge 3w-3$. Applying (*) to $w=x,y,z$ and adding, then using the conditions on x,y,z, we get

$$0\ge x^3+y^3+z^3+x^4+y^4+z^4-2(x^2+y^2+z^2)$$
$$\ge 3(x+y+z)-9\ge 0.$$

Thus, for such x,y,z, we must have equalities in the (*) inequality for x,y,z. So $x=y=z=1$ is the only solution.

Comments: For the idea behind this solution, we refer the readers to the article on the tangent line method (see *Math Excalibur*, *vol*. 10, *no*. 5, *page* 1). For those who do not know this method, we provide the

Proposer's solution. Suppose (x,y,z) is a solution. Let $s=x+y+z$ and $S=x^2y+y^2z+z^2x+xy^2+yz^2+zx^2$. By expansion, we have $s(x^2+y^2+z^2)-S=x^3+y^3+z^3$. Hence,

$$s(x^2+y^2+z^2)-S+x^4+y^4+z^4 \le 2(x^2+y^2+z^2),$$

which is equivalent to

$$(s-2)(x^2+y^2+z^2)+x^4+y^4+z^4 \le S. \qquad (*)$$

Since S is the dot product of the vectors $v=(x^2,y^2,z^2,x,y,z)$ and $w=(y,z,x,y^2,z^2,x^2)$, by the Cauchy Schwarz inequality,

$$S \le x^2+y^2+z^2+x^4+y^4+z^4. \qquad (**)$$

Combining (*) and (**), we conclude $(s-3)(x^2+y^2+z^2)\le 0$. Since $s\ge 3$, we get $s=3$ and (*) and (**) are equalities. Hence, vectors v and w are scalar multiple of each other. Since x,y,z are not all zeros, simple algebra yields $x=y=z=1$. This is the only solution.

Comments: Some solvers overlooked the possibility that x or y or z may be negative in applying the Cauchy Schwarz inequality!

Other commended solvers: **U. BATZORIG** (National University of Mongolia) and **Shaarvdorj** (11th High School of UB, Mongolia), **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Thien NGUYEN** (Luong The Vinh High School, Dong Nai, Vietnam), **Anna PUN Ying** (HKU Math), **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **WONG Sze Nga** (Diocesan Girls' School).

Problem 360. (*Due to Terence ZHU, Affiliated High School of Southern China Normal University*) Let n be a positive integer. We call a set S of at least n distinct positive integers a *n-divisible* set if among every n elements of S, there always exist two of them, one is divisible by the other.

Determine the least integer m (in terms of n) such that every n-divisible set S with m elements contains n integers, one of them is divisible by all the remaining $n-1$ integers.

Solution. **Anna PUN Ying** (HKU Math) and the proposer independently.

The smallest m is $(n-1)^2+1$. First choose distinct prime numbers $p_1, p_2, \ldots, p_{n-1}$. For i from 1 to $n-1$, let

$$A_i=\{p_i,p_i^2,\ldots,p_i^{n-1}\}$$

and let A be any nonempty subset of their union. Then A is n-divisible because among every n of the elements, by the pigeonhole principle, two of them will be in the same A_i, then one is divisible by the other. However, among n elements, two of them will also be in different A_i's and neither one is divisible by the other. So $m \le (n-1)^2$ will not work.

If $m \ge (n-1)^2+1$ and S is a n-divisible set with m elements, then let k_1 be the largest element in S and let B_1 be the subset of S consisted of all the divisors of k_1 in S. Let k_2 be the largest element in S and not in B_1. Let B_2 be the subset of S consisted of all the divisors of k_2 in S and not in B_1. Repeat this to get a partition of S.

Assume there are at least n of these B_i set. For i from 1 to n, let j_i be the largest element in B_i. However, by the definition of the B_i sets, $\{j_1,j_2,\ldots,j_n\}$ contradicts the n-divisiblity of S. So there are at most $n-1$ B_i's.

Since $m \ge (n-1)^2+1$, one of the B_i must have at least n elements. Then for S, we can choose n elements from this B_i with k_i included so that k_i is divisible by all the remaining $n-1$ integers. Therefore, the least m is $(n-1)^2+1$.

Other commended solvers: **WONG Sze Nga** (Diocesan Girls' School).

Olympiad Corner

(continued from page 1)

Problem 3. Let A be a finite set of real numbers. $A_1,A_2,\ldots,A_n$ are nonempty subsets of A satisfying the following conditions:

(1) the sum of all elements in A is 0;
(2) for every $x_i\in A_i$ ($i=1,2,\ldots,n$), we have $x_1+x_2+\cdots+x_n > 0$.

Prove that there exist $1\le i_1<i_2<\cdots<i_k\le n$ such that

$$\left|A_{i_1}\cup A_{i_2}\cup\cdots\cup A_{i_k}\right|<\frac{k}{n}|A|.$$

Here $|X|$ denotes the number of elements in the finite set X.

Problem 4. Let n be a positive integer, set $S=\{1,2,\ldots,n\}$. For nonempty finite sets A and B of real numbers, find the minimum of $|A\ \Delta\ S|+|B\ \Delta\ S|+|C\ \Delta\ S|$, where $C=A+B=\{a+b \mid a\in A, b\in B\}$, $X\ \Delta\ Y=\{x \mid x$ belongs to exactly one of X or $Y\}$, $|X|$ denotes the number of elements in the finite set X.

Problem 5. Let $n\ge 4$ be a given integer. For nonnegative real numbers $a_1,a_2,\ldots,a_n$, $b_1,b_2,\ldots,b_n$ satisfying $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n > 0$, find the maximum of

$$\frac{\sum_{i=1}^{n}a_i(a_i+b_i)}{\sum_{i=1}^{n}b_i(a_i+b_i)}.$$

Problem 6. Prove that for every given positive integers m,n, there exist infinitely many pairs of coprime positive integers a,b such that

$$a+b \mid am^a+bn^b.$$

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2011 Canadian Math Olympiad, which was held on March 23, 2011.

Problem 1. Consider 70-digit numbers n, with the property that each of the digits 1, 2, 3, …, 7 appears in the decimal expansion of n ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.

Problem 2. Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

Problem 3. Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 29, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Harmonic Series (I)

Leung Tat-Wing

A series of the form

$$\frac{1}{m}+\frac{1}{m+d}+\frac{1}{m+2d}+\cdots,$$

where m, d are numbers such that the denominators are never zero, is called a *harmonic series*. For example, the series

$$H(n)=H(1,n)=1+\frac{1}{2}+\ldots+\frac{1}{n}$$

is a harmonic series, or more generally

$$H(m,n)=\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n}$$

is also a harmonic series. Below we always assume $1 \le m < n$. There are many interesting properties concerning this kind of series.

Example 1: $H(1,n)$ is unbounded, i.e. for any positive number A, we can find n big enough, so that $H(1,n) \ge A$.

Solution For any positive integer r, note

$$\frac{1}{r+1}+\frac{1}{r+2}+\cdots+\frac{1}{2r}\ge\frac{1}{2},$$

which can be proved by induction. Hence we can take enough pieces of these fractions to make $H(1,n)$ as large as possible.

Example 2: $H(m,n)$ is never an integer.

Solution (i) For the special case $m = 1$, let s be such that $2^s \le n < 2^{s+1}$. We then multiply $H(1,n)$ by $2^{s-1}Q$, where Q is the product of all odd integers in $[1, n]$. All terms in $H(1,n)$ will become an integer except the term 2^s will become an integer divided by 2 (a half integer). This implies $H(1,n)$ is not an integer.

(ii) Alternatively, for the case $m = 1$, let p be the greatest prime number not exceeding n. By Bertrand's postulate there is a prime q with $p < q < 2p$. Therefore we have $n < 2p$. If $H(1,n)$ is an integer, then

$$n!H(n)=\sum_{i=1}^{n}\frac{n!}{i}$$

is an integer divisible by p. However the term $n!/p$ (an addend) is not divisible by p but all other addends are.

(iii) We deal with the case $m > 1$. Suppose $2^\alpha \mid k$ but $2^{\alpha+1}$ does not divide k (write this as $2^\alpha \parallel k$), then we call α the "parity order" of k. Now observe 2^α, $3{\cdot}2^\alpha$, $5{\cdot}2^\alpha$, ⋯ all have the same parity order. Between these numbers, there are $2{\cdot}2^\alpha$, $4{\cdot}2^\alpha$, $6{\cdot}2^\alpha$, ⋯, all have greater parity orders. Hence, between any two numbers of the same parity order, there is one with greater parity order. This implies among m, m+1, …, n, there is a unique integer with the greatest parity order, say q of parity order μ. Now multiply

$$\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n}$$

by $2^\mu L$, where L is the product of all odd integers in $[m, n]$. Then $2^\mu L{\cdot}H(m,n)$ is an odd number. Hence

$$H(m,n)=\frac{2r+1}{2^\mu L}=\frac{q}{p},$$

where p is even and q is odd and so is not an integer.

Example 3 (APMO 1997): Given that

$$S=1+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{1}{3}+\frac{1}{6}}+\ldots+\frac{1}{1+\frac{1}{3}+\frac{1}{6}+\ldots+\frac{1}{1993006}},$$

where the denominators contains partial sum of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

Solution Let T_n be the nth triangular number. Then $T_n = n(n+1)/2$ and hence

$$\frac{1}{T_1}+\frac{1}{T_2}+\ldots+\frac{1}{T_n}=\frac{2}{1\cdot2}+\frac{2}{2\cdot3}+\ldots+\frac{2}{n(n+1)}$$
$$=2(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1})=2(1-\frac{1}{n+1})=\frac{2n}{n+1}.$$

Since 1993006=1996·1997/2, we get

$$S=\frac{1}{2}\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{1997}{1996}\right)$$
$$>\frac{1}{2}\left(1996+1+\frac{1}{2}+\left(\frac{1}{3}+\cdots+\frac{1}{1024}\right)\right).$$

Hence, S > (1996+6)/2=1001 using example 1 that $H(r+1,2r) \ge 1/2$ for r = 2, 4, 8, 16, 32, 64, 128, 256, 512.

Congruence relations of harmonic series are of some interest. First, let us look at an example.

Example 4 (IMO 1979): Let p, q be natural numbers such that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that p is divisible by 1979.

Solution We will prove the famous Catalan identity (due to N. Botez (1872) and later used by Catalan):

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

It is proved as follows:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

Thus

$$\frac{p}{q}=\frac{1}{660}+\frac{1}{661}+\ldots+\frac{1}{1318}+\frac{1}{1319}$$

$$=\frac{1}{2}\left(\frac{1}{660}+\frac{1}{1319}+\frac{1}{661}+\frac{1}{1318}+\ldots+\frac{1}{1319}+\frac{1}{660}\right)$$

$$=\frac{1}{2}\left(\frac{1979}{660\cdot1319}+\frac{1979}{661\cdot1318}+\ldots+\frac{1979}{1319\cdot660}\right)$$

$$=1979\cdot\frac{A}{B},$$

where B is the product of some positive integers less than 1319. However, 1979 is prime, hence $1979|p$.

For another proof using congruence relations, observe that if $(k,1979)=1$, then by Fermat's little theorem, $k^{1978}\equiv 1 \pmod{1979}$. Hence, we can consider $1/k\equiv k^{1977} \pmod{1979}$. Then

$$\sum_{k=1}^{1319}(-1)^{k-1}\frac{1}{k}\equiv\sum_{k=1}^{1319}(-1)^{k-1}k^{1977}$$

$$=\sum_{k=1}^{1319}k^{1977}-2\sum_{k=1}^{659}(2k)^{1977}$$

$$=\sum_{k=1}^{1319}k^{1977}-2\cdot2^{1977}\sum_{k=1}^{659}k^{1977}$$

$$\equiv\sum_{k=1}^{1319}k^{1977}-\sum_{k=1}^{659}k^{1977}$$

$$=\sum_{k=660}^{1319}k^{1977}=\sum_{k=660}^{989}\left(k^{1977}+(1979-k)^{1977}\right)$$

$$\equiv\sum_{k=660}^{989}\left(k^{1977}+(-k)^{1977}\right)=0 \pmod{1979}.$$

Note that $1/k \pmod p$ (as well as many fraction mod p) makes sense if $k\not\equiv 0 \pmod p$. Also, as a generalization, we have

Example 5: If $H(m,n)=q/p$ and $m+n$ is an odd prime number, then $m+n \mid q$.

Solution Note that $H(m,n)$ has an even number of terms and it equals

$$\sum_{j=0}^{(n-m-1)/2}\left(\frac{1}{m+j}+\frac{1}{n-j}\right)$$

$$=\sum_{j=0}^{(n-m-1)/2}\frac{m+n}{(m+j)(n-j)}=(m+n)\frac{s}{r}.$$

where $\gcd(s,r)=1$. Since $m+n$ is prime, $\gcd(r;m+n)=1$. Then $q/p=(m+n)s/r$ and $m+n \mid q$.

The Catalan identity is also used in the following example.

Example 6 (Rom Math Magazine, July 1998): Let

$$A=\frac{1}{1\cdot2}+\frac{1}{3\cdot4}+\ldots+\frac{1}{2011\cdot2012}$$

and

$$B=\frac{1}{1007\cdot2012}+\frac{1}{1008\cdot2011}+\ldots+\frac{1}{2012\cdot1007}.$$

Evaluate A/B.

Solution

$$A=\sum_{k=1}^{1006}\frac{1}{(2k-1)2k}=\sum_{k=1}^{1006}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$$

$$=\frac{1}{1007}+\frac{1}{1008}+\cdots+\frac{1}{2012}$$

$$=\frac{1}{2}\left(\frac{1}{1007}+\frac{1}{2012}+\frac{1}{1008}+\frac{1}{2011}+\cdots+\frac{1}{2012}+\frac{1}{1007}\right)$$

$$=\frac{1}{2}\left(\frac{3019}{1007\cdot2012}+\frac{3019}{1008\cdot2011}+\cdots+\frac{3019}{2012\cdot1007}\right)$$

$$=\frac{3019B}{2}.$$

Hence $\dfrac{A}{B}=\dfrac{3019}{2}.$

Example 7: Given any proper fraction m/n, where m, n are positive integers satisfying $0<m<n$, then prove it is the sum of fractions of the form

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k},$$

where $x_1, x_2, \ldots, x_k$ are distinct positive integers.

Solution We use the "greedy method". Let x_1 be the positive integer such that

$$\frac{1}{x_1}\le\frac{m}{n}<\frac{1}{x_1-1},$$

i.e. x_1 is the least integer greater than or equal to n/m. If $1/x_1=m/n$, then the problem is done. Otherwise

$$\frac{m}{n}-\frac{1}{x_1}=\frac{mx_1-n}{nx_1}=\frac{m_1}{nx_1},$$

where $m_1=mx_1-n<m$ (due to $m/n<1/(x_1-1)$) and obviously $nx_1>n$. Let x_2 be another positive integer such that

$$\frac{1}{x_2}\le\frac{m_1}{nx_1}<\frac{1}{x_2-1}.$$

The procedure can be repeated until $m>m_1>m_2>\cdots>m_k>0$ and

$$\frac{m}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k},$$

where $1\le k\le m$. (Note: writing

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)},$$

we observe actually there are infinitely many ways of writing any proper fractions as sum of fractions of this kind. These fractions are called *unit fractions* or *Egyptian fractions*.)

Example 8: Remove those terms in

$$1+\frac{1}{2}+\cdots+\frac{1}{n}+\cdots$$

such that its denominator in decimal expansion contains the digit "9", then prove that the sequence is bounded.

Solution The integers without the digit 9 in the interval $[10^{m-1}, 10^m-1]$ are m-digit numbers. The first digit from the left cannot be the digits "0" and "9", (8 choices), the other digits cannot contain "9", hence nine choices 0, 1, 2, 3, 4, 5, 6, 7 and 8. Altogether there are $8\cdot9^{m-1}$ such integers. The sum of their reciprocals is less than

$$\frac{8\cdot9^{m-1}}{10^{m-1}}=8\left(\frac{9}{10}\right)^{m-1}.$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 29, 2011.***

Problem 366. Let n be a positive integer in base 10. For i =1,2,…,9, let $a(i)$ be the number of digits of n that equal i. Prove that

$$2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)} \le n+1$$

and determine all equality cases.

Problem 367. For n = 1,2,3,…, let x_n and y_n be positive real numbers such that

$$x_{n+2} = x_n + x_{n+1}^2$$

and

$$y_{n+2} = y_n^2 + y_{n+1}.$$

If x_1, x_2, y_1, y_2 are all greater than 1, then prove that there exists a positive integer N such that for all $n > N$, we have $x_n > y_n$.

Problem 368. Let C be a circle, A_1, A_2, …, A_n be distinct points inside C and B_1, B_2, …, B_n be distinct points on C such that no two of the segments A_1B_1, A_2B_2,…, A_nB_n intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment A_tB_t ($t \ne r,s$). Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v.

Problem 369. ABC is a triangle with $BC > CA > AB$. D is a point on side BC and E is a point on ray BA beyond A so that $BD=BE=CA$. Let P be a point on side AC such that E, B, D, P are concyclic. Let Q be the intersection point of ray BP and the circumcircle of $\triangle ABC$ different from B. Prove that $AQ+CQ=BP$.

Problem 370. On the coordinate plane, at every lattice point (x,y) (these are points where x, y are integers), there is a light. At time $t = 0$, exactly one light is turned on. For n = 1, 2, 3, …, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was turned on at time $t = n - 1$. Prove that every light at a lattice point will eventually be turned on at some time.

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Solutions

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Problem 361. Among all real numbers a and b satisfying the property that the equation $x^4+ax^3+bx^2+ax+1=0$ has a real root, determine the minimum possible value of a^2+b^2 with proof.

Solution. **U. BATZORIG** (National University of Mongolia) and **Evangelos MOUROUKOS** (Agrinio, Greece).

Consider all a,b such that the equation has x as a real root. The equation implies $x \ne 0$. Using the Cauchy-Schwarz inequality (or looking at the equation as the line $(x^3 + x)a + x^2b + (x^4 + 1) = 0$ in the (a,b)-plane and computing its distance from the origin), as

$$(a^2+2b^2+a^2)\left(x^6+\frac{x^4}{2}+x^2\right)$$

$$\ge (ax^3+bx^2+ax)^2 = (x^4+1)^2,$$

we get $a^2+b^2 \ge \dfrac{(x^4+1)^2}{2x^6+x^4+2x^2}$ with equality if and only if $x = \pm 1$ (at which both sides are 4/5). For $x = 1$, $(a,b) = (-4/5, -2/5)$. For $x = -1$, $(a,b) = (-2/5,4/5)$. Finally,

$$\frac{(x^4+1)^2}{2x^6+x^4+2x^2} \ge \frac{4}{5}$$

by calculus or rewriting it as

$$5(x^4+1)^2 - 4(2x^6+x^4+2x^2)$$
$$= (x^2-1)^2(5x^4+2x^2+5) \ge 0.$$

So the minimum of $a^2 + b^2$ is 4/5.

Other commended solvers: **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Raymond LO** (King's College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 362. Determine all positive rational numbers x,y,z such that

$$x+y+z,\ \ xyz,\ \ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

are integers.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, Raymond LO** (King's College), **Anna PUN Ying** (HKU Math) and **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School).

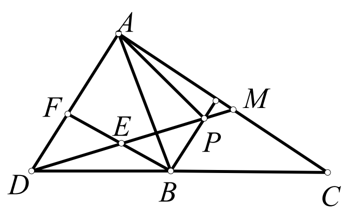
Let $A = x + y + z$, $B = xyz$ and $C = 1/x + 1/y + 1/z$, then A, B, C are integers. Since $xy + yz + zx = BC$, so x,y,z are the roots of the equation $t^3 - At^2 + BCt - B = 0$. Since the coefficients are integers and the coefficient of t^3 is 1, by Gauss lemma or the rational root theorem, the roots x, y, z are integers.

Since they are positive, without loss of generality, we may assume $z \ge y \ge x \ge 1$. Now $1 \le 1/x + 1/y + 1/z \le 3/x$ lead to x=1, 2 or 3. For x = 1, $1/y + 1/z$ = 1 or 2, which yields (y,z) = (1,1) or (2,2). For x = 2, $1/y + 1/z = 1/2$, which yields (y,z) = (3,6) or (4,4). For x = 3, $1/y + 1/z = 2/3$, which yields (y,z) = (3,3). So the solutions are (x,y,z) = (1,1,1), (1,2,2), (2,3,6), (2,4,4), (3,3,3) and permutations of coordinates.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB=AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP = \angle ACB$.

Solution. **Raymond LO** (King's College).



Construct line $BF \parallel$ line CA with F on line AD. Let DM intersect BF at E.

Since $BD=AB$, we get $\angle BDF = \angle BAF = ½\angle ABC = \angle ABP = \angle CBP$. Then line $FD \parallel$ line PB. Hence, $\triangle DFE$ is similar to $\triangle PBE$.

Since $BF \parallel CA$ and M is the midpoint of

AC, so E is the midpoint of FB, i.e. $FE=BE$. Then $\triangle DFE$ is congruent to $\triangle PBE$. Hence, $FD=PB$.

This along with $DB = BA$ and $\angle BDF = \angle ABP$ imply $\triangle BDF$ is congruent to $\triangle ABP$. Therefore, $\angle BAP = \angle DBF = \angle ACB$.

Other commended solvers: **U. BATZORIG** (National University of Mongolia), **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, Abby LEE Shing Chi** (SKH Lam Woo Memorial Secondary School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School), **Ercole SUPPA** (Liceo Scientifico Statale E.Einstein, Teramo, Italy) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

***Solution*.** **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Raymond LO** (King's College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Let n be the least number of locks required. If for every group of 5 robbers, we put a new lock on the box and give a key to each of 6 other robbers only, then the plan works. Thus

$$n \le \binom{11}{5} = 462.$$

Conversely, in the case when there are n locks, for every group G of 5 robbers, there exists a lock $L(G)$, which they do not have the key, but the other 6 robbers all have keys to $L(G)$. Assume there exist $G \neq G'$ such that $L(G)=L(G')$. Then there is a robber in G and not in G'. Since G is one of the 6 robbers not in G', he has a key to $L(G')$, which is $L(G)$, contradiction. So $G \neq G'$ implies $L(G) \neq L(G')$. Then the number of locks is at least as many groups of 5 robbers. So $n \ge \binom{11}{5} = 462$. Therefore, $n = 462$.

Problem 365. For nonnegative real numbers a,b,c satisfying $ab+bc+ca = 1$, prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c} \ge 2.$$

Solution. **CHAN Long Tin** (Diocesan Boys' School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Since $a, b, c \ge 0$ and $ab+bc+ca = 1$, none of the denominators can be zero. Multiplying both sides by $a+b+c$, we need to show

$$\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}+2 \ge 2(a+b+c).$$

This follows from using the Cauchy-Schwarz inequality and expanding $(c+a+b-2)^2 \ge 0$ as shown below

$$2\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right)$$
$$=((a+b)c+(b+c)a+(c+a)b)\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right)$$
$$\ge (c+a+b)^2$$
$$\ge 4(a+b+c)-4.$$

Other commended solvers: **Andrea FANCHINI** (Cantu, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Anna PUN Ying** (HKU Math) and **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

Problem 4. Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a which is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Problem 5. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, where for any k, $\varepsilon_k = 1$ or $\varepsilon_k = -1$, such that

$$S = \varepsilon_1(1+d)^2 + \varepsilon_2(1+2d)^2 + \varepsilon_3(1+3d)^2 + \cdots + \varepsilon_n(1+nd)^2.$$

Harmonic Series (I)

(continued from page 2)

The sum of reciprocals of all such numbers is therefore less than

$$\sum_{m=0}^{\infty} 8\left(\frac{9}{10}\right)^m = \frac{8}{1-\frac{9}{10}} = 80.$$

__Example 9__: Let $m > 1$ be a positive integer. Show that $1/m$ is the sum of consecutive terms in the sequence

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)}.$$

__Solution__ Since

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1},$$

the problem is reduced to finding integers a and b such that

$$\frac{1}{m} = \frac{1}{a} - \frac{1}{b} \quad (*).$$

One obvious solution is $a = m-1$ and $b = m(m-1)$. To find other solutions of (*), we note that $1/a > 1/m$, so $m > a$.

Let $a = m-c$, then $b = (m^2/c)-m$. For each c satisfying $c \mid m^2$ and $1 \le c \le m$, there exists one and only one pair of a and b satisfying (*), and because $a < b$, the representation is unique. Let $d(n)$ count the number of factors of n. Now consider all factors of m^2 except m, there are $d(m^2)-1$ of them. If c is one of them, then exactly one of c or m^2/c will be less than m. Hence the number of solutions of (*) is $[d(m^2)-1]/2$.

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Olympiad Corner

Below are the problems of the 2011 Asia Pacific Math Olympiad, which was held in March 2011.

Problem 1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers a^2+b+c, b^2+c+a, c^2+a+b to be perfect squares.

Problem 2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

Problem 3. Let ABC be an acute triangle with $\angle BAC=30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***June 25, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Harmonic Series (II)

Leung Tat-Wing

As usual, for integers a, b, n (with $n > 0$), we write $a\equiv b \pmod n$ to mean $a-b$ is divisible by n. If <u>$b\neq 0$ and n are relatively prime</u> (i.e. they have no common prime divisor), then $0, b, 2b, \ldots, (n-1)b$ are distinct (mod n) because for $0\le s<r<n$, $rb\equiv sb \pmod n$ implies $(r-s)b = kn$. Since b, n have no common prime divisor, this means b divides k. Then $0 < (k/b)n = r-s < n$, contradicting $b \le k$. Hence, there is a unique r among $1, \ldots, n-1$ such that $rb\equiv 1 \pmod n$. We will denote this r as b^{-1} or $1/b$ (mod n). Further, we can extend (mod n) to fractions by defining $a/b \equiv ab^{-1} \pmod n$. We can easily check that the usual properties of fractions holds in mod n arithmetic.

Next, we will introduce Wolstenholme's theorem, which is an important relation concerning harmonic series.

<u>Theorem (Wolstenholme)</u>: For a prime $p \ge 5$,

$$H(p-1)=1+\frac{1}{2}+\ldots+\frac{1}{p-1}\equiv 0 \pmod{p^2}.$$

(More precisely, for a prime $p\ge 5$, if

$$H(p-1)=1+\frac{1}{2}+\ldots+\frac{1}{p-1}=\frac{a}{b},$$

then $p^2 \mid a$.)

<u>Example</u> We have

$$H(10)=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}=\frac{7381}{2520}$$

and $11^2 \mid 7381$.

<u>First proof</u> We have

$$H(p-1)=1+\frac{1}{2}+\cdots+\frac{1}{p-1}$$
$$=\sum_{n=1}^{(p-1)/2}\left(\frac{1}{n}+\frac{1}{p-n}\right)=p\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}.$$

So we need to prove

$$\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}\equiv 0 \pmod p.$$

Now $\displaystyle\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}\equiv-\sum_{n=1}^{(p-1)/2}\frac{1}{n^2} \pmod p$.

Since every $1/n^2$ is congruent to exactly one of the numbers $1^2, 2^2, \ldots, [(p-1)/2]^2$ (mod p) and $1/n^2$ are all distinct for $n=1,2,\ldots,(p-1)/2$, we have when $p\ge 5$,

$$\sum_{n=1}^{(p-1)/2}\frac{1}{n^2}\equiv\sum_{k=1}^{(p-1)/2}k^2=\frac{(p^2-1)p}{24}\equiv 0 \pmod p.$$

Wolstenholme's theorem follows.

<u>Second proof</u> (using polynomials mod p) We use a theorem of Lagrange, which says if $f(x) = c_0 + c_1x + \cdots + c_nx^n$ is a polynomial of degree n, with integer coefficients, and if $f(x) \equiv 0 \pmod p$ has more than n solutions, where p is prime, then every coefficient of $f(x)$ is divisible by p. The proof is not hard. It can be done basically by induction and the division algorithm mod p. The statement is false if p is not prime. For instance, $x^2 - 1 \equiv 0 \pmod 8$ has 4 solutions. Here is the other proof.

From Fermat's Little theorem, $x^{p-1}\equiv 1 \pmod p$ has $1, 2, \ldots, p-1$ as solutions. Thus $x^{p-1}-1 \equiv (x-1)(x-2)\cdots(x-p+1) \pmod p$. Let

$$(x-1)(x-2)\cdots(x-p+1)$$
$$=x^{p-1}-s_1x^{p-2}+\cdots-s_{p-2}x+s_{p-1}. \quad (*)$$

By Wilson's theorem, $s_{p-1}=(p-1)!\equiv -1 \pmod p$. Thus

$$0\equiv s_1x^{p-2}+\cdots-s_{p-2}x \pmod p.$$

The formula is true for every integer x. By Lagrange's theorem, p divides each of $s_1, s_2, \ldots, s_{p-2}$. Putting $x = p$ in (*), we get $(p-1)!=p^{p-1}-s_1p^{p-2}+\cdots-s_{p-2}p+s_{p-1}$. Canceling out $(p-1)!$ and dividing both sides by p, we get

$$0=p^{p-2}-s_1p^{p-3}+\cdots+s_{p-3}p-s_{p-2}.$$

As $p\ge 5$, each of the terms is congruent to 0 (mod p^2). Hence, we have $s_{p-2}\equiv 0 \pmod{p^2}$. Finally,

$$s_{p-2}=(p-1)!\left(1+\frac{1}{2}+\ldots+\frac{1}{p-1}\right)=(p-1)!\frac{a}{b}.$$

This proves Wolstenholme's theorem.

Using Wolstenholme's theorem and setting $x = kp$ in (*), we get

$$\begin{aligned}&(kp-1)(kp-2)\cdots(kp-p+1)\\ &= (kp)^{p-1}-s_1(kp)^{p-2}+\cdots\\ &\qquad +s_{p-3}(kp)^2-s_{p-2}kp+s_{p-1}\\ &\equiv s_{p-3}(kp)^2 - s_{p-2}kp + s_{p-1}\\ &\equiv (p-1)! \pmod{p^3}.\end{aligned}$$

Upon dividing by $(p-1)!$, we have

$$\binom{kp-1}{p-1} \equiv 1 \pmod{p^3}, \quad k = 1,2,\ldots.$$

This result may in fact be taken as the statement of Wolstenholme's theorem.

Here are a few further remarks. Wolstenholme's theorem on the congruence of harmonic series is related to the Bernoulli numbers B_n. For instance, we have

$$\binom{kp-1}{p-1} \equiv 1 - \frac{1}{3}(k^2-k)p^3 B_{p-3} \pmod{p^4},$$

which is usually called <u>*Glaisher's congruence*</u>. These numbers are related to Fermat's Last Theorem. It is known that for any prime $p \ge 5$,

$$\binom{kp-1}{p-1} \equiv 1 \pmod{p^3}.$$

Are there primes satisfying

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}?$$

These primes are called <u>*Wolstenholme primes*</u>. (So far, we only know 16843 and 2124679 are such primes). In another direction, one can ask if there exist composite numbers n such that

$$\binom{kn-1}{n-1} \equiv 1 \pmod{n^3}?$$

All these are very classical questions.

<u>**Example 10 (APMO 2006)**</u>: Let $p \ge 5$ be a prime and let r be the number of ways of placing p checkers on a $p\times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5.

<u>**Solution**</u> Observe that

$$r = \binom{p^2}{p} - p = p\left(\frac{(p^2-1)\cdots(p^2-(p-1))}{(p-1)!} - 1\right).$$

Hence it suffices to show that

$$(p^2-1)(p^2-2)\cdots(p^2-(p-1)) - (p-1)! \equiv 0 \pmod{p^4} \qquad (1)$$

Now let

$$\begin{aligned} f(x) &= (x-1)(x-2)\cdots(x-(p-1))\\ &= x^{p-1} + s_1x^{p-2} + \cdots + s_{p-2}x + s_{p-1}. \quad (2)\end{aligned}$$

Thus the first congruence relation is the same as $f(p^2) - (p-1)! \equiv 0 \pmod{p^4}$. Therefore it suffices to show that $s_{p-2}p^2 \equiv 0 \pmod{p^4}$ or $s_{p-2} \equiv 0 \pmod{p^2}$, which is exactly Wolstenholme's theorem.

<u>**Example 11 (Putnam 1996)**</u>: Let p be a prime number greater than 3 and $k = [2p/3]$. Show that

$$\binom{p}{1} + \binom{p}{2} + \ldots + \binom{p}{k} \equiv 0 \pmod{p^2}$$

For example,

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 98 \equiv 0 \pmod{7^2}.$$

<u>**Solution**</u> Recall

$$\binom{p}{i} = \frac{p(p-1)\ldots(p-i+1)}{1\cdot 2\cdot\ldots i}.$$

This is a multiple of p if $1 \le i \le p-1$. Modulo p, the right side after divided by p is congruent to

$$\frac{(-1)\cdots(-(i-1))}{1\cdot 2\cdots i} = (-1)^{i-1}\frac{1}{i}.$$

Hence, to prove the congruence, it suffices to show

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{k-1}\frac{1}{k} \equiv 0 \pmod{p}.$$

Now observe that

$$-\frac{1}{2i} \equiv \frac{1}{2i} + \frac{1}{p-i} \pmod{p}.$$

This allows us to replace the sum by

$$1 + \frac{1}{2} + \ldots + \frac{1}{p-1} \equiv 0 \pmod{p},$$

which is Wolstenholme's theorem.

We can also give a more detailed proof as follow. Let

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and

$$P(n) = 1 - \frac{1}{2} + \ldots + (-1)^{n-1}\frac{1}{n}.$$

Then the problem is reduced to showing that for any $p > 3$, p divides the numerator of $P([2p/3])$. First we note that p divides the numerator of $H(p-1)$ because

$$\begin{aligned}&2H(p-1)\\ &= \left(1+\frac{1}{p-1}\right) + \left(\frac{1}{2}+\frac{1}{p-2}\right) + \cdots + \left(\frac{1}{p-1}+1\right)\\ &= \frac{p}{p-1} + \frac{p}{2(p-2)} + \cdots + \frac{p}{p-1} \equiv 0 \pmod{p}.\end{aligned}$$

Next we have two cases.

<u>*Case 1*</u> ($p = 3n+1$) Then $[2p/3] = 2n$. So we must show p divides the numerator of $P(2n)$. Now

$$\begin{aligned}&H(3n) - P(2n)\\ &= 2\left(1+\frac{1}{2}+\ldots+\frac{1}{2n}\right) + \left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{3n}\right)\\ &= \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) + \left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{3n}\right)\\ &= \left(1+\frac{1}{p-1}\right) + \left(2+\frac{1}{p-2}\right) + \ldots + \left(\frac{1}{n}+\frac{1}{p-n}\right)\\ &= \frac{p}{p-1} + \frac{p}{2(p-2)} + \ldots + \frac{p}{n(p-n)}.\end{aligned}$$

So p divides the numerators of both $H(3n)$ and $H(3n) - P(2n)$, hence also the numerator of $P(2n)$.

<u>*Case 2*</u> ($p = 3n+2$) Then $[2p/3] = 2n+1$. So we must show p divides the numerator of $P(2n+1)$. Now

$$\begin{aligned}&H(3n+1) - P(2n+1)\\ &= 2\left(1+\frac{1}{2}+\ldots+\frac{1}{2n}\right) + \left(\frac{1}{2n+2}+\frac{1}{2n+2}+\ldots+\frac{1}{3n+1}\right)\\ &= \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) + \left(\frac{1}{2n+2}+\frac{1}{2n+2}+\ldots+\frac{1}{3n+1}\right)\\ &= \left(1+\frac{1}{p-1}\right) + \left(2+\frac{1}{p-2}\right) + \ldots + \left(\frac{1}{n}+\frac{1}{p-n}\right)\\ &= \frac{p}{p-1} + \frac{p}{2(p-2)} + \ldots + \frac{p}{n(p-n)}.\end{aligned}$$

So, p divides the numerator of $H(3n+1)-P(2n+1)$, and hence $P(2n+1)$.

<u>**Example 12:**</u> Let $p \ge 5$ be a prime, show that if

$$1 + \frac{1}{2} + \ldots + \frac{1}{p} = \frac{a}{b},$$

then $p^4 | ap - b$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***June 25, 2011.***

Problem 371. Let a_1, a_2, a_3, … be a sequence of nonnegative rational numbers such that $a_m+a_n=a_{mn}$ for all positive integers *m, n*. Prove that there exist two terms that are equal.

Problem 372. (*Proposed by Terence ZHU)* For all $a,b,c>0$ and $abc=1$, prove that

$$\frac{1}{a(a+1)+ab(ab+1)}+\frac{1}{b(b+1)+bc(bc+1)}+\frac{1}{c(c+1)+ca(ca+1)}\geq\frac{3}{4}.$$

Problem 373. Let x and y be the sums of positive integers $x_1,x_2,\ldots,x_{99}$ and $y_1,y_2,\ldots,y_{99}$ respectively. Prove that there exists a 50 element subset S of $\{1,2,\ldots,99\}$ such that the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Problem 374. O is the circumcenter of acute ΔABC and T is the circumcenter of ΔAOC. Let M be the midpoint of side AC. On sides AB and BC, there are points D and E respectively such that $\angle BDM=\angle BEM=\angle ABC$. Prove that $BT\perp DE$.

Problem 375. Find (with proof) all odd integers $n>1$ such that if a, b are divisors of n and are relatively prime, then $a+b-1$ is also a divisor of n.

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Solutions

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Problem 366. Let n be a positive integer in base 10. For i=1,2,…,9, let $a(i)$ be the number of digits of n that equal i. Prove that

$$2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)}\leq n+1$$

and determine all equality cases.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2).

Let $f(n)=2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)}$. If n is a number with one digit, then $f(n)=n+1$. Suppose all numbers A with k digits satisfy the given inequality $f(A)\leq A+1$. For any (k+1) digit number, it is of the form $10A+B$, where A is a k digit number and $0\leq B\leq 9$. We have

$$\begin{aligned}f(10A+B)&=(B+1)f(A)\leq(B+1)(A+1)\\&=(B+1)A+B+1\leq 10A+B+1.\end{aligned}$$

Equality holds if and only if $f(A)=A+1$ and $B=9$. By induction, the inequality holds for all positive integers n and equality holds if and only if all but the leftmost digits of n are 9's.

Other commended solvers: **CHAN Long Tin** (Diocesan Boys' School), **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School), **Gordon MAN Siu Hang** (CCC Ming Yin College) and **YUNG Fai**.

Problem 367. For n = 1,2,3,…, let x_n and y_n be positive real numbers such that

$$x_{n+2}=x_n+x_{n+1}^2$$

and

$$y_{n+2}=y_n^2+y_{n+1}.$$

If x_1, x_2, y_1, y_2 are all greater than 1, then prove that there exists a positive integer N such that for all $n>N$, we have $x_n>y_n$.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2) and **Gordon MAN Siu Hang** (CCC Ming Yin College).

Since x_1, x_2, y_1, y_2 are all greater than 1, by induction, we can get $x_{n+1}>x_n^2>1$ and $y_{n+1}>1+y_n>n$ for $n\geq 2$. Then $x_{n+2}=x_n+x_{n+1}^2>x_{n+1}^2>x_n^4$ and $y_{n+2}=y_n^2+y_{n+1}=y_n^2+y_{n-1}^2+y_n<3y_n^2<y_n^3$ for all $n\geq 4$.

Hence, $\log x_{n+2}>4\log x_n$ and $\log y_{n+2}<3\log y_n$. So for $n\geq 4$,

$$\frac{\log x_{n+2}}{\log y_{n+2}}>\frac{4}{3}\left(\frac{\log x_n}{\log y_n}\right).\qquad (*)$$

As $4/3>1$, by taking logarithm, we can solve for a positive integer k satisfying the inequality

$$\left(\frac{4}{3}\right)^k\min\left\{\frac{\log x_4}{\log y_4},\frac{\log x_5}{\log y_5}\right\}>1.$$

Let $N=2k+3$. If $n>N$, then either $n=2m+4$ or $n=2m+5$ for some integer $m\geq k$.

Applying (*) m times, we have

$$\frac{\log x_n}{\log y_n}>\left(\frac{4}{3}\right)^m\min\left\{\frac{\log x_4}{\log y_4},\frac{\log x_5}{\log y_5}\right\}>1.$$

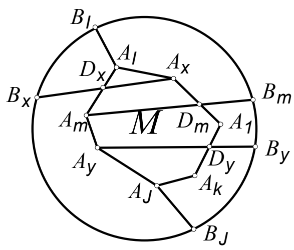
This implies $x_n>y_n$.

Other commended solvers: **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School) and **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 368. Let C be a circle, A_1, A_2, …, A_n be distinct points inside C and B_1, B_2, …, B_n be distinct points on C such that no two of the segments A_1B_1, A_2B_2,…, A_nB_n intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment A_tB_t ($t\neq r,s$). Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v.

Solution. **William PENG**.

The cases n = 1 or 2 are clear. Suppose $n\geq 3$. By reordering the pairs A_i, B_i, we may suppose the convex hull of A_1, A_2, …, A_n is the polygonal region M with vertices A_1, A_2, …, A_k ($k\leq n$). For $1\leq m\leq k$, if every A_mB_m intersects M only at A_m, then the n-th case follows by removing two pairs of A_m, B_m separately and applying case $n-1$.



Otherwise, there exists a segment A_mB_m intersecting M at more than 1 point. Let it intersect the perimeter of M again at D_m. Since A_iB_i's do not intersect, so A_jD_j's (being subsets of A_iB_i's) do not intersect. In particular, D_m is not a vertex of M.

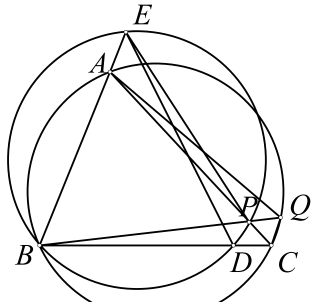
Now A_mD_m divides the perimeter of M into two parts. Moving from A_m to D_m clockwise on the perimeter of M, there are points A_x, D_x such that there is no D_w between them. As D_x is not a vertex, there is a vertex A_I between A_x and D_x. Then A_IB_I only intersect M at A_I. Also, moving from A_m to D_m anti-clockwise on the perimeter of M, there is A_J such that A_JB_J only intersects M at A_J. Then A_IB_I and A_JB_J do not intersect any diagonal of M with endpoints different from A_I and A_J.

Removing A_I, B_I and applying case $n-1$, the grasshopper can jump between any two of the points $A_1,\ldots,A_{I-1}, A_{I+1},\ldots,A_n$. Also, removing A_J, B_J and applying case $n-1$, the grasshopper can jump between any two of the points $A_1,\ldots,A_{J-1}, A_{J+1},\ldots,A_n$. Using these two cases, we see the grasshopper can jump from any A_u to any A_v via A_t ($t\neq I,J$).

Other commended solvers: **T. h. G**.

Problem 369. ABC is a triangle with $BC > CA > AB$. D is a point on side BC and E is a point on ray BA beyond A so that $BD=BE=CA$. Let P be a point on side AC such that E, B, D, P are concyclic. Let Q be the intersection point of ray BP and the circumcircle of $\triangle ABC$ different from B. Prove that $AQ+CQ=BP$.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Giorgos KALANTZIS** (Demenica's Public High School, Patras, Greece) and **LAU Chun Ting** (St. Paul's Co-educational College, Form 2).



Since A,B,C,Q are concyclic and E,P,D,B are concyclic, we have

$$\angle AQC = 180^\circ - \angle ABC = \angle EPD$$

and

$$\angle PED = \angle PBD = \angle QAC.$$

Hence, $\triangle AQC$ and $\triangle EPD$ are similar. So we have $AQ/AC=PE/DE$ and $CQ/AC = PD/DE$. Cross-multiplying and adding these two equations, we get

$$(AQ+CQ)\times DE = (PE+PD)\times AC. \quad (*)$$

For cyclic quadrilateral $EPDB$, by the Ptolemy theorem, we have

$$\begin{aligned} BP\times DE &= PE\times BD+PD\times BE \\ &= (PE+PD)\times AC \qquad (**) \end{aligned}$$

Comparing (*) and (**), we have $AQ+CQ=BP$.

Other commended solvers: **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School).

Problem 370. On the coordinate plane, at every lattice point (x,y) (these are points where x, y are integers), there is a light. At time $t = 0$, exactly one light is turned on. For $n = 1, 2, 3, \ldots$, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was on at time $t = n - 1$. Prove that every light at a lattice point will eventually be turned on at some time.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2), **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School), **Gordon MAN Siu Hang** (CCC Ming Yin College) and **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy).

We may assume the light that was turned on at $t = 0$ was at the origin.

Let $z = 2005 = 5\times401 = (2^2+1^2)(20^2+1^2) = |(2+i)(20+i)|^2 = |41+22i|^2 = 41^2+22^2$. Let $x = 41^2-22^2=1037$ and $y = 2\times41\times22 = 1716$. Then $x^2+y^2 = z^2$.

By the Euclidean algorithm, we get $\gcd(1037, 1716) = 1$. By eliminating the remainders in the calculations, we get $84\times1716-139\times1037=1$.

Let V_1, V_2, V_3, V_4, V_5 be the vectors from the origin to (2005,0), (1037, 1716), (1037, −1716), (1716, 1037), (1716, −1037) respectively. We have $V_2+V_3 = (2\times1037,0)$ and $V_4+V_5 = (2\times1716,0)$. Then we can get $(1,0) = 1003[84(V_4+V_5)-139(V_2+V_3)]-V_1$.

So, from the origin, following these vector movements, we can get to the point (1,0). Similarly, we can get to the point (0,1). As $(a,b) = a(1,0) + b(0,1)$, we can get to any lattice point.

Olympiad Corner

(continued from page 1)

Problem 4. Let n be a fixed positive odd integer. Take $m+2$ distinct points P_0, P_1, …, P_{m+1} (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:

(1) $P_0=(0,1)$, $P_{m+1}=(n+1,n)$, and for each integer i, $1 \le i \le m$, both x- and y-coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).

(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.

(3) For each pair i,j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

Determine the maximum possible value that m can take.

Problem 5. Find all functions $f:\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following 2 conditions:

(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.

(2) For every pair of real numbers x and y, $f(xf(y)) + yf(x) = xf(y) + f(xy)$ is satisfied.

Harmonic Series (II)

(continued from page 2)

__Solution__ By Wolstenholme's theorem,

$$p^2 \left| (p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right)\right..$$

So,

$$1+\frac{1}{2}+\ldots+\frac{1}{p-1}=p^2\frac{x}{y},$$

where x, y are integers with y not divisible by p. So we have

$$\frac{a}{b}-\frac{1}{p}=p^2\frac{x}{y},$$

which implies $ap-b = p^3bx/y$. Finally,

$$\frac{a}{b}=\frac{2\cdot3\cdots p+1\cdot3\cdot4\cdots p+\cdots+1\cdot2\cdots(p-1)}{p!}$$

and the numerator of the right side is of the form $mp+(p-1)!$. Hence, it is not divisible by p. So $p \mid b$ and $p^4 \mid p^3bx/y = ap-b$.

__Example 13__: Let p be an odd prime, then prove that

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots+(-1)^{p-2}\frac{1}{(p-1)^2}\equiv 0 \pmod{p}.$$

__Solution__ The proof is not hard. Indeed,

$$\begin{aligned} &\sum_{k=1}^{p-1}(-1)^{k-1}\frac{1}{k^2} \\ &= -\sum_{k=1}^{(p-1)/2}\left((-1)^k\frac{1}{k^2}+(-1)^{p-k}\frac{1}{(p-k)^2}\right) \\ &\equiv -\sum_{k=1}^{(p-1)/2}\left((-1)^k\frac{1}{k^2}+(-1)^{1-k}\frac{1}{(-k)^2}\right) = 0 \pmod{p}. \end{aligned}$$

Mathematical Excalibur

Volume 16, Number 2 June - October 2011

Olympiad Corner

Below are the problems of the 28th Balkan Math Olympiad, which was held in May 6, 2011. Time allowed was 4½ hours.

Problem 1. Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

Problem 2. Given real numbers x, y, z such that $x+y+z=0$, show that

$$\frac{x(x+2)}{2x^2+1}+\frac{y(y+2)}{2y^2+1}+\frac{z(z+2)}{2z^2+1}\ge 0.$$

When does equality hold?

Problem 3. Let S be a finite set of positive integers which has the following property: if x is a member of S, then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ration y/x is a power of a prime number.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 10, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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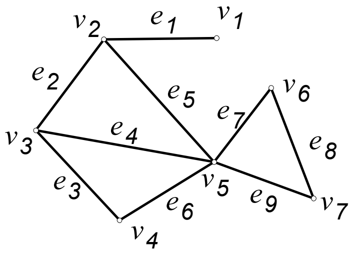
Euler's Planar Graph Formula

Kin Y. Li

A <u>*graph*</u> G is consisted of a *nonempty* set $V(G)$ (its elements are called <u>*vertices*</u>) and a set $E(G)$ (its elements are called <u>*edges*</u>), where an edge is to be thought of as a continuous curve joining a vertex u in $V(G)$ to a vertex v in $V(G)$. A graph G is <u>*finite*</u> if and only if $V(G)$ is a finite set. It is <u>*simple*</u> if and only if each edge in $E(G)$ joins some pair of distinct vertices in $V(G)$ and no other edge joins the same pair. <u>*In this article, all graphs are understood to be finite and simple*</u>.

A graph is <u>*connected*</u> if and only if for every pair of distinct vertices a, b, there is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for i from 1 to n, edge e_i joins v_i and v_{i+1} with $v_1 = a$ and $v_{n+1} = b$. A graph is <u>*planar*</u> if and only if it can be drawn on a plane with no pair of edges intersect at any point other than a vertex of the graph. A planar graph divides the plane into regions (bounded by edges) called <u>*faces*</u>.



In the graph above, there are 7 vertices (labeled v_1 to v_7), 9 edges (labeled e_1 to e_9) and 4 faces (the 3 triangular regions and the outside region bounded by $e_1, e_5, e_7, e_8, e_9, e_6, e_3, e_2, e_1$). The following theorem due to Euler relates the number of vertices, the number of edges and the number of regions for a connected planar graph and is the key tool in solving some interesting problems.

<u>Euler's Theorem on Planar Graphs</u>

Let V, E, F denote the number of vertices, the number of edges, the number of faces respectively for a connected planar (finite simple) graph. Then $V - E + F = 2$, which we will called <u>*Euler's formula*</u>.

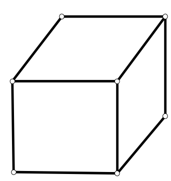
We will *sketch* the usual mathematical induction proof on E. If $E = 0$, then since $V(G)$ is nonempty and G is connected, we have $V = 1$ and $F = 1$. So $V-E+F=2$. Also, if $E=1$, then $V = 2$, $F=1$ and again the formula is true.

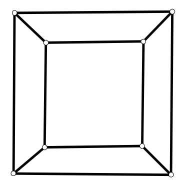
Suppose the cases $E < k$ are true. For the case $E=k$, either there is a cycle (that is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for i from 1 to n, edge e_i joins v_i and v_{i+1} with $v_1 = v_{n+1}$) or no cycle.

In the former case, removing e_n will result in a connected graph with E decreases by 1, V stays the same and F decreases by 1 (since the two regions sharing e_n in their boundaries will become one). The formula still holds.

In the latter case, we call these graphs <u>*trees*</u>. It can be proved that they satisfy $E=V-1$ and $F=1$ (which implies Euler's formula). Basically, removing any edge will split such a graph into two connected graphs with each having no cycle. This observation would allow us to do the induction on E.

Before presenting some examples, we remark that Euler's formula also applies to convex polyhedrons. These are the boundary surfaces of three dimensional convex solids obtained by intersecting finitely many (half-spaces on certain sides of) planes. For example, take the surface of a cube, $V=8$, $E=12$, $F=6$ so that $V - E + F = 2$. For any convex polyhedron, we can obtain a connected planar graph by choosing a face as base, stretching the base sufficiently big and taking a top view projection onto the plane containing the base. The following is a cube and a planar graph for its boundary surface.





Example 1. There are $n > 3$ points on a circle. Each pair of them is connected by a chord such that no three of these chords intersect at the same point inside the circle. Find the number of regions formed inside the circle.

Solution. Removing the n arcs on the circle, we get a *simple* connected planar graph, where the vertices are the n points on the circle and the intersection points inside the circle. For every 4 of the n points, we can draw two chords intersecting at a point inside the circle. So the number of vertices is $V = n + {}_nC_4$.

Since there are $n-1$ edges incident with each of the n points on the circle, 4 edges incident with every intersection point inside the circle and each edge is counted twice, so the number of edges is $E = (n(n-1)+4\ {}_nC_4)/2$.

By Euler's formula, the number of faces for this graph is $F = 2 - V + E$. Excluding the outside face and adding the n regions having the n arcs as boundary, the number of regions inside the circle is $F - 1 + n = n + 1 - V + E = 1 + {}_nC_4 + n(n-1)/2$.

For the next few examples, we define the degree of a vertex v in a graph to be the number of edges meeting at v. Below $d(v)$ will denote the degree of v. *The sum of degrees of all vertices equals twice the number of edges since each edge is counted twice at its two endponts.*

Example 2. A square region is partitioned into n convex polygonal regions. Find the maximal number of edges in the figure.

Solution. Let V, E, F be the number of vertices, edges, faces respectively in the graph. Euler's formula yields

$$n+1 = F = 2-V+E \quad \text{or} \quad V = E + 1 - n.$$

Let A, B, C, D be the vertices of the square, then $t = d(A) + d(B) + d(C) + d(D) \ge 8$ as each term is at least 2.

Let W be the set of vertices inside the square. For any v in W, we have $d(v) \ge 3$ since angles of convex polygons are less than 180°. Let s be the sum of $d(v)$ for all v in W. Since there are $V-4$ vertices in W, we have $s \ge 3(V-4)$.

Now summing degree of all vertices, we get $s + t = 2E$. Then

$$2E-8 \ge 2E - t = s \ge 3(V-4) = 3(E-3-n),$$

which simplifies to $E \le 3n+1$.

Finally, the case $E = 3n+1$ is possible by partitioning the square region into n rectangles using $n - 1$ line segments parallel to a side of the square. So the maximum possible value of E is $3n+1$.

Example 3. (*2000 Belarussian Math Olympiad*) In a convex polyhedron with m triangular faces (and possibly faces of other shapes), exactly four edges meet at each vertex. Find the minimum possible value of m.

Solution. Let V, E, F be the number of vertices, edges, faces respectively on such a polyhedron. Since each vertex is met by 4 distinct edges, summing all degrees, we have $2E = 4V$.

Next, summing the number of edges in the F faces and observing that each edge is counted twice on the 2 faces sharing it, we get $2E \ge 3m+4(F-m)$.

By Euler's formula, we have

$$2 = V - E + F = (E/2) - E + F = F - E/2,$$

which implies

$$4F - 8 = 2E \ \ge\ 3m + 4(F-m).$$

This simplifies to $m \ge 8$. A regular octahedron is an example of the case $m = 8$. So the minimum possible m is 8.

Example 4. (*1985 IMO proposal by Federal Republic of Germany*) Let M be the set of edge-lengths of an octahedron whose faces are congruent quadrilaterals. Prove that M has at most three elements.

Solution. The octahedron has $(4\times 8)/2=16$ edges. By Euler's formula, it has $V = 2 + E - F = 2 + 16 - 8 = 10$ vertices.

Next, let n_i be the number of vertices v with $d(v) = i$. Then, counting vertices and edges respectively in terms of n_i's, we have

$$V = n_3 + n_4 + n_5 + \cdots = 10$$

and

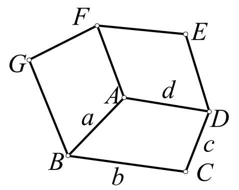
$$2E = 3n_3 + 4n_4 + 5n_5 + \cdots = 2\times 16.$$

Eliminating n_3, we get

$$n_4 + 2n_5 + 3n_6 + \cdots = 2.$$

Hence, $n_4 \le 2$, $n_5 \le 1$ and $n_i = 0$ for $i \ge 6$. Then $n_3 = 10 - n_4 - n_5 > 0$.

Let A be a vertex with degree 3. Assume M has 4 distinct elements a, b, c, d. Then the 3 faces about A are like the figure below, where we may take $AB = a$, $BC = b$, $CD = c$ and $DA = d$.



Since $ABCD$ and $ABGF$ are congruent, so $AF = b$ or d. Also, since $ABCD$ and $AFED$ are congruent, so $AF = a$ or c. Hence, two of a, b, c, d must be equal, contradiction. Therefore, M has at most 3 elements.

Example 5. Let n be a positive integer. A convex polyhedron has $10n$ faces. Prove that n of the faces have the same number of edges.

Solution. Let V be the number of vertices of this polyhedron. For the $10n$ faces, let these faces be polygons with $a_1, a_2, \ldots, a_{10n}$ sides respectively, where the a_i's are arranged in ascending order. Then the number of edges of the polyhedron is $E = (a_1 + a_2 + \cdots + a_{10n})/2$. By Euler's formula, we have

$$V - \frac{a_1 + a_2 + \cdots + a_n}{2} + 10n = 2. \quad (*)$$

Also, since the degree of every vertex is at least 3, we get

$$a_1+a_2+\cdots+a_{10n} \ge 3V. \qquad (**)$$

Using (*) and (**), we can eliminate V and solve for $a_1+a_2+\cdots+a_{10n}$ to get

$$a_1+a_2+\cdots+a_{10n} \le 60n - 12. \quad (***)$$

Assume no n faces have equal number of edges. Then we have $a_1, a_2, \ldots, a_{n-1} \ge 3$, $a_n, a_{n+1}, \ldots, a_{2n-2} \ge 4$ and so on. This leads to

$$\begin{aligned} & a_1 + a_2 + \cdots + a_{10n} \\ \ge\ & (3 + 4 + \cdots + 12)(n-1) + 13\times 10 \\ =\ & 75n + 55. \end{aligned}$$

Comparing with (***), we get $75n + 55 \le 60n - 12$, which is false for n.

Example 6. (*1975 Kiev Math Olympiad and 1987 East German Math Olympiad*) An arrowhead is drawn on every edge of a convex polyhedron H such that at every vertex, there are at least one arrowhead pointing toward the vertex and another arrowhead pointing away from the vertex. Prove that there exist at least two faces of H, the arrowheads on each of its boundary form a (clockwise or counterclockwise) cycle.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***November 10, 2011.***

Problem 376. A polynomial is monic if the coefficient of its greatest degree term is 1. Prove that there exists a monic polynomial $f(x)$ with integer coefficients such that for every prime p, $f(x) \equiv 0 \pmod{p}$ has solutions in integers, but $f(x) = 0$ has no solution in integers.

Problem 377. Let n be a positive integers. For i=1,2,…,n, let z_i and w_i be complex numbers such that for all 2^n choices of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ equal to ±1, we have

$$\left|\sum_{i=1}^{n}\varepsilon_i z_i\right| \le \left|\sum_{i=1}^{n}\varepsilon_i w_i\right|.$$

Prove that $\sum_{i=1}^{n}|z_i|^2 \le \sum_{i=1}^{n}|w_i|^2$.

Problem 378. Prove that for every positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct positive prime divisors.

Problem 379. Let ℓ be a line on the plane of ΔABC such that ℓ does not intersect the triangle and none of the lines AB, BC, CA is perpendicular to ℓ.

Let A', B' C' be the feet of the perpendiculars from A, B, C to ℓ respectively. Let A'', B'', C'' be the feet of the perpendiculars from A', B', C' to lines BC, CA, AB respectively.

Prove that lines $A'A''$, $B'B''$, $C'C''$ are concurrent.

Problem 380. Let S = {1,2,…,2000}. If A and B are subsets of S, then let $|A|$ and $|B|$ denote the number of elements in A and in B respectively. Suppose the product of $|A|$ and $|B|$ is at least 3999. Then prove that sets $A-A$ and $B-B$ contain at least one common element, where $X-X$ denotes $\{s-t : s, t \in X \text{ and } s \ne t\}$.

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Solutions

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Problem 371. Let $a_1, a_2, a_3, \ldots$ be a sequence of nonnegative rational numbers such that $a_m+a_n= a_{mn}$ for all positive integers m, n. Prove that there exists two terms that are equal.

Solution. **U. BATZORIG** (National University of Mongolia), **CHUNG Kwan** (King's College) and **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School).

Let p and q be distinct primes. If a_p and a_q are zeros, then we are done. Otherwise, consider

$$m = p^{Na_q} \quad and \quad n = q^{Na_p},$$

where N is a positive integer that makes both Na_q and Na_p integers. Obviously, we have $m \ne n$ and

$$a_m = (Na_q)a_p = (Na_p)a_q = a_n.$$

Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil).

Problem 372. (*Proposed by Terence ZHU*) For all $a,b,c > 0$ and abc=1, prove that

$$\frac{1}{a(a+1)+ab(ab+1)}+\frac{1}{b(b+1)+bc(bc+1)}$$

$$+\frac{1}{c(c+1)+ca(ca+1)} \ge \frac{3}{4}.$$

Solution. **V. ADIYASUREN** (National University of Mongolia) and **B. SANCHIR** (Mathematics Institute of the National University of Mongolia), **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School) and **Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Substituting $a = z/y$, $b = x/z$, $c = y/x$ (say by choosing x=ab=$1/c$, y=1, z=a) into the inequality and simplifying, we get

$$\sum_{cyc} f(x,y,z) \ge \frac{3}{4},$$

where

$$f(x,y,z) = \frac{y^2}{z(z+y)+x(x+y)} \quad \text{and}$$

$$\sum_{cyc} f(x,y,z) = f(x,y,z)+f(y,z,x)+f(z,x,y).$$

Let $g(x,y,z) = y^2(z^2 + zy + x^2 + xy)$. By the Cauchy-Schwarz inequality, we have

$$\sum_{cyc} f(x,y,z)\sum_{cyc} g(x,y,z) \ge \left(\sum_{cyc} y^2\right)^2.$$

So it is enough to prove

$$\left(\sum_{cyc} y^2\right)^2 \Big/ \left(\sum_{cyc} g(x,y,z)\right) \ge \frac{3}{4}. \quad (*)$$

Expanding and factorizing, we get

$$4\left(\sum_{cyc} y^2\right)^2 - 3\left(\sum_{cyc} g(x,y,z)\right)$$

$$= 4\sum_{cyc} y^4 + 2\sum_{cyc} x^2y^2 - 3\sum_{cyc} xy(x^2+y^2)$$

$$= 3\sum_{cyc}(x-y)^2(x^2+y^2) + \sum_{cyc}(x^2-y^2)^2 \ge 0.$$

This implies (*), which implies the desired inequality.

Other commended solvers: **CHUNG Kwan** (King's College), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 373. Let x and y be the sums of positive integers $x_1, x_2, \ldots, x_{99}$ and $y_1, y_2, \ldots, y_{99}$ respectively. Prove that there exists a 50 element subset S of {1,2,…,99} such that the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Solution. **William Peng** and **Jeff Peng**.

Arrange the numbers $x_1, x_2, \ldots, x_{99}$ in descending order, say $x_{n(1)} \ge x_{n(2)} \ge \cdots \ge x_{n(99)}$ so that

$$\{n(1), n(2), \ldots, n(99)\} = \{1,2,\ldots,99\}.$$

Let $A = \{n(2), n(4), \ldots, n(98)\}$ and $B = \{n(3), n(5), \ldots, n(99)\}$. We have

$$x_{n(1)} + \sum_{j\in B} x_j > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j.$$

If $\sum_{i\in A} y_i \ge \sum_{j\in B} y_j$, then let $S = A \cup \{n(1)\}$.

Now S has 50 elements. Also,

$$\sum_{i\in S} x_i > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j$$

and

$$\sum_{i\in S} y_i > \sum_{i\in A} y_i \ge \sum_{j\in B} y_j.$$

So the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

If $\sum_{i\in A} y_i < \sum_{j\in B} y_j$, then let $S=B\cup\{n(1)\}$.

Again S has 50 elements. Now

$$\sum_{i\in S} x_i = x_{n(1)} + \sum_{j\in B} x_j > \sum_{i\in A} x_i \geq \sum_{j\in B} x_j$$

and

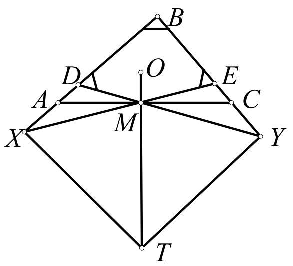
$$\sum_{i\in S} y_i > \sum_{j\in B} y_j > \sum_{i\in A} y_i.$$

So the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Other commended solvers: **U. BATZORIG** (National University of Mongolia) and **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School),

Problem 374. O is the circumcenter of acute ΔABC and T is the circumcenter of ΔAOC. Let M be the midpoint of side AC. On sides AB and BC, there are points D and E respectively such that $\angle BDM=\angle BEM=\angle ABC$. Prove that $BT\perp DE$.

Solution. **William Peng** and **Jeff Peng**.



By the exterior angle theorem, $\angle ABC = \angle BDM > \angle BAM$ and also $\angle ABC = \angle BEM > \angle BCM$. So $\angle ABC$ is the largest angle in ΔABC. Then we have $60° < \angle ABC < 90°$. This implies O is on the same side of line AC as B. Then T will be on the opposite side of line AC as O. Also, O, M, T are on the perpendicular bisector of line AC.

Let X be the intersection of lines AB and ME. Let Y be the intersection of lines CB and MD. Now

$$\begin{aligned}\angle DXE &= 180° - \angle XBE - \angle BEX\\ &= 180° - 2\angle ABC\end{aligned}$$

and similarly $\angle EYD = 180° - 2\angle ABC$. So $\angle DXE = \angle EYD$, which implies D, X, Y, E are concyclic.

Next, since T is the circumcenter of ΔAOC, so

$$\begin{aligned}\angle ATM &= \angle ATO = 2\angle ACO\\ &= 2(90° - \angle BXE)\\ &= 180° - 2\angle ABC\\ &= \angle BXE = \angle AXM.\end{aligned}$$

This implies A, M, T, X are concyclic. So $\angle AXT = 180° - \angle AMT = 90°$. Similarly, $\angle CYT = 90°$. Then $\angle BXT = \angle BYT$, which implies B, X, T, Y are concyclic. So

$$\angle TBY = \angle TXY = 90° - \angle BXY. \quad (*)$$

Since D, X, Y, E are concyclic,

$$\begin{aligned}&\angle BED + \angle TBE\\ =\ &\angle BXY + \angle TBY\\ =\ &90° \qquad \text{by } (*),\end{aligned}$$

which implies $BT\perp DE$.

Other commended solvers: **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School),

Problem 375. Find (with proof) all odd integers $n > 1$ such that if a, b are divisors of n and are relatively prime, then $a+b-1$ is also a divisor of n.

Solution. **U. BATZORIG** (National University of Mongolia), **William Peng** and **Jeff Peng**.

For such odd n, let p be its <u>*least*</u> prime divisor. Then $n = p^m a$, where m is the exponent of p in the prime factorization of n. We will show $a = 1$.

Assume $a > 1$. Then every prime divisors of a is at least $p+2$. Also $c = a+p-1$ $(> p)$ is a divisor of n. Since

$$\gcd(c,a) = \gcd(c-a,a) = \gcd(p-1,a) = 1,$$

this implies $c=p^r$ with $r \geq 2$. Then $d = a+p^2-1$ $(> p^2)$ is also a divisor of n. Similarly,

$$\gcd(d,a) = \gcd(d-a,a)= \gcd(p^2-1,a)= 1.$$

So $d=p^s$ with $s \geq 3$. Finally, $p^r-p = c-p = a-1= d-p^2$, which is divisible by p^2, while p^r-p is not. Therefore, $a = 1$.

It is easy to check all $n=p^m$ with p an odd prime and m a positive integer indeed satisfy the condition.

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) A non-empty subset T of S is <u>*bad*</u> if whenever $x, y\in T$ and $x < y$, the ration y/x is not a power of a prime number. We agree that a singleton subset of S is both good and bad. Let k be the largest possible size of a good subset of S. Prove that k is also the smallest number of pairwise-disjoint bad subsets whose union is S.

Problem 4. Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of one of these two triangles is at least 3/2.

Euler's Planar Graph Formula

(continued from page 2)

<u>*Solution.*</u> Call $\{a,b\}$ a <u>*hook*</u> if a, b are two consecutive edges on the boundary of some face of H. Call a hook $\{a,b\}$ <u>*traversible*</u> if the arrowheads on a and b are both counterclockwise or both clockwise.

Note every hook is part of the boundary of a unique face. Let E be the number of edges on H and h be the number of hooks on H. As each edge on H is a part of 4 hooks, we get $h = 2E$.

Next at every vertex v, $d(v) \geq 3$. By the given condition on the vertices, there must be at least 2 traversible hooks through every vertex. Let V be the number of vertices on H, then there are at least $2V$ traversible hooks on H.

Let h_+ and h_- be the number of traversible and non-traversible hooks respectively on H. Then $h_+ \geq 2V$.

In every face where the boundary arrowheads do not form a cycle, there are at least two changes in directions on the boundary, which result in at least two non-traversible hooks. Let F be the number of faces on H. Let f_+ be the number of faces the boundary arrowheads form cycles. Let $f_-=F-f_+$. Then $h_- \geq 2f_-$.

By Euler's formula, $V - E + F = 2$. Then

$$\begin{aligned}2f_+ &= 2F - 2f_-\\ &= (4 + 2E - 2V) - 2f_-\\ &\geq 4 + h - h_+ - 2f_-\\ &= 4 + h_- - 2f_- \geq 4,\end{aligned}$$

which implies $f_+ \geq 2$. This gives the desired conclusion.

Mathematical Excalibur

Volume 16, Number 3 *November 2011 -January 2012*

Olympiad Corner

Below are the problems of the 2011 International Math Olympiad.

Problem 1. Given any set $A=\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1+a_2+a_3+a_4$ by s_A. Let n_A denote the number of pairs (i,j) with $1\le i<j\le 4$ for which a_i+a_j divides s_A. Find the sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 2. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely.

Show that we can choose a point P in S and a line ℓ going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Remarks on IMO 2011

Leung Tat-Wing

The 52nd IMO was held in Amsterdam, Netherlands, on 12-24, July, 2011. Contestants took two 4½ hour exams during the mornings of July 18 and 19. Each exam was consisted of 3 problems of varying degree of difficulty. The problems were first shortlisted by the host country, selected from problems submitted earlier by various countries. Leaders from 101 countries then picked the 2011 IMO problems (see Olympiad Corner). Traditionally an easy pair was selected (Problems 1 and 4), then a hard pair (Problems 3 and 6), with Problem 6 usually selected as the "anchor problem", and finally the intermediate pair (Problems 2 and 5). I would like to discuss first the problems selected, aim to provide something extra besides those which were provided by the solutions. However I would discuss the problems by slightly different grouping.

Problems 1 and 4

First the easy pair, problems 1 and 4. The problem selection committee thought that both problems were quite easy. It was nice to select one as a problem of the contest. But if both problems were selected, then the paper would be too easy (or even disastrous). Indeed eventually both problems were selected. But it was not enough for anyone to get a bronze medal even if he could solve both problems (earning 14 points) as the cut-off for bronze was 16.

In my opinion problem 1 is the easier of the pair. Indeed we may without loss of generality assume $a_1 < a_2 < a_3 < a_4$. So if the sum of one pair of the a_i's divides s_A, then it will also divide the sum of the other pair. But clearly a bigger pair cannot divide a smaller pair, so it is impossible that $a_3 + a_4$ dividing $a_1 + a_2$, nor is it possible that $a_2 + a_4$ dividing $a_1 + a_3$. Therefore the maximum possible value of n_A can only be 4. To achieve this, it suffices to consider divisibility conditions among the other pairs.

Now as we need $a_1 + a_4$ dividing $a_2 + a_3$ and also $a_2 + a_3$ dividing $a_1 + a_4$, we must have $a_1 + a_4 = a_2 + a_3$. Putting $a_4= a_2+a_3-a_1$ into the equations $a_3 + a_4 = m(a_1 + a_2)$ and $a_2 + a_4 = n(a_1 + a_3)$ with $m > n > 1$, we eventually get $(m,n) = (3,2)$ or $(4,2)$. Finally we get $(a_1, a_2, a_3, a_4) = (k, 5k, 7k, 11k)$ or $(k, 11k, 19k, 29k)$, where k is a positive integer. As the derivation of the answers is rather straight-forward, it does not pose any serious difficulty.

For problem 4, it is really quite easy if one notes the proper recurrence relation. Indeed the weights $2^0, 2^1, 2^2, \ldots, 2^{n-1}$ form a "super-increasing sequence", any weight is heavier than the sum of all lighter weights. Denote by $f(n)$ the number of ways of placing the weights. We consider first how to place the lightest weight (weight 1). Indeed if it is placed in the first move, then it has to be in the left pan. However if it is placed in the second to the last move, then it really doesn't matter where it goes, using the "super-increasing property". Hence altogether there are $2n-1$ possibilities of placing the weight of weight 1. Now placing the weights $2^1, 2^2, \ldots, 2^{n-1}$ clearly is the same as placing the weights $2^0, 2^1, \ldots, 2^{n-2}$. There are $f(n-1)$ ways of doing this. Thus we establish the recurrence relation $f(n) = (2n-1)f(n-1)$. Using $f(1) = 1$, by induction, we get

$$f(n) = (2n-1)(2n-3)(2n-5)\cdots 1.$$

The problem becomes a mere exercise of recurrence relation if one notices how to place the lightest weight (minimum principle).

It is slightly harder if we consider how to place the heaviest weight. Indeed if the heaviest weight is to be placed in the i^{th} move, then it has to be placed in the left pan. There are $\binom{n-1}{i-1}$ ways of choosing the previous $i-1$ weights and there are $f(i-1)$ ways of placing them. After the heaviest weight is placed, it doesn't matter how to place the other weights, and there are $(n-i)!\times 2^{n-i}$ ways of placing the remaining weights. Thus

$$f(n)=\sum_{i=1}^{n}\binom{n-1}{i-1}f(i-1)(n-i)!2^{n-i}.$$

Replacing n by $n-1$ and by comparing the two expressions we again get $f(n) = (2n-1)f(n-1)$. We have no serious difficulty with this problem.

Problems 3 and 5

In my opinion both problems 3 and 5 were of similar flavor. Both were "functional equation" type of problems. Problem 3 was slightly more involved and problem 5 more number theoretic. One can of course put in many values and obtain some equalities or inequalities. But the important thing is to substitute some suitable values so that one can derive important relevant properties that can solve the problem.

In problem 5, indeed the condition $f(m-n) \mid (f(m) - f(n))$ (*) poses very serious restrictions on the image of $f(x)$. Putting n=0, one gets $f(m) \mid (f(m) - f(0))$, thus $f(m) \mid f(0)$. Since $f(0)$ can only have finitely many factors, the image of $f(x)$ must be finite. Putting m=0, one gets $f(-n) \mid f(n)$, and by interchanging n and $-n$, one gets $f(n) = f(-n)$. Now $f(n) \mid (f(2n) - f(n))$, hence $f(n) \mid f(2n)$, and by induction $f(n) \mid f(mn)$. Put $n = 1$ into the relation. One gets $f(1) \mid f(m)$. The image of $f(x)$ is therefore a finite sequence $f(1) = a_1 < a_2 < \cdots < a_k = f(0)$. One needs to show $a_i \mid a_{i+1}$. To complete the proof, one needs to analyze the sequence more carefully, say one may proceed by induction on k. But personally I like the following argument. Let $f(x) = a_i$ and $f(y) = a_{i+1}$. We have $f(x-y) \mid (f(y) - f(x)) < f(y)$ and $f(y) - f(x)$ is positive, hence $f(x - y)$ is in the image of $f(x)$ and therefore $f(x-y) \le a_i = f(x)$. Now if $f(x-y) < f(x)$, then $f(x) - f(x-y) > 0$. Thus $f(y) = f(x-(x-y)) \mid (f(x) - f(x-y))$.
In this case the right-hand side is positive. We have $f(y) \le f(x) - f(x-y)) < f(x) < f(y)$, a contradiction. So we have $f(x-y) = f(x)$. Thus $f(x) \mid f(y)$ as needed.

It seems that Problem 3 is more involved. However, by making useful and clever substitutions, it is possible to solve the problem in a relatively easy way. The following solution comes from one of our team members. Put $y = z-x$ into the original equation $f(x+y) \le yf(x) + f(f(x))$, one gets $f(z) \le z f(x) - xf(x) + f(f(x))$. By letting $z = f(k)$ in the derived inequality one gets $f(f(k)) \le f(k) f(x) - xf(x) + f(f(x))$.
Interchanging k and x one then gets $f(f(x)) \le f(k) f(x) - kf(k) + f(f(k))$. Hence

$$\begin{aligned} f(x+y) &\le y f(x) + f(f(x)) \\ &\le f(x)f(k) - kf(k) + f(f(k)). \end{aligned}$$

Letting $y = f(k) - x$ in the inequality, we get

$$\begin{aligned} f(f(k)) \le{}& f(k) f(x) - xf(x) + \\ & f(k) f(x) - kf(k) + f(f(k)) \end{aligned}$$

or $0 \le 2 f(k) f(x) - xf(x) - k f(k)$. Finally letting $k = 2 f(x)$ and simplifying, we arrive at the important and essential (hidden) inequality $0 \le -xf(x)$. This means for $x > 0$, $f(x) \le 0$, and for $x < 0$, $f(x) \ge 0$. But if there is an $x_0 < 0$ such that $f(x_0) > 0$, then putting $x = x_0$ and $y = 0$ into the original equation, we gets $0 < f(x_0) \le f(f(x_0))$. However if $f(x_0) > 0$, then $f(f(x_0)) \le 0$, hence a contradiction. This means for all $x < 0, f(x) = 0$. Finally one has to prove $f(0) = 0$. We suppose first $f(0) > 0$. Put $x = 0$ and $y < 0$ sufficiently small into the original equation, one gets $f(y) < 0$, a contradiction. Suppose $f(0) < 0$. Take $x, y < 0$. We get

$$\begin{aligned} 0 = f(x+y) &\le y f(x) + f(f(x)) \\ &= y f(x) + f(0) = f(0) < 0, \end{aligned}$$

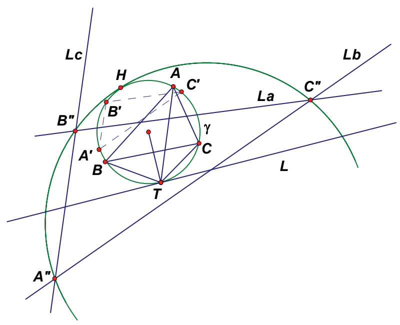
again contradiction! This implies $f(0) = 0$.

Problem 2

To me, problem 2 was one of a kind. The problem was considered as "intermediate" and should not be too hard. However at the end only 21 out of 564 contestants scored full marks. It was essentially a problem of computational geometry. We know that if there is a line that goes through two or more of the points and such that all other points are on the line or only on one side of the points, then by repeatedly turning angles as indicated in the problem, the convex hull of the point set will be constructed (so-called Jarvis' march). Therefore some points may be missed. So in order to solve the problem, we cannot start from the "boundary". Thus it is natural that we start from the "center", or a line going through a point that separates the other points into equal halves (or differ by one). Indeed this idea is correct. The hard part is how to substantiate the argument. Many contestants found it hard. Induction argument does not work because adding or deleting one point may change the entire route. The proposer gives the following "continuity argument". We consider only the case that there are an odd number of points on the plane. Let l be a line that goes through one of the points and that separates the other points into two equal halves. Note that such line clearly exists. Color one half-plane determined by the line orange (for Netherlands) and the other half-plane blue. The color of the plane changes accordingly while the line is turning. Note also that when the line moves to another pivot, the number of points on the two sides remain the same, except when two points are on the line during the change of pivots. So consider what happen when the line turns 180°, (turning while changing pivots). The line will go through the same original starting point. Only the colors of the two sides of the line interchange! This means all the points have been visited at least once! A slightly modified argument works for the case there are an even number of points on the plane.

Problem 6

This was the most difficult problem of the contest (the anchor problem), only 6 out of more than 564 contestants solved the problem. Curiously these solvers were not necessarily from the strongest teams. The problem is hard and beautiful, and I feel that it may be a known problem because it is so nice. However, I am not able to find any further detail. It is not convenient to reproduce the full solution here. But I still want to discuss the main idea used in the first official solution briefly.



From $\triangle ABC$ and the tangent line L at T, we produce the reflecting lines L_a, L_b, and L_c. The reflecting lines meet at A'', B'' and C'' respectively. Now from A, we draw a circle of radius AT, meeting the circumcircle γ of ABC at A'. Likewise we have $BT=BB'$ and $CT=CC'$ (see the figure).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 28, 2012.***

Problem 381. Let k be a positive integer. There are 2^k balls divided into a number of piles. For every two piles A and B with p and q balls respectively, if $p \ge q$, then we may transfer q balls from pile A to pile B. Prove that it is always possible to make finitely many such transfers so as to have all the balls end up in one pile.

Problem 382. Let $v_0 = 0$, $v_1 = 1$ and

$$v_{n+1} = 8v_n - v_{n-1} \quad \text{for } n = 1,2,3,\ldots.$$

Prove that v_n is divisible by 3 if and only if v_n is divisible by 7.

Problem 383. Let O and I be the circumcenter and incenter of $\triangle ABC$ respectively. If $AB \neq AC$, points D, E are midpoints of AB, AC respectively and $BC=(AB+AC)/2$, then prove that the line OI and the bisector of $\angle CAB$ are perpendicular.

Problem 384. For all positive real numbers a,b,c satisfying $a + b + c = 3$, prove that

$$\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-bc)}+\frac{c^2+3a^2}{ca^2(4-ca)} \ge 4.$$

Problem 385. To prepare for the IMO, in everyday of the next 11 weeks, Jack will solve at least one problem. If every week he can solve at most 12 problems, then prove that for some positive integer n, there are n consecutive days in which he can solve a total of 21 problems.

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Solutions

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Problem 376. A polynomial is <u>*monic*</u> if the coefficient of its greatest degree term is 1. Prove that there exists a monic polynomial $f(x)$ with integer coefficients such that for every prime p, $f(x) \equiv 0 \pmod p$ has solutions in integers, but $f(x) = 0$ has no solution in integers.

Solution. **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania), **Koopa KOO** and **Andy LOO** (St. Paul's Co-educational College)**.**

Let $f(x)=(x^2-2)(x^2-3)(x^2-6)$. Then $f(x) = 0$ has no solution in integers. For $p = 2$ or 3, $f(6) \equiv 0 \pmod p$. For a prime $p > 3$, if there exists x such that $x^2 \equiv 2$ or $3 \pmod p$, then $f(x) \equiv 0 \pmod p$ has solutions in integers. Otherwise, from Euler's criterion, it follows that there will be x such that $x^2 \equiv 6 \pmod p$ and again $f(x) \equiv 0 \pmod p$ has solutions in integers.

Comments: For readers not familiar with Euler's criterion, we will give a bit more details. For c relatively prime to a prime p, by Fermat's little theorem, we have

$$(c^{(p-1)/2}-1)(c^{(p-1)/2}+1) = c^{p-1}-1 \equiv 0 \pmod p,$$

which implies $c^{(p-1)/2} \equiv 1$ or $-1 \pmod p$.

If there exists x such that $x^2 \equiv c \pmod p$, then $c^{(p-1)/2} \equiv x^{p-1} \equiv 1 \pmod p$. Conversely, if $c^{(p-1)/2} \equiv 1 \pmod p$, then there is x such that $x^2 \equiv c \pmod p$. [This is because there is a primitive root $g \pmod p$ (see *vol.* 15, *no.* 1, *p.* 1 of <u>*Math Excalibur*</u>), so we get $c \equiv g^i \pmod p$ for some positive integer i, then $g^{i(p-1)/2} \equiv 1 \pmod p$. Since g is a primitive root $\pmod p$, so $i(p-1)/2$ is a multiple of $p-1$, then i must be even, hence $c \equiv (g^{i/2})^2 \pmod p$.] In above, if 2 and 3 are not squares $\pmod p$, then $6^{(p-1)/2}=2^{(p-1)/2}3^{(p-1)/2} \equiv (-1)^2 = 1 \pmod p$, hence 6 is a square $\pmod p$.

Problem 377. Let n be a positive integer. For $i=1,2,\ldots,n$, let z_i and w_i be complex numbers such that for all 2^n choices of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ equal to ± 1, we have

$$\left|\sum_{i=1}^n \varepsilon_i z_i\right| \le \left|\sum_{i=1}^n \varepsilon_i w_i\right|.$$

Prove that $\displaystyle\sum_{i=1}^n |z_i|^2 \le \sum_{i=1}^n |w_i|^2$.

Solution. **William PENG** and **Jeff PENG** (Dallas,Texas, USA).

The case $n = 1$ is clear. Next, recall the parallelogram law $|a+b|^2+|a-b|^2=2|a|^2+2|b|^2$, which follows from adding the + and − cases of the identity

$$(a \pm b)(\bar{a} \pm \bar{b}) = a\bar{a} \pm a\bar{b} \pm b\bar{a} + b\bar{b}.$$

For $n = 2$, we have

$$|z_1+z_2| \le |w_1+w_2| \text{ and } |z_1-z_2| \le |w_1-w_2|.$$

Squaring both sides of these inequalities, adding them and applying the parallelogram law, we get the desired inequality. Next assume the case $n=k$ holds. Then for the $n=k+1$ case, we use the 2^k choices with $\varepsilon_1 = \varepsilon_2$ to get from the $n=k$ case that

$$\begin{aligned} &|z_1+z_2|^2+|z_3|^2+\cdots+|z_{k+1}|^2 \\ \le\ &|w_1+w_2|^2+|w_3|^2+\cdots+|w_{k+1}|^2. \end{aligned}$$

Similarly, using the other 2^k choices with $\varepsilon_1 = -\varepsilon_2$, we get

$$\begin{aligned} &|z_1-z_2|^2+|z_3|^2+\cdots+|z_{k+1}|^2 \\ \le\ &|w_1-w_2|^2+|w_3|^2+\cdots+|w_{k+1}|^2. \end{aligned}$$

Adding the last two inequalities and applying the parallelogram law, we get the $n=k+1$ case.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School),**Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania), **O Kin Chit, Alex** (G.T.(Ellen Yeung) College) and **Mohammad Reza SATOURI** (Bushehr, Iran).

Problem 378. Prove that for all positive integers m and n, there exists a positive integer k such that 2^k-m has at least n distinct positive prime divisors.

Solution. **William PENG** and **Jeff PENG**(Dallas,Texas, USA).

For the case m is odd, we will prove the result by inducting on n. If $n=1$, then just choose k large so that the odd number $2^k -m$ is greater than 1. Next assume there exists a positive integer k such that $j = 2^k-m$ has at least n distinct positive prime divisors. Let $s= k+\varphi(j^2)$, where $\varphi(j^2)$ is the number of positive integers at most j^2 that are relatively prime to j^2. Since j is odd, by Euler's theorem,

$$2^s - m \equiv 2^k \times 1 - m = j \pmod{j^2}.$$

Then $2^s - m$ is of the form $j+tj^2$ for some positive integer t. Hence it is divisible by j and $(2^s - m)/j$ is relatively prime to j. Therefore, $2^s - m$ has at least $n+1$ distinct prime divisors.

For the case m is even, write $m=2^i r$, where i is a nonnegative integer and r is odd. Then as proved above there is k such that $2^k - r$ has at least n distinct prime divisors and so is $2^{i+k} - m$.

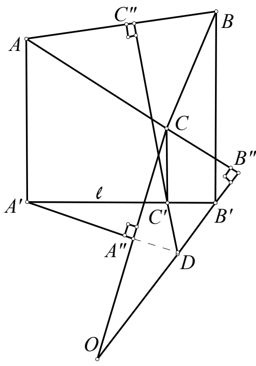
Other commended solvers: **Maxim BOGDAN** (“Mihai Eminescu” National College, Botosani, Romania)

Problem 379. Let ℓ be a line on the plane of ΔABC such that ℓ does not intersect the triangle and none of the lines *AB, BC, CA* is perpendicular to ℓ.

Let *A’, B’, C’* be the feet of the perpendiculars from *A, B, C* to ℓ respectively. Let *A”, B”, C”* be the feet of the perpendiculars from *A’, B’, C’* to lines *BC, CA, AB* respectively.

Prove that lines *A’A”, B’B”, C’C”* are concurrent.

***Solution*.** **William PENG** and **Jeff PENG** (Dallas, Texas, USA) and **ZOLBAYAR Shagdar** (9th Grade, Orchlon Cambridge International School, Mongolia).



Let lines *B’B”* and *C’C”* intersect at *D*. To show line *A’A”* also contains *D*, since $\angle CA''A' = 90°$, it suffices to show $\angle CA''D = 90°$.

Let lines *BC* and *B’B”* intersect at *O*. We claim that $\Delta DOA''$ is similar to $\Delta COB''$. (Since $\angle OB''C = 90°$, the claim will imply $\angle OA''D = 90°$, which is the same as $\angle CA''D = 90°$.)

For the claim, first note $\angle AC''D = 90° = \angle AB''D$, which implies *A,C”,B”,D* are concyclic. So $\angle C''AB'' = \angle B''DC''$. Next, $\angle BC''D = 90° = \angle DA''B$ implies *B,C”,A”,D* are concyclic. So $\angle C''BA'' = \angle A''DC''$. Then

$$\begin{aligned}\angle ODA'' &= 180° - (\angle A''DC'' + \angle B''DC'')\\ &= 180° - (\angle C''BA'' + \angle C''AB'')\\ &= \angle ACB\\ &= \angle OCB''.\end{aligned}$$

This along with $\angle DOA'' = \angle COB''$ yield the claim and we are done.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School) and **Maxim BOGDAN** (“Mihai Eminescu” National College, Botosani, Romania).

Problem 380. Let $S = \{1,2,\ldots,2000\}$. If *A* and *B* are subsets of *S*, then let $|A|$ and $|B|$ denote the number of elements in *A* and in *B* respectively. Suppose the product of $|A|$ and $|B|$ is at least 3999. Then prove that sets $A-A$ and $B-B$ contain at least one common element, where $X-X$ denotes $\{s-t : s, t \in X \text{ and } s \neq t\}$.
(*Source*: 2000 *Hungarian-Israeli Math Competition*)

Solution. **Maxim BOGDAN** (“Mihai Eminescu” National College, Botosani, Romania) and **William PENG** and **Jeff PENG** (Dallas,Texas, USA).

Note that the set $T=\{(a,b)$: $a \in A$ and $b \in B\}$ has $|A|\times|B| \geq 3999$ elements. Also, the set $W=\{a+b$: $a \in A$ and $b \in B\}$ is a subset of $\{2,3,\ldots 4000\}$. If $W = \{2,3,\ldots,4000\}$, then 2 and 4000 in W imply sets *A* and *B* both contain 1 and 2000. This leads to $A-A$ and $B-B$ both contain 1999.

If $W \neq \{2,3,\ldots 4000\}$, then W has less than 3999 elements. By the pigeonhole principle, there would exist $(a,b) \neq (a',b')$ in T such that $a+b=a'+b'$. This leads to $a-a'=b'-b$ in both $A-A$ and $B-B$.

Olympiad Corner

(continued from page 1)

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq y f(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Problem 4. Let $n > 0$ be an integer. We are given a balance and n weights of weigh 2^0, 2^1, …, 2^{n-1}. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all the weights have been placed.

Determine the number of ways in which this can be done.

Problem 5. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m)-f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Problem 6. Let ABC be an acute triangle with circumcircle γ. Let L be a tangent line to γ, and let L_a, L_b and L_c be the line obtained by reflecting L in the lines *BC, CA* and *AB*, respectively. Show that the circumcircle of the triangle determined by the lines L_a, L_b and L_c is tangent to the circle γ.

Remarks on IMO 2011

(continued from page 2)

The essential point is to observe that *A”B”C”* is in fact homothetic to *A’B’C’*, with the homothetic center at *H*, a point on γ, i.e. *A”B”C”* is an expansion of *A’B’C’* at *H* by a constant centre. This implies the circumcircle of *A”B”C”* is tangent to γ at *H*.

A lot of discussions were conducted concerning changing the format of the Jury system during the IMO. At present the leaders assemble to choose six problems from the short-listed problems. There are issues concerning security and also financial matter (to house the leaders in an obscure place far away from the contestants can be costly). Many contestants need good results to obtain scholarships and enter good universities and the leaders have incentive for their own good to obtain good results for their teams. For me I am inclined to let the Jury system remains as such. The main reason is simply the law of large numbers, a better paper may be produced if more people are involved. Indeed both the Problem Selection Group and the leaders may make mistakes. But we get a better chance to produce a better paper after detailed discussion. In my opinion we generally produce a more balanced paper. The discussion is still going on. Perhaps some changes are unavoidable, for better or for worse.

Here are some remarks concerning the performance of the teams. We keep our standard or perhaps slightly better than the last few years. I am glad that some of our team members are able to solve the harder problems. Although the Chinese team is still ranked first (unofficially), they are not far better than the other strong teams (USA, Russia, etc). In particular, the third rank performance of the Singaporean team this time is really amazing.

Mathematical Excalibur

Volume 16, Number 4 *February 2012*

Olympiad Corner

Below are the problems of the 2011-2012 British Math Olympiad Round 1 held on 2 December 2011.

Problem 1. Find all (positive or negative) integers n for which $n^2+20n+11$ is a perfect square.

Problem 2. Consider the numbers 1, 2, ⋯, n. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

Problem 3. Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

Problem 4. Initially there are m balls in one bag, and n in the other, where m, $n > 0$. Two different operations are allowed:

a) Remove an equal number of balls from each bag;
b) Double the number of balls in one bag.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 28, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Zsigmondy's Theorem

Andy Loo (St. Paul's Co-educational College)

In recent years, a couple of "hard" number theoretic problems in the IMO turn out to be solvable by simple applications of deep theorems. For instances, IMO 2003 Problem 6 and IMO 2008 Problem 3 are straight forward corollaries of the Chebotarev density theorem and a theorem of Deshouillers and Iwaniec respectively. In this article we look at yet another mighty theorem, which was discovered by the Austro-Hungarian mathematician Karl Zsigmondy in 1882 and which can be used to tackle many Olympiad problems at ease.

Zsigmondy's theorem

First part: If a, b and n are positive integers with $a>b$, gcd(a, b)=1 and $n\geq 2$, then a^n-b^n has at least one prime factor that does not divide a^k-b^k for all positive integers $k<n$, with the exceptions of:

i) 2^6-1^6 and

ii) $n=2$ and $a+b$ is a power of 2.

Second part: If a, b and n are positive integers with $a>b$ and $n\geq 2$, then a^n+b^n has at least one prime factor that does not divide a^k+b^k for all positive integers $k<n$, with the exception of 2^3+1^3.

The proof of this theorem is omitted due to limited space. Interested readers may refer to [2].

To see its power, let us look at how short solutions can be obtained using Zsigmondy's theorem to problems of various types.

Example 1 (Japanese MO 2011). Find all quintuples of positive integers (a,n,p,q,r) such that

$$a^n-1 = (a^p-1)(a^q-1)(a^r-1).$$

Solution. If $a\geq 3$ and $n\geq 3$, then by Zsigmondy's theorem, a^n-1 has a prime factor that does not divide a^p-1, a^q-1 and a^r-1 (plainly $n>p,q,r$), so there is no solution. The remaining cases ($a<3$ or $n<3$) are easy exercises for the readers.

Example 2 (IMO Shortlist 2000). Find all triplets of positive integers (a,m,n) such that $a^m+1|(a+1)^n$.

Solution. Note that $(a,m,n)=(2,3,n)$ with $n\geq 2$ are solutions. For $a>1$, $m\geq 2$ and $(a,m)\neq(2,3)$, by Zsigmondy's theorem, a^m+1 has a prime factor that does not divide $a+1$, and hence does not divide $(a+1)^n$, so there is no solution. The cases ($a=1$ or $m=1$) lead to easy solutions.

Example 3 (Math Olympiad Summer Program 2001) Find all quadruples of positive integers (x,r,p,n) such that p is a prime, $n,r>1$ and $x^r-1=p^n$.

Solution. If x^r-1 has a prime factor that does not divide $x-1$, then since x^r-1 is divisible by $x-1$, we deduce that x^r-1 has at least two distinct prime factors, a contradiction *unless* (by Zsigmondy's theorem) we have the exceptional cases $x=2$, $r=6$ and $r=2$, $x+1$ is a power of 2. The former does not work. For the latter, obviously $p=2$ since it must be even. Let $x+1=2^y$. Then

$$2^n = x^2-1 = (x+1)(x-1) = 2^y(2^y-2).$$

It follows that $y=2$ (hence $x=3$) and $n=3$.

Example 4 (Czech-Slovak Match 1996). Find all positive integral solutions to $p^x-y^p=1$, where p is a prime.

Solution. The equation can be rewritten as $p^x=y^p+1$. Now $y=1$ leads to $(p,x)=(2,1)$ and $(y,p)=(2,3)$ leads to $x=2$. For $y>1$ and $p\neq 3$, by Zsigmondy's theorem, y^p+1 has a prime factor that does not divide $y+1$. Since y^p+1 is divisible by $y+1$, it follows that y^p+1 has at least two prime factors, a contradiction.

Remark. Alternatively, the results of Examples 3 and 4 follow from Catalan's conjecture (proven in 2002), which guarantees that the only positive integral solution to the equation $x^a-y^b=1$ with $x,y,a,b>1$ is $x=3$, $a=2$, $y=2$, $b=3$.

Example 5 (Polish MO 2010 Round 1). Let p and q be prime numbers with $q>p>2$. Prove that $2^{pq}-1$ has at least three distinct prime factors.

Solution. Note that 2^p-1 and 2^q-1 divide $2^{pq}-1$. By Zsigmondy's theorem, $2^{pq}-1$ has a prime factor p_1 that does not divide 2^p-1 and 2^q-1. Moreover, 2^q-1 has a prime factor p_2 that does not divide 2^p-1. Finally, 2^p-1 has a prime factor p_3.

The next example illustrates a more involved technique of applying Zsigmondy's theorem to solve a class of Diophantine equations.

Example 6 (Balkan MO 2009). Solve the equation $5^x - 3^y = z^2$ in positive integers.

Solution. By considering (mod 3), we see that x must be even. Let $x=2w$. Then $3^y = 5^{2w}-z^2 = (5^w-z)(5^w+z)$. Note that

$$\begin{aligned}(5^w-z,5^w+z) &= (5^w-z,2z)\\ &= (5^w-z,z)\\ &= (5^w,z) = 1,\end{aligned}$$

so $5^w - z = 1$ and $5^w + z = 3^a$ for some positive integer $a \ge 2$. Adding, $2(5^w) = 3^a + 1$. For $a = 2$, we have $w = 1$, corresponding to the solution $x = 2$, $y = 2$ and $z = 4$. For $a \ge 3$, by Zsigmondy's theorem, 3^a+1 has a prime factor p that does not divide $3^2 + 1 = 10$, which implies $p\ne 2$ or 5, so there is no solution in this case.

Example 7. Find all positive integral solutions to $p^a-1=2^n(p-1)$, where p is a prime.

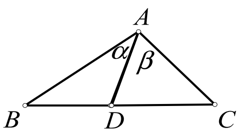
Solution. The case $p=2$ is trivial. Assume p is odd. If a is not a prime, let $a=uv$. Then p^u-1 has a prime factor that does not divide $p-1$. Since p^u-1 divides $p^a-1=2^n(p-1)$, this prime factor of p^u-1 must be 2. But by Zsigmondy's theorem, p^a-1 has a prime factor that does not divide p^u-1 and $p-1$, a contradiction to the equation. So a is a prime. The case $a=2$ yields $p=2^n-1$, i.e. the Mersenne primes. If a is an odd number, then by Zsigmondy's theorem again, $p^a-1=2^n(p-1)$ has a prime factor that does not divide $p-1$; this prime factor must be 2. However, 2 divides $p-1$, a contradiction.

(continued on page 4)

A Geometry Theorem

Kin Y. Li

The following is a not so well known, but useful theorem.



Subtended Angle Theorem. D is a point inside $\angle BAC$ (<180°). Let $\alpha=\angle BAD$ and $\beta=\angle CAD$. D is on side BC if and only if

$$\frac{\sin(\alpha+\beta)}{AD}=\frac{\sin\alpha}{AC}+\frac{\sin\beta}{AB} \qquad (*).$$

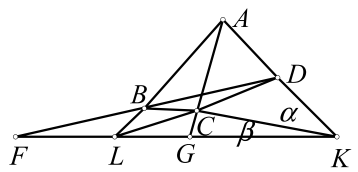
Proof. Note D is on segment BC if and only if the area of $\triangle ABC$ is the sum of the areas of $\triangle ABD$ and $\triangle ACD$. This is

$$\frac{AB\cdot AC\sin(\alpha+\beta)}{2}=\frac{AB\cdot AD\sin\alpha}{2}+\frac{AC\cdot AD\sin\beta}{2}.$$

Multiplying by $2/(AB\cdot AC\cdot AD)$ yields (*).

Below, we will write $PQ\cap RS=X$ to mean lines PQ and RS intersect at point X.

Example 1. Let $AD\cap BC=K$, $AB\cap CD=L$, $BD\cap KL=F$ and $AC\cap KL=G$. Prove that $1/KL = ½(1/KF +1/KG)$.



Solution. Applying the subtended angle theorem to $\triangle KAL$, $\triangle KDL$, $\triangle KDF$ and $\triangle KAG$, we get

$$\frac{\sin(\alpha+\beta)}{KB}=\frac{\sin\alpha}{KL}+\frac{\sin\beta}{KA},\ \frac{\sin(\alpha+\beta)}{KC}=\frac{\sin\alpha}{KL}+\frac{\sin\beta}{KD}$$

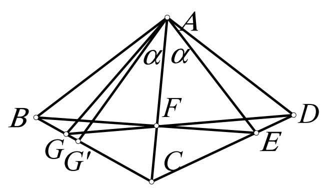
$$\frac{\sin(\alpha+\beta)}{KB}=\frac{\sin\alpha}{KF}+\frac{\sin\beta}{KD},\ \frac{\sin(\alpha+\beta)}{KC}=\frac{\sin\alpha}{KG}+\frac{\sin\beta}{KA}$$

Call these (1), (2), (3), (4) respectively. Doing (1)+(2)−(3)−(4), we get

$$0=\frac{2\sin\alpha}{KL}-\frac{\sin\alpha}{KF}-\frac{\sin\alpha}{KG},$$

which implies the desired equation.

Example 2. (*1999 Chinese National Math Competition*) In the convex quadrilateral $ABCD$, diagonal AC bisects $\angle BAD$. Let E be on side CD such that $BE\cap AC=F$ and $DF\cap BC=G$. Prove that $\angle GAC = \angle EAC$.



Solution. Let $\angle BAC=\angle DAC=\theta$ and G' be on segment BC such that $\angle G'AC = \angle EAC=\alpha$. We will show G', F, D are collinear, which implies $G'=G$. Applying the subtended angle theorem to $\triangle ABE$, $\triangle ABC$ and $\triangle ACD$ respectively, we get

$$(1)\ \frac{\sin(\theta+\alpha)}{AF}=\frac{\sin\alpha}{AB}+\frac{\sin\theta}{AE},$$

$$(2)\ \frac{\sin\theta}{AG'}=\frac{\sin\alpha}{AB}+\frac{\sin(\theta-\alpha)}{AC}\ \text{and}$$

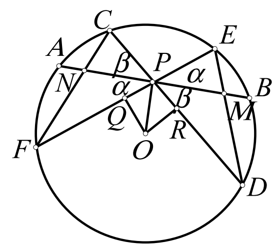
$$(3)\ \frac{\sin\theta}{AE}=\frac{\sin\alpha}{AD}+\frac{\sin(\theta-\alpha)}{AC}.$$

Doing (1)−(2)+(3), we get

$$\frac{\sin(\theta+\alpha)}{AF}=\frac{\sin\alpha}{AD}+\frac{\sin\theta}{AG'}.$$

By the subtended angle theorem, G', F, D are collinear. Therefore, $G = G'$.

Example 3. (*Butterfly Theorem*) Let A,C,E,B,D,F be points in cyclic order on a circle and $CD\cap EF=P$ is the midpoint of AB. Let $M = AB\cap DE$ and $N = AB\cap CF$. Prove that $MP = NP$.



Solution. By the intersecting chord theorem, $PC\cdot PD=PE\cdot PF$, call this x. Applying the subtended angle theorem to $\triangle PDE$ and $\triangle PCF$, we get

$$\frac{\sin(\alpha+\beta)}{PM}=\frac{\sin\alpha}{PD}+\frac{\sin\beta}{PE},$$
$$\frac{\sin(\alpha+\beta)}{PN}=\frac{\sin\alpha}{PC}+\frac{\sin\beta}{PF}.$$

Subtracting these equations, we get

$$\sin(\alpha+\beta)\left(\frac{1}{PM}-\frac{1}{PN}\right) \qquad (*)$$
$$=\sin\beta\frac{PF-PE}{x}-\sin\alpha\frac{PD-PC}{x}.$$

Let Q and R be the midpoints of EF and CD respectively. Since $OP\perp AB$, we have $PF-PE = 2PQ = 2OP\cos(90°-\alpha) = 2OP\sin\alpha$. Then similarly we have $PD-PC = 2OP\sin\beta$. Hence, the right side of (*) is zero. So the left side of (*) is also zero. Since $0 < \alpha+\beta < 180°$, we get $\sin(\alpha+\beta) \ne 0$. Then $PM = PN$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***March 28, 2012.***

Problem 386. Observe that $7+1=2^3$ and $7^7+1= 2^3\times113\times911$. Prove that for $n = 2, 3, 4,\ldots$, in the prime factorization of $A_n = 7^{7^n}+1$, the sum of the exponents is at least $2n+3$.

Problem 387. Determine (with proof) all functions $f: [0,+\infty) \to [0,+\infty)$ such that for every $x \ge 0$, we have $4f(x) \ge 3x$ and $f(4f(x) - 3x) = x$.

Problem 388. In $\triangle ABC$, $\angle BAC=30°$ and $\angle ABC=70°$. There is a point M lying inside $\triangle ABC$ such that $\angle MAB=\angle MCA=20°$. Determine $\angle MBA$ (with proof).

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

$$x^2y^2 = z^2(z^2 - x^2 - y^2).$$

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Solutions

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Problem 381. Let k be a positive integer. There are 2^k balls divided into a number of piles. For every two piles A and B with p and q balls respectively, if $p \ge q$, then we may transfer q balls from pile A to pile B. Prove that it is always possible to make finitely many such transfers so as to have all the balls end up in one pile.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3), **LO Shing Fung** (F3E, Carmel Alison Lam Foundation Secondary School) and **Andy LOO** (St. Paul's Co-educational College).

We induct on k. For k=1, we can merge the 2 balls in at most 1 transfer.

Suppose the case k=n is true. For k=n+1, since 2^k is even, considering (odd-even) parity of the number of balls in each pile, we see the number of piles with odd numbers of balls is even. Pair up these piles. In each pair, after 1 transfer, both piles will result in even number of balls.

So we need to consider only the situation when all piles have even number of balls. Then in each pile, pair up the balls. This gives altogether 2^n pairs. Applying the case k=n with the paired balls, we solve the case k=n+1.

Problem 382. Let $v_0 = 0$, $v_1 = 1$ and

$$v_{n+1} = 8v_n - v_{n-1} \quad \text{for } n = 1,2,3,\ldots.$$

Prove that v_n is divisible by 3 if and only if v_n is divisible by 7.

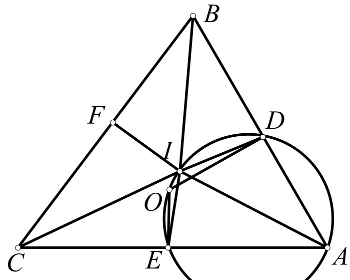
Solution. **Alumni 2011** (Carmel Alison Lam Foundation Secondary School) and **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia) and **Mihai STOENESCU** (Bischwiller, France).

For $n = 1,2,3,\ldots$, $v_{n+2} = 8(8v_n - v_{n-1}) - v_n = 63v_n - 8v_{n-1}$. Then $v_{n+2} \equiv v_{n-1} \pmod 3$ and $v_{n+2} \equiv -v_{n-1} \pmod 7$. Since $v_0= 0$, $v_1=1$, $v_2 = 8$, so $v_{3k+1}, v_{3k+2} \not\equiv 0$ (mod 3) and (mod 7) and $v_{3k} \equiv 0$ (mod 3) and (mod 7).

Other commended solvers: **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **CHAN Long Tin** (Diocesan Boys' School), **CHAN Yin Hong** (St. Paul's Co-educational College), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Andy LOO** (St. Paul's Co-educational College), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (G.T.(Ellen Yeung) College), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Yan Yin WANG** (City University of Hong Kong, Computing Math, Year 2), **ZOLBAYAR Shagdar** (Orchlon School, Ulaanbaatar, Mongolia),**Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 383. Let O and I be the circumcenter and incenter of $\triangle ABC$ respectively. If $AB\neq AC$, points D, E are midpoints of AB, AC respectively and $BC=(AB+AC)/2$, then prove that the line OI and the bisector of $\angle CAB$ are perpendicular.

***Solution* 1. Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3).



From $BC = (AB+AC)/2 = BD+CE$, we see there exists a point F be on side BC such that $BF=BD$ and $CF=CE$. Since BI bisects $\angle FBD$, by SAS, $\triangle IBD \cong \triangle IBF$. Then $\angle BDI=\angle BFI$. Similarly, $\angle CEI = \angle CFI$. Then

$$\begin{aligned}&\angle ADI + \angle AEI\\ &= (180° - \angle BDI) + (180° - \angle CEI)\\ &= 360° - \angle BFI - \angle CFI = 180°.\end{aligned}$$

So A,D,I,E are concyclic.

Since $OD \perp AD$ and $OE \perp AE$, so A,D,O,E are also conyclic. Then A,D,I,O are concyclic. So $\angle OIA = \angle ODA$=90°.

***Solution* 2. AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Ercole SUPPA** (Teramo, Italy), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Let $a=BC$, $b=CA$, $c=AB$ and let R, r, s be the circumradius, the inradius and the semiperimeter of $\triangle ABC$ respectively. By the famous formulas $OI^2 = R^2-2Rr$, $s-a = AI\cos(A/2)$, $Rr = abc/(4s)$ and $\cos^2(A/2) = s(s-a)/(bc)$, we get

$$AI^2 = \frac{(s-a)^2}{\cos^2(A/2)} = \frac{bc(s-a)}{s},$$

$$OI^2 = R^2 - 2Rr = R^2 - \frac{abc}{2s}.$$

If $a = (b+c)/2$, then we get $2s = 3a$ and $bc(s-a)/s = abc/(2s)$. So $AI^2+OI^2 = R^2 = OA^2$. By the converse of Pythagoras' Theorem, we get $OI\perp AI$.

Comment: In the last paragraph, all steps may be reversed so that $OI\perp AI$ if and only if $a = (b+c)/2$.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Andy LOO** (St. Paul's Co-educational College), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **Mihai STOENESCU** (Bischwiller, France) and **ZOLBAYAR Shagdar** (Orchlon School, Ulaanbaatar, Mongolia).

Problem 384. For all positive real numbers a,b,c satisfying $a + b + c = 3$, prove that

$$\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-bc)}+\frac{c^2+3a^2}{ca^2(4-ca)}\geq 4.$$

***Solution*. William PENG**.

Let

$$A=\frac{a}{b^2(4-ab)}+\frac{b}{c^2(4-bc)}+\frac{c}{a^2(4-ca)},$$

$$B=\frac{1}{a(4-ab)}+\frac{1}{b(4-bc)}+\frac{1}{c(4-ca)},$$

$$C=\frac{4-ab}{a}+\frac{4-bc}{b}+\frac{4-ca}{c}$$

and $D=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Then $A+3B$ is the left side of the desired inequality. Now since $a + b + c = 3$, we have $C = 4D - 3$. By the Cauchy-Schwarz inequality, we have $(a+b+c)D \geq 3^2$, $AC \geq D^2$ and $BC \geq D^2$. The first of these gives us $D \geq 3$ so that $(D-3)(D-1)\geq 0$, which implies $D^2 \geq 4D-3$. The second and third imply

$$A+3B\geq\frac{4D^2}{C}=\frac{4D^2}{4D-3}\geq 4.$$

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Andy LOO** (St. Paul's Co-educational College), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 385. To prepare for the IMO, in everyday of the next 11 weeks, Jack will solve at least one problem. If every week he can solve at most 12 problems, then prove that for some positive integer n, there are n consecutive days in which he can solve a total of 21 problems.

***Solution.* AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Andy LOO** (St. Paul's Co-educational College) and **Yan Yin WANG** (City University of Hong Kong, Computing Math, Year 2).

Let S_i be the total number of problems Jack solved from the first day to the end of the i-th day. Since he solves at least one problem everyday, we have $0 < S_1 < S_2 < S_3 < \cdots < S_{77}$. Since he can solve at most 12 problems every week, we have $S_{77} \leq 12\times 11 = 132$.

Consider the two strictly increasing sequences $S_1, S_2, \cdots, S_{77}$ and $S_1+21, S_2+21, \cdots, S_{77}+21$. Now these 154 integers are at least 1 and at most 132+21=153. By the pigeonhole principle, since the two sequences are strictly increasing, there must be $m < k$ such that $S_k=S_m+21$. Therefore, Jack solved a total of 21 problems from the $(m+1)$-st day to the end of the k-th day.

Olympiad Corner

(continued from page 1)

Problem 4 (cont). Is it possible to empty both bags after a finite sequence of operations?

Operation b) is now replaced with

b') Triple the number of balls in one bag.

Is it now possible to empty both bags after a finite sequence of operations?

Problem 5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

Problem 6. Let ABC be an acute-angled triangle. The feet of all the altitudes from A, B and C are D, E and F respectively. Prove that $DE+DF \leq BC$ and determine the triangles for which equality holds.

Zsigmondy's Theorem

(continued from page 2)

Example 8. Find all positive integral solutions to

$$(a+1)(a^2+a+1)\cdots(a^n+a^{n-1}+\cdots+1)$$
$$= a^m+a^{m-1}+\cdots+1.$$

Solution. Note that $n = m = 1$ is a trivial solution. Other than that, we must have $m>n$. Write the equation as

$$\frac{a^2-1}{a-1}\cdot\frac{a^3-1}{a-1}\cdot\ldots\cdot\frac{a^{n+1}-1}{a-1}=\frac{a^{m+1}-1}{a-1},$$

then rearranging we get

$$\left(a^2-1\right)\left(a^3-1\right)\ldots\left(a^{n+1}-1\right)$$
$$=\left(a^{m+1}-1\right)\left(a-1\right)^{n-1}.$$

By Zsigmondy's theorem, we must have $a = 2$ and $m + 1 = 6$, i.e. $m = 5$ (otherwise, a^m-1 has a prime factor that does not divide a^2-1, a^3-1, …, $a^{n+1}-1$, a contradiction), which however does not yield a solution for n.

The above examples show that Zsigmondy's theorem can instantly reduce many number theoretic problems to a handful of small cases. We should bear in mind the exceptions stated in Zsigmondy's theorem in order not to miss out any solutions.

Below are some exercises for the readers.

Exercise 1 (1994 Romanian Team Selection Test). Prove that the sequence $a_n=3^n-2^n$ contains no three terms in geometric progression.

Exercise 2. Fermat's last theorem asserts that for a positive integer $n \geq 3$, the equation $x^n+y^n=z^n$ has no integral solution with $xyz\neq 0$. Prove this statement when z is a prime.

Exercise 3 (1996 British Math Olympiad Round 2). Determine all sets of non-negative integers x, y and z which satisfy the equation $2^x+3^y=z^2$.

Reference

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Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2011 IMO Team Selection Contest from Estonia.

Problem 1. Two circles lie completely outside each other. Let A be the point of intersection of internal common tangents of the circles and let K be the projection of this point onto one of their external common tangents. The tangents, different from the common tangent, to the circles through point K meet the circles at M_1 and M_2. Prove that the line AK bisects angle M_1KM_2.

Problem 2. Let n be a positive integer. Prove that for each factor m of the number $1+2+\cdots+n$ such that $m \ge n$, the set $\{1,2,\cdots,n\}$ can be partitioned into disjoint subsets, the sum of the elements of each being equal to m.

Problem 3. Does there exist an operation * on the set of all integers such that the following conditions hold simultaneously:

(1) for all integers x, y and z,

$$(x*y)*z = x*(y*z);$$

(2) for all integers x and y,

$$x*x*y = y*x*x = y\ ?$$

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 11, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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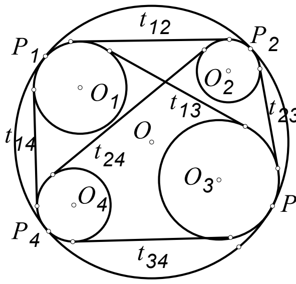
Casey's Theorem

Kin Y. Li

We recall *Ptolemy's theorem*, which asserts that for four noncollinear points *A, B, C, D* on a plane, we have

$$AB\cdot CD + AD\cdot BC = AC\cdot BD$$

if and only if $ABCD$ is a cyclic quadrilateral (cf *vol.* 2, *no.* 4 of <u>*Math Excalibur*</u>). In this article, we study a generalization of this theorem known as

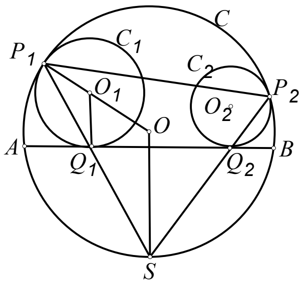


<u>Casey's Theorem.</u> If circles C_1, C_2, C_3, C_4 with centers O_1, O_2, O_3, O_4 are internally tangent to a circle C with center O at points P_1, P_2, P_3, P_4 in cyclic order respectively, then

$$t_{12}\cdot t_{34} + t_{14}\cdot t_{23} = t_{13}\cdot t_{24}, \qquad (*)$$

where t_{ik} denote the length of an external common tangent of circle C_i and C_k.

To prove this, consider the following figure.



Let line AB be an external common tangent to C_1, C_2 intersecting C_1 at Q_1, C_2 at Q_2. Let line P_1Q_1 intersect C at S. Let r_1, r be the respective radii of C_1, C. Then the isosceles triangles $P_1O_1Q_1$ and P_1OS are similar. So $O_1Q_1 \parallel OS$. Since $O_1Q_1\perp AB$, so $OS\perp AB$, hence S is the midpoint of arc AB. Similarly, line P_2Q_2 passes through S. Now $\angle SQ_1Q_2 = \angle P_1Q_1A = ½\angle P_1O_1Q_1 = ½\angle P_1OS = \angle SP_2P_1$. Then ΔSQ_1Q_2 and ΔSP_2P_1 are similar. So

$$\frac{Q_1Q_2}{P_2P_1}=\frac{SQ_1}{SP_2}=\frac{SQ_2}{SP_1}=\sqrt{\frac{SQ_1\cdot SQ_2}{SP_1\cdot SP_2}}=\sqrt{\frac{OQ_1\cdot OQ_2}{OP_1\cdot OP_2}}$$

$$t_{12}=Q_1Q_2=P_1P_2\frac{\sqrt{(r-r_1)(r-r_2)}}{r}. \qquad (**)$$

The expressions for the other t_{ik}'s are similar. Since $P_1P_2P_3P_4$ is cyclic, by Ptolemy's theorem,

$$P_1P_2\cdot P_3P_4 + P_1P_4\cdot P_2P_3 = P_1P_3\cdot P_2P_4.$$

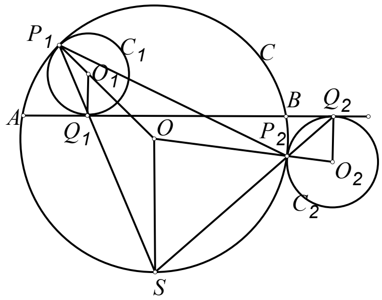
Multiplying all terms by

$$\frac{\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}}{r^2}$$

and using (**), we get (*).

Casey's theorem can be *<u>extended</u>* to cover cases some C_k's are externally tangent to C. For this, define t_{ik} more generally to be the length of the external (resp. internal) common tangent of circles C_i and C_k when the circles are on the same (resp. opposite) side of C.

In case C_k is externally tangent to C, consider the following figure. The proof is the same as before except the factor $r-r_k$ should be replaced by $r+r_k$.



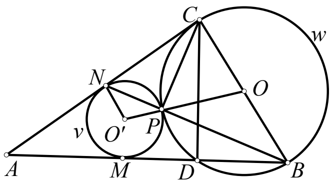
The converse of Casey's theorem and its extension are also true. However, the proofs are harder, longer and used inversion in some cases. For the curious readers, a proof of the converse can be found in *Roger A. Johnson*'s book <u>Advanced Euclidean Geometry</u>, published by Dover.

Next we will present some geometry problems that can be solved by Casey's theorem and its converse.

Example 1. (*2009 China Hong Kong Math Olympiad*) Let $\triangle ABC$ be a right-angled triangle with $\angle C=90°$. CD is the altitude from C to AB, with D on AB. w is the circumcircle of $\triangle BCD$. v is a circle situated in $\triangle ACD$, it is tangent to the segments AD and AC at M and N respectively, and is also tangent to circle w.

(i) Show that $BD \cdot CN + BC \cdot DM = CD \cdot BM$.

(ii) Show that $BM = BC$.



Solution. (i) Think of B, C, D as circles with radius 0 externally tangent to w. Then $t_{BD} = BD$, $t_{Cv} = CN$, $t_{BC} = BC$, $t_{Dv} = DM$, $t_{CD} = CD$ and $t_{Bv} = BM$. By Casey's theorem, (*) yields

$$BD \cdot CN + BC \cdot DM = CD \cdot BM.$$

(ii) Let circles v and w meet at P. Then $\angle BPC=90°$. Let O and O' be centers of circles w and v. Then O, P, O' are collinear. So

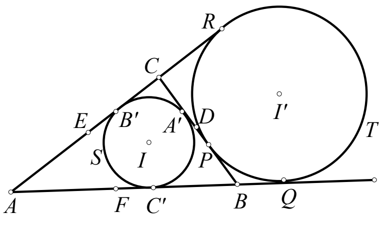
$$\angle PNC+\angle PCN = ½(\angle PO'N+\angle POC) = ½(360°-\angle O'NC-\angle OCN) = 90°.$$

So $\angle NPC = 90°$. Hence, B, P, N are collinear. By the power-of-a-point theorem, $BM^2=BP\cdot BN$. Also $\angle C=90°$ and $CP\perp BN$ imply $BC^2 = BP\cdot BN$. Therefore, $BM=BC$.

Example 2. (*Feuerbach's Theorem*) Let D,E,F be the midpoints of sides AB, BC, CA of ΔABC respectively.

(i) Prove that the inscribed circle S of ΔABC is tangent to the (nine-point) circle N through D, E, F.

(ii) Prove that the described circle T on side BC is also tangent to N.



Solution. (1) We consider D, E, F as circles of radius 0. Let A', B', C' be the points of tangency of S to sides BC, CA, AB respectively.

First we recall that the two tangent segments from a point to a circle have the same length. Let $AB' = x = C'A$, $BC' = y = A'B$, $CA' = z = B'C$ and $s = (a+b+c)/2$, where $a=BC$, $b=CA$, $c=AB$. From $y+x = BA = c$, $z+y = CB = a$ and $x+z = AC = b$, we get $x = (c+b-a)/2 = s-a$, $y=s-b$, $z=s-c$. By the midpoint theorem, $t_{DE} = DE = ½BA = c/2$ and

$$t_{FS} = FC' = |FB-BC'| = |(c/2)-y| = |c-2(s-b)|/2 = |b-a|/2.$$

Similarly, $t_{EF} = a/2$, $t_{DS} = |c-b|/2$, $t_{FD} = b/2$ and $t_{ES}=|a-c|/2$. Without loss of generality, we may assume $a\le b\le c$. Then

$$\begin{aligned} t_{DE} \cdot t_{FS} + t_{EF} \cdot t_{DS} &= c(b-a)/4+a(c-b)/4 \\ &= b(c-a)/4 \\ &= t_{FD} \cdot t_{ES}. \end{aligned}$$

By the converse of Casey's theorem, we get S is tangent to the circle N through D,E,F.

(2) Let I' be the center of T, let P,Q,R be the points of tangency of T to lines BC, AB, CA respectively. As in (1), $t_{DE} = c/2$.

To find t_{FT}, we need to know BQ. First note $AQ=AR$, $BP=BQ$ and $CR=CP$. So $2AQ=AQ+AR=AB+BP+CP+AC=2s$. So $AQ=s/2$. Next $BQ=AQ-AB=s-c$. Hence, $t_{FT}=FQ=FB+BQ = (c/2)+(s-c) = (b+a)/2$. Similarly, $t_{ET} = (a+c)/2$. Now $t_{DT} = DP = DB-BP = DB-BQ = (a/2) - (s-c) = (c-b)/2$. Then

$$\begin{aligned} t_{FD} \cdot t_{ET} + t_{EF} \cdot t_{DT} &= b(a+c)/4+a(c-b)/4 \\ &= c(b+a)/4 \\ &= t_{DE} \cdot t_{FT}. \end{aligned}$$

By the converse of Casey's theorem, we get T is tangent to the circle N through D,E,F.

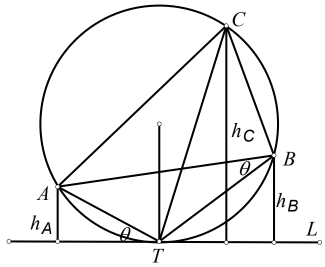
Example 3. (*2011 IMO*) Let ABC be an acute triangle with circumcircle Γ. Let L be a tangent line to Γ, and let L_a, L_b and L_c be the line obtained by reflecting L in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines L_a, L_b and L_c is tangent to the circle Γ.

Solution. (*Due to **CHOW Chi Hong**, 2011 Hong Kong IMO team member*)

Below for brevity, we will write $\angle A$, $\angle B$, $\angle C$ to denote $\angle CAB$, $\angle ABC$, $\angle BCA$ respectively.

Lemma. In the figure below, L is a tangent line to Γ, T is the point of tangency. Let h_A, h_B, h_C be the length of the altitudes from A, B, C to L respectively. Then

$$\sqrt{h_A}\sin\angle A+\sqrt{h_B}\sin\angle B=\sqrt{h_C}\sin\angle C.$$



Proof. By Ptolemy's theorem and sine law,

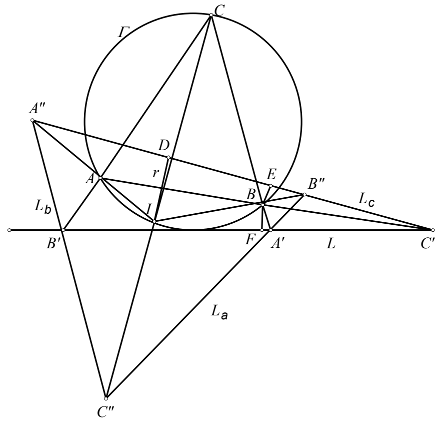
$$AT\cdot BC + BT\cdot CA = CT\cdot BC \quad \text{(or}$$

$AT \sin\angle A + BT\sin\angle B = CT \sin\angle C$).

Let θ be the angle between lines AT and L as shown. Then $AT = h_A / \sin\theta = h_A(2k/AT)$, where k is the circumradius of ΔABC. Solving for AT (then using similar argument for BT and CT), we get

$$AT=\sqrt{2kh_A},BT=\sqrt{2kh_B},CT=\sqrt{2kh_C}.$$

Substituting these into (*), the result follows. This finishes the proof of the lemma.



For the problem, let $L_a\cap L=A'$, $L_b\cap L = B'$, $L_c\cap L=C'$, $L_a\cap L_b=C''$, $L_b\cap L_c=A''$, $L_c\cap L_a=B''$. Next

$$\begin{aligned} \angle A''C''B'' &= \angle A''B'A' - \angle C''A'B' \\ &= 2\angle CB'A' - (180° - 2\angle CA'B') \\ &= 180° - 2\angle C. \end{aligned}$$

Similarly, $\angle A''B''C'' = 180° - 2\angle B$ and $\angle B''A''C'' = 180° - 2\angle A$. (***)

Consider $\Delta A'C'B''$. Now $A'B$ bisects $\angle B'A'B''$ and $C'B$ bisects $\angle A'C'B''$. So B is the excenter of $\Delta A'C'B''$ opposite C'. Hence $B''B$ bisects $\angle A''B''C''$. Similarly, $A''A$ bisects $\angle B''A''C''$ and $C''C$ bisects $\angle B''C''A''$. Therefore, they intersect at the incenter I of $\Delta A''B''C''$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 11, 2012.***

Problem 391. Let $S(x)$ denote the sum of the digits of the positive integer x in base 10. Determine whether there exist distinct positive integers a, b, c such that $S(a+b)<5$, $S(b+c)<5$, $S(c+a)<5$, but $S(a+b+c)>50$ or not.

Problem 392. Integers $a_0, a_1, \cdots, a_n$ are all greater than or equal to -1 and are not all zeros. If

$$a_0+2a_1+2^2a_2+\cdots+2^na_n = 0,$$

then prove that $a_0+a_1+a_2+\cdots+a_n>0$.

Problem 393. Let p be a prime number and $p \equiv 1 \pmod 4$. Prove that there exist integers x and y such that

$$x^2 - py^2 = -1.$$

Problem 394. Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$. The bisector of $\angle BAC$ meets the circumcircle Γ of $\triangle ABC$ at D. Let E be the mirror image of D with respect to line BC. Let F be on Γ such that DF is a diameter. Let lines AE and FH meet at G. Let M be the midpoint of side BC. Prove that $GM \perp AF$.

Problem 395. One frog is placed on every vertex of a $2n$-sided regular polygon, where n is an integer at least 2. At a particular moment, each frog will jump to one of the two neighboring vertices (with more than one frog at a vertex allowed).

Find all n such that there exists a jumping of these frogs so that after the moment, all lines connecting two frogs at different vertices do not pass through the center of the polygon.

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Solutions

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Problem 386. Observe that $7+1=2^3$ and $7^7+1= 2^3\times113\times911$. Prove that for $n = 2, 3, 4,\ldots$, in the prime factorization of $A_n = 7^{7^n}+1$, the sum of the exponents is at least $2n+3$.

Solution. **Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG**.

The case $n = 0$ is given. Suppose the result is true for n. Let $x = A_n-1$. Then $A_{n+1} = x^7+1 = (x+1)P = A_nP$, where $P = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$. Comparing P with $(x+1)^6$, we find

$$\begin{aligned} P &= (x+1)^6 - 7x(x^4+2x^3+3x^2+2x+1) \\ &= (x+1)^6 - 7x(x^2+x+1)^2. \end{aligned}$$

Now $7x=7^{2m}$,where $m=(7^n+1)/2$. Then $P = [(x+1)^3+7^m(x^2+x+1)][(x+1)^3-7^m(x^2+x+1)]$. Next, $x > 7^m \geq 7$, $x^2+x+1 > (x+1)^2$ and

$$(x+1)^3-7^m(x^2+x+1) > (x+1)^2(x+1-7^m) > 1.$$

So P is the product of at least 2 more primes. Therefore, the result is true for $n+1$.

Problem 387. Determine (with proof) all functions $f : [0,+\infty) \to [0,+\infty)$ such that for every $x \geq 0$, we have $4f(x) \geq 3x$ and $f(4f(x) - 3x) = x$.

Solution. **Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG**.

We can check $f(x) = x$ is a solution. Assume there is another solution such that $f(c) \neq c$ for some $c \geq 0$. Let $x_0 = f(c)$, $x_1 = c$ and

$$x_{n+2} = 4x_n - 3x_{n+1} \quad \text{for } n = 0,1,2,\ldots.$$

From the given conditions, we can check by math induction that $x_n = f(x_{n+1}) \geq 0$ for $n = 0,1,2,\ldots$. Since $z^2+3z-4 = (z-1)(z+4)$, we see $x_n=\alpha+(-4)^n\beta$ for some real α and β. Taking $n = 0$ and 1, we get $f(c) = \alpha+\beta$ and $c = \alpha-4\beta$. Then $\beta = (f(c)-c)/5 \neq 0$.

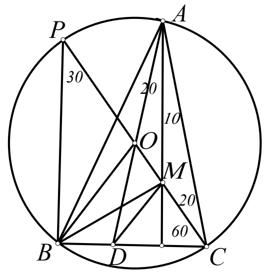
If $\beta > 0$, then $x_{2k+1}= \alpha+(-4)^{2k+1}\beta \to -\infty$ as $k \to\infty$, a contradiction. Similarly, if $\beta < 0$, then $x_{2k}= \alpha+(-4)^{2k}\beta \to -\infty$ as $k \to\infty$, yet another contradiction.

Other commended solvers: **CHAN Yin Hong** (St. Paul's Co-educational College) and **YEUNG Sai Wing** (Hong Kong Baptist University, Math, Year 1).

Problem 388. In $\triangle ABC$, $\angle BAC=30°$ and $\angle ABC=70°$. There is a point M lying inside $\triangle ABC$ such that $\angle MAB=\angle MCA =20°$. Determine $\angle MBA$ (with proof).

Solution 1. **CHOW Chi Hong** (Bishop Hall Jubilee Schol) and **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia).

Extend CM to meet the circumcircle Γ of $\triangle ABC$ at P.



Then we have $\angle BPC = \angle BAC = 30°$ and $\angle PBC=180° - \angle BPC - \angle BCM = 90°$. So line CM passes through center O of Γ.

Let lines AO and BC meet at D. Then $\angle AOB = 2\angle ACB=160°$. Now $OA = OB$ implies $\angle OAB=10°$. Then $\angle MAO = 10° = \angle MAC$ and $\angle ADC = 180° - 100°= 80° = \angle ACD$. These imply AM is the perpendicular bisector of CD. Then $MD = MC$. This along with $OB = OC$ and $\angle BOC = 60$ imply $\triangle OCB$ and $\triangle MCD$ are equilateral, hence $BOMD$ is cyclic. Then $\angle DBM = \angle DOM = 2\angle OAC = 40°$. So $\angle MBA = \angle ABC - \angle DBM = 30°$.

Solution 2. **CHAN Yin Hong** (St. Paul's Co-educational College), **Mathematics Group** (Carmel Alison Lam Foundation Secondary School), **O Kin Chit Alex** (G.T.(Ellen Yeung) College) and **Mihai STOENESCU** (Bischwiller, France).

Let $x = \angle MBA$. Applying the sine law to $\triangle ABC$, $\triangle ABM$, $\triangle AMC$ respectively, we get

$$\frac{AB}{AM}=\frac{\sin(20^\circ+x)}{\sin x},\ \frac{AB}{AC}=\frac{\sin 80^\circ}{\sin 70^\circ},\ \frac{AC}{AM}=\frac{\sin 30^\circ}{\sin 20^\circ}.$$

Multiplying the last 2 equations, we get

$$\frac{\sin(20^\circ+x)}{\sin x}=\frac{AB}{AM}=\frac{\sin 80^\circ}{\sin 70^\circ}\cdot\frac{\sin 30^\circ}{\sin 20^\circ}. \quad (\dagger)$$

Multiplying

$$\frac{\sin 80^\circ}{\sin 40^\circ}=2\cos 40^\circ=\frac{\sin 50^\circ}{\sin 30^\circ},$$

$$\frac{\sin 40^\circ}{\sin 20^\circ}=2\cos 20^\circ=\frac{\sin 70^\circ}{\sin 30^\circ},$$

we see (†) can be simplified to $\sin(20°+x)/\sin x = \sin 50°/\sin 30°$. Since the left side is equal to $\sin 20° \cot x + \cos 20°$, which is strictly decreasing (hence injective) for x between 0° to 70°, we must have $x=30°$.

Comments: One can get a similar equation as (†) directly by using the trigonometric form of Ceva's theorem.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **NG Ho Man** (La Salle

College, Form 5), **Bobby POON** (St. Paul's College), **St. Paul's College Mathematics Team**, **Aliaksei SEMCHANKAU** (Secondary School No.41, Minsk, Belarus) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia),

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

(*Source*: 2004 *Turkish MO*.)

***Solution*. William PENG**.

Below we denote the number of elements in a set S by $|S|$.

To show $k\geq 27$, take cities $A_1, A_2, \cdots, A_{28}$. For i=1,2,⋯,27, design flights between A_i and A_{i+1}. For the remaining 52 cities, partition them into pairwise disjoint subsets $Y_0, Y_1, \cdots, Y_9$ so $|Y_0|=6=|Y_9|$ and the other $|Y_k|=5$. Let $Z_0=\{A_1,A_2\}\cup Y_0$, $Z_9=\{A_{27},A_{28}\}\cup Y_9$ and for $1\leq m\leq 8$, let $Z_m=\{A_{3m},A_{3m+1},A_{3m+2}\}\cup Y_m$. Then design flights between each pair of cities in Z_m for $1\leq m\leq 8$. In this design, from A_1 to A_{28} requires 27 flights.

Assume $k > 27$. Then there would exist two cities A_1 and A_{29} the shortest connection between them would involve a sequence of 28 flights from cities A_i to A_{i+1} for i=1,2,…,28. Due to the shortest condition, each of A_1 and A_{29} has flights to 6 other cities not in $B=\{A_2,A_3,\ldots,A_{28}\}$. Each A_i in B has flights to 5 other cities not in $C = \{A_1,A_2,\ldots,A_{29}\}$.

Next for each A_i in $\{A_1,A_4,A_7,A_{10},A_{13},A_{16},A_{19},A_{22},A_{25},A_{29}\}$, let X_i be the set of cities not in C that have a flight to A_i. We have $|X_1|\geq 6$, $|X_{29}|\geq 6$ and the other $|X_i|\geq 5$. Now every pair of X_i's is disjoint, otherwise we can shorten the sequence of flights between A_1 and A_{29}. However, the union of C and all the X_i's would yield at least 29+6×2+ 5×8 =81 cities, contradiction. So k = 27.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

$$x^2y^2 = z^2(z^2 - x^2 - y^2).$$

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Ioan Viorel CODREANU** (Satulung Secondary School, Maramure, Romania) and **Aliaksei SEMCHANKAU** (Secondary School No.41, Minsk, Belarus).

Lemma. The system $a^2-b^2=c^2$ and $a^2+b^2=w^2$ has no solution in positive integers.

Proof. Assume there is a solution. Then consider a solution with minimal a^2+b^2. Due to minimality, gcd(a,b)=1. Also $2a^2 = w^2+c^2$. Considering (mod 2), we see $w+c$ and $w-c$ are even. Then $a^2=r^2+s^2$, where $r=(w+c)/2$ and $s=(w-c)/2$.

Let d=gcd(a,r,s). Then d divides a and $r+s=w$. Since $a^2+b^2=w^2$, d divides b. As gcd(a,b)=1, we get d=1. By the theorem on Pythagorean triples, there are relatively prime positive integers m,n with $m>n$ such that $\{r,s\}=\{m^2-n^2, 2mn\}$ and $a=m^2+n^2$. Now $b^2 = (w^2-c^2)/2 = 2rs = 4mn(m^2-n^2)$ implies b is an even integer, say $b=2k$. Then $k^2=mn(m+n)(m-n)$. As gcd(m,n)=1, we see $m, n, m+n, m-n$ are pairwise relatively prime integers. Hence, there exist positive integers d,e,f,g such that $m=d^2$, $n=e^2$, $m+n=f^2$ and $m-n=g^2$. Then $d^2-e^2=g^2$ and $d^2+e^2=f^2$, but

$$d^2+e^2=m+n < 4mn(m^2-n^2)=b^2 < a^2+b^2,$$

contradicting a^2+b^2 is minimal. The lemma is proved.

Now for the problem, the equation may be rearranged as $z^4-(x^2+y^2)z^2-x^2y^2=0$. If there is a solution (x,y,z) in positive integers, then considering discriminant, we see $x^4+6x^2y^2+y^4=w^2$ for some integer w. This can be written as $(x^2-y^2)^2+2(2xy)^2 = w^2$. Also, we have $(x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2$. Letting $c = |x^2-y^2|$, $b = 2xy$ and $a = x^2+y^2$. Then we have $c^2+b^2=a^2$ and $c^2+2b^2 =w^2$ (or $a^2+b^2=w^2$). This contradicts the lemma above. So there is no solution.

Other commended solvers: **Mathematics Group** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(continued from page 1)

Problem 4. Let a, b, c be positive real numbers such that $2a^2+b^2=9c^2$. Prove that

$$\frac{2c}{a}+\frac{c}{b}\geq\sqrt{3}.$$

Problem 5. Prove that if n and k are positive integers such that $1 < k < n-1$, then the binomial coefficient $\binom{n}{k}$ is divisible by at least two different primes.

Problem 6. On a square board with m rows and n columns, where $m \leq n$, some squares are colored black in such a way that no two rows are alike. Find the biggest integer k such that for every possible coloring to start with one can always color k columns entirely red in such a way that no two rows are still alike.

Casey's Theorem

(continued from page 2)

We have $\angle IAB = \angle AA''C' + \angle AC'A'' = ½(\angle B'A''C' + \angle B'C'A'') = ½\,\angle A'B'C''$ and similarly $\angle IBA = ½\,\angle B'A'C''$. So

$$\begin{aligned}\angle AIB &= 180° - \angle IA''B'' - \angle IB''A''\\ &= 180° - ½\angle C''A''B'' - ½\angle C''B''A''\\ &= 90° + ½\angle A''C''B''\\ &= 90° + ½(180° - 2\angle C)\ \text{ by (***)}\\ &= 180° - \angle ACB.\end{aligned}$$

Hence, I lies on Γ.

Let D be the foot of the perpendicular from I to $A''B''$, then $ID=r$ is the inradius of $\Delta A''B''C''$. Let E, F be the feet of the perpendiculars from B to $A''B''$, $B'A'$ respectively. Then $BE = BF = h_B$.

Let $T(X)$ be the length of tangent from X to Γ, where X is outside of Γ. Since $\angle A''B''I = ½\ \angle A''B''C'' = 90° - \angle B$ by (***), we get

$$\begin{aligned}T(B'') &= \sqrt{B''B\cdot B''I}\\ &= \sqrt{\frac{BE}{\sin(90^\circ-\angle B)}\cdot\frac{ID}{\sin(90^\circ-\angle B)}}\\ &= \frac{\sqrt{h_B r}}{\cos B}.\end{aligned}$$

Let R be the circumradius of $\Delta A''B''C''$. Then

$$\begin{aligned}T(B'')\cdot C''A'' &= \frac{\sqrt{h_B r}}{\cos B}2R\sin(180^\circ-2\angle B)\\ &= 4R\sqrt{r}\sqrt{h_B}\sin B.\end{aligned}$$

Similarly, we can get expressions for $T(A'')\cdot B''C''$ and $T(C'')\cdot A''B''$. Using the lemma, we get

$$T(A'')\cdot B''C'' + T(B'')\cdot C''A'' = T(C'')\cdot A''B''.$$

By the converse of Casey's theorem, we have the result.

Mathematical Excalibur

Volume 17, Number 1 *May-August 2012*

Olympiad Corner

Below are the problems of the 2012 International Math Olympiad.

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This circle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 2. Let $n \geq 3$ be an integer, and let $a_2,a_3,\ldots,a_n$ be positive real numbers such that $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Problem 3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***September 20, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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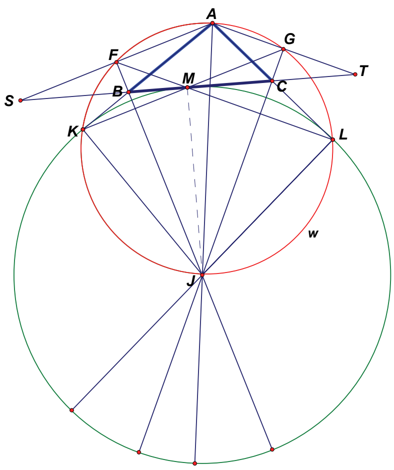
IMO 2012 (Leader Perspective)

Tat-Wing Leung

As leader, I arrived Mar del Plata, Argentina (the IMO 2012 site) four days earlier than the team. Despite cold weather, jet lag and delay of luggage, I managed to get myself involved in choosing the problems for the contest. Once the "easy" pair was selected, the jury did not have much choice but to choose problems of possibly other topics for the "medium" and the "difficult" pairs. The two papers of the contest were then set. We had to decide the various official versions and the marking scheme of the contest. After that, I just had to wait for the contestants to finish the contest and get myself involved in the coordination to decide the points obtained by our team. Here I would like to discuss the problems. (Please see Olympiad Corner for the statements of the problems.)



Problem 1. Really problem 1 is quite easy, merely a lot of angle chasings and many angles of 90° (tangents) and similar triangles, etc, and no extra lines or segments needed to be constructed. First note that $\angle AKJ = \angle ALJ = 90°$, hence A,K,L,J lie on the circle ω with diameter AJ. The idea is to show that F and G also lie on the same circle. Looking at angles around B, we see that $4\angle MBJ + 2\angle ABC = 360°$. Thus $\angle MBJ = 90° - ½\angle ABC$. Also, $\angle BMF = \angle CML = ½\angle ACB$ (as $CM=CL$). Then $\angle LFJ = \angle MBJ - \angle BMF = ½\angle BAC$. Thus $\angle LFJ = \angle LAJ$. Hence, F lies on ω. By the same token, so is G. Now AB and SB are symmetric with respect to the external bisector of $\angle ABC$, so is BK and BM. Now $SM = SB+BM = AB+BK = AK$. Similarly, $TM=AL$. So $SM=TM$.

It is relatively easy to tackle the problem using coordinate geometry. For instance, we can let the excircle be the unit circle with J=(0,0), M=(0,1), BC is aligned so that B=(b,1) and C=(c,1). Coordinates of other points are then calculated to verify the required property. But one must be really careful if he tries to use coordinate method. It was somehow decided that if a contestant cannot get a full solution using coordinate method, then he will be "seriously penalized"!

Problem 2. As it turned out, this problem caused quite a bit of trouble and many students didn't know how to tackle the problem at all. More sophisticated inequalities such as Muirhead do not work, since the expression is not "homogeneous". The Japanese leader called the problem a disaster. There were trivial questions such as "why is there no a_1?" A more subtle issue is how to isolate $a_2,a_3,\ldots,a_n$.

Clearly$(1+a_2)^2 \geq 2^2a_2$ by the AM-GM inequality. But how about $(1+a_3)^3$? Indeed the trick is to apply AM-GM inequality to get for k=2 to $n-1$,

$$(1+a_{k+1})^{k+1} = \left(\frac{1}{k}+\cdots+\frac{1}{k}+a_{k+1}\right)^{k+1}$$

$$\geq \left((k+1)\sqrt[k+1]{\frac{a_{k+1}}{k^k}}\right)^{k+1} = \frac{(k+1)^{k+1}a_{k+1}}{k^k}.$$

By multiplying the inequalities, the constants cancelled out and we get the final inequality. That the inequality is strict is trivial using the conditions of AM equals GM. The above inequality can also be used as the inductive step of proving the equivalent inequality

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^na_2a_3\cdots a_n.$$

Problem 3. Comparing with problem 6, I really found this problem harder to approach! Nevertheless there were still 8 contestants who completely solved the problem. Among them three were from the US team. That was an amazing achievement!

We can deal with this combinatorial probabilistic problem as follows. Ask repeatedly if x is 2^k. If A answers *no* k+1 times in a row, then the answer is honest and $x\neq 2^k$. Otherwise B stops asking about 2^k at the first time answer *yes*. He then asks, for each i=1,2,…,k, if the binary representation of x has a 0 in the i-th digit. Whatever the answer is, they are all inconsistent with a certain number y in the set $\{0,1,2,\cdots, 2^k-1\}$. The answer *yes* to 2^k is also inconsistent with y. Hence $x\neq y$. Otherwise the last k+1 answers are not honest and that is impossible. So we find y and it can be eliminated. Or we can eliminate corresponding numbers with nonzero digits at higher end. Notice we may need to do some re-indexing and asking more questions about the *indices* of the numbers subsequently. With these questions, we can reduce the size of the set that x lies until it lies in a set of size 2^k.

Part 2 makes use of a function so that using the function, A can devise a strategy (to lie or not to lie, but lying not more than k times consecutively) so that no extra information will be provided to B and hence B cannot eliminate anything for sure. Due to limit of space, I cannot provide all details here.

It was decided that part 1 answered correctly alone was worth 3 points and part 2 alone worth 5 points. But altogether a problem is worth at most 7 points. So 3 + 5 = 7! At the end it really did not matter. After all, not too many students did the problem right.

The problem is noted to be related to the Lovasz Local Lemma. See N. Alon et al, _The Probabilistic Methods_, Wiley, 1992. In the book it seems that there is an example that deals with similar things. One may check how the lemma and the problem are related!

Problem 4. Despite being regarded as an easy problem, this problem is not at all easy. It is much more involved than expected. Also this problem eventually caused more trouble because of the disputes about the marking scheme. First, by putting $a=b=c=0$, one gets $f(0)=0$. By putting $b=-a$ and $c=0$, one gets $f(a) = f(-a)$. More importantly, by putting $c=-(a+b)$ and solving $f(a+b) = f(-(a+b))$ as a quadratic equation of $f(a)$ and $f(b)$, one gets

$$f(a+b)=f(a)+f(b)\pm 2\sqrt{f(a)f(b)}.$$

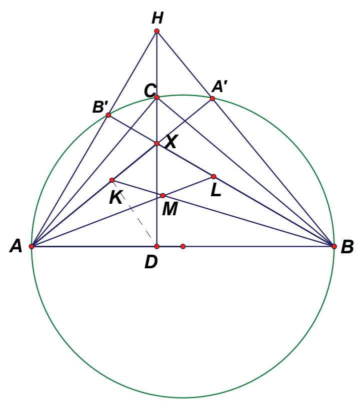
Putting $a=b$ and $c=-2a$ into the original equation, one gets $f(2a)=0$ or $f(2a)=4f(a)$. Now the problem becomes getting all possible solutions from these two relations. Using the two conditions, one checks that there are four types of solution:

(i) $f_1(x)\equiv 0$, (ii) $f_2(x) = kx^2$,

(iii) $f_3(x)=\begin{cases}0, x \text{ even}\\ k, x \text{ odd}\end{cases}$ and

(iv) $f_4(x)=\begin{cases}0, x\equiv 0\,(\text{mod}\,4)\\ k, x\equiv \pm1\,(\text{mod}\,4)\\ 4k, x\equiv 2\,(\text{mod}\,4)\end{cases}.$

The "k" in the solutions is essentially $f(1)$. Indeed if $f(1)=0$, then $f(2)=0$, one then show by induction $f(x)=0$ for all x. (Or by showing $f(x)$ is periodic of period 1.) Now if $f(1)=k$, using the condition $f(2a)=0$, one can show again by induction $f(x)$ is k for x odd and is 0 for x even. Now if $f(1)=k$ and $f(2)=4k$, then $f(4)=0$ or $16k$. In the first case we get a function with period 4 and arrive at the solution $f_4(x)$. In the second case we get $f_2(x)$. (One needs to verify the details.) By checking the values of a, b and c mod 2 or 4, or other possible forms, one can check the solutions are indeed valid. Eventually if a contestant claimed that all the solutions are easy to check, but without checking, one point would be deducted. If a contestant says nothing about the solutions satisfy the functional equation and check nothing, then two points would be deducted!



Problem 5. The following solution was obtained by one of our team members.

Extend AX to meet the circumcircle of ABC at A', likewise extend BX to meet the circle at B'. Now extend AB' and BA' to meet at H, which is exactly the orthocentre of ABX and it lies on the extension of DC.

Since $BK^2 = BC^2 = BD{\cdot}BA$, we have $\Delta ABK\sim\Delta KBD$, so $\angle BKD = \angle BAK = \angle BHD$, which implies B, D, K, H concyclic. So $\angle BKH = \angle BDH$=90°. This implies $HK^2 = BH^2 - BK^2 = BH^2 - BD{\cdot}BA = BH^2 - BA'{\cdot}BH = HA'{\cdot}HB$. Similarly $HL^2 = HB'{\cdot}HA$. But $HA'{\cdot}HB = HB'{\cdot}HA$. Hence $HK=HL$. Using similar arguments as above, we have $\angle ALH$ = 90° ($=\angle BKH$.) Along with $HK=HL$, we see $\Delta MKH\cong\Delta MLH$. Therefore, $MK=ML$.

Problem 6. Clearing denominators of

$$\frac{1}{3^{a_1}}+\frac{2}{3^{a_2}}+\ldots+\frac{n}{3^{a_n}}=1,$$

one gets $x_1+2x_2+\cdots+nx_n=3^a$, where $x_1,x_2,\ldots,x_n$ are non-negative integer powers of 3. Taking mod 2, one gets $n(n+1)/2 \equiv 1(\text{mod } 2)$. This is the case only when $n \equiv 1, 2 \pmod 4$. The hard part is to prove the converse also holds. The cases n=1 or 2 are easy. By trials, for n=5, $(a_1,\ldots,a_5)$=(2,2,2,3,3) works. The official solution gave a systematic analysis of how to obtain solutions by using identities $1/2^a=1/2^{a+1}+1/2^{a+1}$ and $w/3^a=u/3^{a+1}+v/3^{a+1}$, where $u+v=3w$. For $n=4k+1\geq 5$, one can arrive at the solution $a_1=2=a_3$, $a_2=k+1$, $a_{4k} = k+2 = a_{4k+1}$ and $a_m = [m/4]+3$ for $4\leq m<4k$. Similarly, for $n=4k+2\geq 6$, one can arrive at the solution $a_1 = 2$, $a_2 = k+1$, $a_3 = a_4 = 3$, $a_{4k+1} = k+2 = a_{4k+2}$ and $a_m = [(m-1)/4]+3$ for $4<m\leq 4k$. One can check these are indeed solutions by math induction on k. In the inductive steps of both cases, just notice a_2, a_{n-1}, a_n are increased by 1 so to balance the new a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4} terms.

This reminds me of the 1978 *USAMO* problem: an integer n is called _good_ if we can write $n=a_1+a_2+\cdots+a_k$, where $a_1,a_2,\ldots,a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}=1.$$

Given 33 to 73 are good, prove that all integer greater than 33 are good. The idea there is to show if n is good, then $2n$+8 and $2n$+9 are good by dividing both sides of the above equation by 2 and adding the terms 1/4+1/4 and 1/3+1/6 respectively.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***September 20, 2012.***

Problem 396. Determine (with proof) all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y, we have

$$f(x^2+xy+f(y))=(f(x))^2+xf(y)+y.$$

Problem 397. Suppose in some set of 133 distinct positive integers, there are at least 799 pairs of relatively prime integers. Prove that there exist a,b,c,d in the set such that $\gcd(a,b)=\gcd(b,c)=\gcd(c,d)=\gcd(d,a)=1$.

Problem 398. Let k be positive integer and m an odd integer. Show that there exists a positive integer n for which the number n^n-m is divisible by 2^k.

Problem 399. Let ABC be a triangle for which $\angle BAC=60°$. Let P be the point of intersection of the bisector of $\angle ABC$ and the side AC. Let Q be the point of intersection of the bisector of $\angle ACB$ and the side AB. Let r_1 and r_2 be the radii of the incircles of triangles ABC and APQ respectively. Determine the radius of the circumcircle of triangle APQ in terms of r_1 and r_2 with proof.

Problem 400. Determine (with proof) all the polynomials $P(x)$ with real coefficients such that for every rational number r, the equation $P(x)=r$ has a rational solution.

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Solutions

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Problem 391. Let $S(x)$ denote the sum of the digits of the positive integer x in base 10. Determine whether there exist distinct positive integers a, b, c such that $S(a+b)<5$, $S(b+c)<5$, $S(c+a)<5$, but $S(a+b+c)>50$ or not.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **LI Jianhui** (CNEC Christian College, F.5), **LO Shing Fung** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (St. Paul's Co-educational College), **YUEN Wai Kiu** (St. Francis' Canossian College) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Yes, we can try a=5,555,554,445 and b=5,554,445,555 and c=4,445,555,555. Then

$$S(a+b)=S(11{,}110{,}000{,}000)=4,$$
$$S(b+c)=S(10{,}000{,}001{,}110)=4,$$
$$S(c+a)=S(10{,}001{,}110{,}000)=4.$$

Finally,

$$S(a+b+c)=S(15{,}555{,}555{,}555)=51.$$

Other commended solvers: **Alice WONG** (Diocesan Girls' School), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 392. Integers $a_0, a_1, \cdots, a_n$ are all greater than or equal to -1 and are not all zeros. If

$$a_0+2a_1+2^2a_2+\cdots+2^na_n=0,$$

then prove that $a_0+a_1+a_2+\cdots+a_n>0$.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Harry NG Ho Man** (La Salle College, Form 5), **SHUM Tsz Hin** (City University of Hong Kong), **Alice WONG** (Diocesan Girls' School), **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

For all the conditions to hold, $n\neq 0$. We will prove by mathematical induction. For n=1, if $a_0+2a_1=0$, then the conditions on a_0 and a_1 imply a_0 is an even positive integer. So $a_0+a_1=a_0/2>0$. Suppose the case n=k is true. For the case n=k+1, the given equation implies a_0 is even, hence $a_0\geq 0$. So a_0=2b, with b a nonnegative integer. Then dividing the equation by 2 on both sides, we get that $(b+a_1)+2a_2+\cdots+2^ka_{k+1}=0$. From the cases n=k and n=1 (in cases $a_2=\cdots=a_{k+1}=0$), we get $a_0+a_1+a_2+\cdots+a_n\geq(b+a_1)+a_2+\cdots+a_n>0$, ending the induction.

Problem 393. Let p be a prime number and $p\equiv 1 \pmod 4$. Prove that there exist integers x and y such that

$$x^2-py^2=-1.$$

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (St. Paul's Co-educational College), **Corneliu MĂNESCU-AVRAM** (Dept of Math, Transportation High School, Ploiesti, Romania), **Alice WONG** (Diocesan Girls' School) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Let (m,n) be the fundamental solution (i.e. the least positive integer solution) of the Pell's equation $x^2-py^2=1$ (see <u>*Math Excal*</u>., *vol*. 6, *no*. 3, *p*.1). Then

$$m^2-n^2\equiv m^2-pn^2=1 \pmod 4.$$

Then m is odd and n is even. Since

$$\frac{m-1}{2}\cdot\frac{m+1}{2}=p\left(\frac{n}{2}\right)^2$$

and $(m-1)/2$, $(m+1)/2$ are consecutive integers (hence relatively prime), either

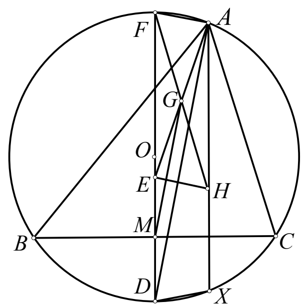
$$\frac{m-1}{2}=pu^2,\frac{m+1}{2}=v^2,n=2uv$$

or $$\frac{m-1}{2}=u^2,\frac{m+1}{2}=pv^2,n=2uv$$

for some positive integers u and v. In the former case, $v^2-pu^2=1$ with $0<v\leq v^2=(m+1)/2<m$ and $0<u=n/(2v)<n$. This contradicts the minimality of (m,n). So the latter case must hold, i.e. $u^2-pv^2=-1$.

Problem 394. Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$. The bisector of $\angle BAC$ meets the circumcircle Γ of $\triangle ABC$ at D. Let E be the mirror image of D with respect to line BC. Let F be on Γ such that DF is a diameter. Let lines AE and FH meet at G. Let M be the midpoint of side BC. Prove that $GM\perp AF$.

***Solution* 1.** **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **Mihai STOENESCU** (Bischwiller, France), **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



As AD bisects $\angle BAC$, D is the midpoint of arc BC. Hence, FD is the perpendicular bisector of BC. Thus, (1) $FE \parallel AH$. Let line AH meet Γ again at X. Since

$$\angle BCX = \angle BAX = 90^\circ - \angle ABC = \angle BCH,$$

H is the mirror image of X with respect to BC. Therefore, $\angle HED = \angle XDE = \angle AFE$. Thus, (2) $AF \parallel HE$. By (1) and (2), $AFEH$ is a parallelogram. Hence, G is the midpoint of AE. As M is also the midpoint of DE, we get $GM \parallel AD$. Since DF is the diameter of Γ, $AD \perp AF$, hence $GM \perp AF$.

***Solution* 2. Andy LOO** (St. Paul's Co-educational College).

Place the figure on the complex plane and let the circumcircle of $\triangle ABC$ be the unit circle centered at the origin. Denote the complex number representing each point by the respective lower-case letter. Without loss of generality we may assume $a = 1$ and that the points A, B and C lie on the circle in anticlockwise order. Let $b = u^2$ and $c = v^2$, where $|u| = |v| = 1$. Then $d = uv$ and hence $f = -uv$. Next, E is the mirror image of D with respect to BC means

$$\frac{e-b}{c-b} = \overline{\left(\frac{d-b}{c-b}\right)},$$

giving $e = u^2 - uv + v^2$. By the Euler line theorem, $h = a+b+c = 1+u^2+v^2$. Now G on lines AE and FH means

$$\frac{g-a}{e-a} = \frac{\overline{g}-\overline{a}}{\overline{e}-\overline{a}} \text{ and } \frac{g-f}{h-f} = \frac{\overline{g}-\overline{f}}{\overline{h}-\overline{f}}.$$

Solving these simultaneously for G, we get $g = (u^2-uv+v^2+1)/2$. Also, $m = (b+c)/2 = (u^2+v^2)/2$.

To show $GM \perp AF$, it suffices to prove that $(m-g)/(f-a)$ is an imaginary number. Indeed, $\dfrac{m-g}{f-a} = \dfrac{1}{2} \cdot \dfrac{1-uv}{1+uv}$ and

$$\overline{\left(\frac{m-g}{f-a}\right)} = \frac{1}{2} \cdot \frac{1 - \frac{1}{u} \cdot \frac{1}{v}}{1 + \frac{1}{u} \cdot \frac{1}{v}} = \frac{1}{2} \cdot \frac{uv-1}{uv+1} = -\frac{m-g}{f-a}$$

as desired.

Other commended solvers: **Simon LEE** (Carmel Alison Lam Foundation Secondary School), and **Alice WONG** (Diocesan Girls' School).

Problem 395. One frog is placed on every vertex of a $2n$-sided regular polygon, where n is an integer at least 2. At a particular moment, each frog will jump to one of the two neighboring vertices (with more than one frog at a vertex allowed).

Find all n such that there exists a jumping of these frogs so that after the moment, all lines connecting two frogs at different vertices do not pass through the center of the polygon.

***Solution.* Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **LI Jianhui** (CNEC Christian College, F.5) and **Andy LOO** (St. Paul's Co-educational College).

If $n \equiv 2 \pmod 4$, say $n=4k+2$, then label the $2n=8k+4$ vertices from 1 to $8k+4$ in clockwise direction. For $j \equiv 1$ or $2 \pmod 4$, let the frog at vertex j jump in the clockwise direction. For $j \equiv 3$ or $4 \pmod 4$, let the frog at vertex j jump in the counter-clockwise direction. After the jump, the frogs are at vertices 2, 6, …, $8k+2$ and 3, 7, …, $8k+3$. No two of these vertex numbers have a difference of the form $2 \pmod 4$. So no line through two different vertices with frogs will go through the center.

If $n \not\equiv 2 \pmod 4$, then assume there is such a jump. We may exclude the cases all frogs jump clockwise or all frogs jump counter-clockwise, which clearly do not work. Hence, in this jump, there is a frog, say at vertex i, jumps in the counter-clockwise direction, then the frog at vertex $i+m(n-2) \pmod{2n}$ must jump in the same direction as the frog at vertex i for $m=1,2,\ldots$.

If n is odd, then $\gcd(n-2,2n) = 1$. So there are integers a and b such that $a(n-2) + b(2n) = 1$. For every integer q in $[1,2n]$, letting $m = (q-i)a$, we have $i+m(n-2) \equiv q \pmod{2n}$. This means all frogs jump in the counter-clockwise direction, which does not work.

If n is divisible by 4, then $\gcd(n-2,2n) = 2$. So there are integers c and d such that $c(n-2)+d(2n)=2$. Letting $m=nc/2$, we have $i+m(n-2) \equiv i+n \pmod{2n}$. Then frogs at vertices i and $i+n$ jump in the counter-clockwise direction and the line after the jump passes through the center, contradiction.

Therefore, the answer is $n \equiv 2 \pmod 4$.

Other commended solvers: **Alice WONG** (Diocesan Girls' School).

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \ge 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \ge 1.99^k$ such that B cannot guarantee a win.

Problem 4. Find all functions $f: Z \to Z$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here Z denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK=BC$. Similarly, let L be the point on the segment BX such that $AL=AC$. Let M be the point of intersection of AL and BK. Show that $MK=ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \ldots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1.$$

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2012 IMO Team Selection Test 1 from Saudi Arabia.

Problem 1. In triangle ABC, points D and E lie on sides BC and AC respectively such that $AD \perp BC$ and $DE \perp AC$. The circumcircle of triangle ABD meets segment BE at point F (other than B). Ray AF meets segment DE at point P. Prove that $DP/PE = CD/DB$.

Problem 2. In an $n \times n$ board, the numbers 0 through n^2-1 are written so that the number in row i and column j is equal to $(i-1)+n(j-1)$ where $1 \le i,j \le n$. Suppose we select n different cells of the board, where no two cells are in the same row or column. Find the maximum possible product of the numbers in the n cells.

Problem 3. Let $\mathbb{Q}$ be the set of rational numbers. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that for all rational numbers x,y,

$$f(f(x)+xf(y)) = x + f(x)y.$$

Problem 4. Find all pairs of prime numbers p, q such that $p^2-p-1 = q^3$.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 20, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2012 (Member Perspective)

Andy Loo

This year's International Mathematical Olympiad (IMO) has been of considerable significance to Hong Kong. At the 1997 IMO held in Mar del Plata, Argentina, shortly after our official transfer of sovereignty, the Hong Kong delegation accomplished the special mission of elucidating Article 149 of its *Basic Law* in light of Annex I of the *Sino-British Joint Declaration*, thereby consolidating the legitimacy of its participation in the IMO. This July, following the 15th anniversary of the establishment of the Special Administrative Region, this annual event returns to Argentina, in exactly the same city as last time's. In addition to battling in the examination hall, the Hong Kong team was endowed with the invigorating task of bringing the IMO to Hong Kong again in 2016.

Joined by 542 young brains from 99 countries, the Hong Kong team comprised the following personnel: Dr. Leung Tat Wing (leader), Mr. Leung Chit Wan (deputy leader) and the team members were Kevin Lau Chun Ting (St. Paul's Co-educational College), Andy Loo (St. Paul's Co-educational College), Albert Li Yau Wing (Ying Wah College), Jimmy Chow Chi Hong (Bishop Hall Jubilee School), Kung Man Kit (SKH Lam Woo Memorial Secondary School) and Alice Wong Sze Nga (Diocesan Girls' School).

This contest bestows certain personal touch upon me, for it not only marks my unprecedented landing on the continent of South America, but is also my first and, in all probability, my last IMO, an ultimate platform for me to display my years of Mathematical Olympiad endeavor in my high school career. Having represented Hong Kong at both the International Physics Olympiad (IPhO) and the IMO is a great responsibility which I feel extremely grateful to have had the unique chance to shoulder.

<u>July 7 and 8</u> Our flights from Hong Kong to Frankfurt and from Frankfurt to Buenos Aires, each over 12 hours long, were predominantly occupied by sleep and math exercises, considering the disappointing fact that our planes turned out to be two of the very few Boeing-74748 models of Lufthansa that lack in-flight entertainment systems. Our amazement at a German flight attendant, who spoke more than fluent Mandarin Chinese, as well as a cozy conversation with a Slovakian neighbor, highlighted the otherwise uneventful journey.

We arrived at the Argentinean capital city early in the morning of July 8 (in winter!), and, after being transported to the domestic airport, employed a time-consuming conglomeration of *Google Translate* effort and sign language to manage to purchase a couple of SIM cards at a tiny store, where the shopkeeper knew literally no English. A Maradona-like bus driver kindly offering us a free ride, we embarked on a tour around the city and enjoyed a beef-dominated meal before returning to the airport in the late afternoon to catch our flight to Mar del Plata, on which I, being absolutely exhausted, slept from the first to the last minute.

<u>July 9</u> The major event of this day was the Opening Ceremony. It was held in the Radio City. I met British team member Josh Lam and congratulated him on his mother's recent promotion to Chief Secretary of Hong Kong. If I were to describe the entire ceremony in one word it would definitely be "Spanish". Almost all the speeches were delivered in Spanish, albeit accompanied by English interpretation. To most, the more exciting parts of the ceremony included the IMO anthem, the parade of nations and the distant waves from the leaders, who were forbidden to communicate with us before the contest as they took part in problem selection.

July 10 On this first day of the contest, we had 3 problems to solve in 4.5 hours. Because questions could only be raised in the first 30 minutes, I had to understand all the problems quickly.

Problem 1 Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

I decided to use my favorite method – complex numbers. Indeed, denote the complex number representing each point by the corresponding small letter. Setting j=0 and m=1, I found s=2k/(k+l) and t=2l/(k+l) after a straightforward computation, and the result followed.

Problem 2 Let $n \ge 3$ be an integer, and let $a_2,a_3,\ldots,a_n$ be positive real numbers such that $a_2a_3\cdots a_n=1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Inequalities were once among the hottest topics on the IMO but totally disappeared in the last three years due to the rising popularity of brute force techniques, e.g. Muirhead's inequality and Schur's inequality. But my firm belief in the revival of inequalities has never been shaken, and instead was only strengthened by Problem 5 of APMO 2012. Consequently I had done appreciable preparation in this area before the Olympiad.

In IMO history, this problem was quite unique. For one, it is an n-variable inequality. For the other, it has no equality case. Both features are unparalleled according to my memory.

I spent about an hour attempting to solve the problem using induction or analysis, with no avail. In despair, I took logarithm and applied Jensen's inequality by appealing to concavity of the log function. Miraculously, it gave precisely the inequality in the problem! After checking that equality case cannot satisfy the condition $a_2a_3\cdots a_n$ =1, I was basically done.

Then on a second thought, I realized that I could actually convert my proof into a logarithm-free one that involves the AM-GM inequality only. So I rewrote my solution in this new form and marked the original as an alternative solution. It turned out that Alice was also able to solve this problem with the AM-GM inequality.

Problem 3 The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any k+1 consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \ge 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \ge 1.99^k$ such that B cannot guarantee a win.

This problem was not only long, but also terribly difficult. In the end, only 8 contestants managed to solve it. Despite my effort, the only thing I was able to do was proving the k = 1 case in Part 1, with the hope of getting slim partial credits.

Finally Day 1 of the contest was over. Our team aced Problem 1. As for Problem 2, Alice and I should be able to get 7's while Albert's partial analytic solution would be subject to vigorous debate. Kit also finished the k = 1 case in Part 1 of Problem 3. Overall I was satisfied with my performance on Day 1.

July 11 The six IMO problems are usually partitioned into the four categories (algebra, combinatorics, geometry and number theory) in the fashion of {1,5}, {2,4}, {3} and {6} (up to permutation). Judging from this pattern I would face an easy algebraic problem, an intermediate geometric problem and a hard number theoretic problem on Day 2. I figured that I would plausibly get a Gold medal for solving two of them, a Silver medal for one and a Bronze medal for none. My strategy was to guarantee Problem 4 and then aim to get Problem 5 by hook or by crook.

To my astonishment, Problem 4 was much more involved than I had expected. On the other hand I felt I could do Problem 5 with analytic tools:

Problem 5 Let ABC be a triangle with $\angle BCA$ = 90°, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that BK=BC. Let L be the point on the segment BX such that AL=AC. Let M be the point of intersection of AL and BK. Show that MK=ML.

I proceeded to do coordinate geometry, only to find out I was doomed after almost one hour. The reason was as follows. The expressions were quadratic in nature (as lengths took part in the formulation of the problem), leading to the prevalence of square roots. (As a side note, this also deterred me from using complex numbers, where one may have difficulty in selecting the correct roots of the quadratic equations.)

As the old Chinese saying goes, one should "drop his cleaver and become a Buddha (放下屠刀，立地成佛)". I decided to abandon Problem 5 for a moment and to reconsider Problem 4:

Problem 4 Find all functions $f: Z \to Z$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$\begin{aligned} &f(a)^2+f(b)^2+f(c)^2 \\ &=2f(a)f(b)+2f(b)f(c)+2f(c)f(a). \end{aligned}$$

(Here Z denotes the set of integers.)

This was a problem with unusual answers. It took me quite a while to write up a tidy solution and to ensure that no point could sneak away from my hands. Thus it was 2.5 hours into Day 2. I still had Problems 5 and 6 left.

Problem 6 Find all positive integers n for which there exist non-negative integers $a_1,a_2,\ldots,a_n$ such that

$$\frac{1}{2^{a_1}}+\frac{1}{2^{a_2}}+\ldots+\frac{1}{2^{a_n}}=\frac{1}{3^{a_1}}+\frac{2}{3^{a_2}}+\ldots+\frac{n}{3^{a_n}}=1.$$

I quickly determined that Problem 6 was hopeless. Turning to Problem 5 again, I spent all the remaining time expanding everything. I was finally able to convince myself that my proof was complete.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***November 20, 2012.***

Problem 401. Suppose all faces of a convex polyhedron are parallelograms. Can it have exactly 2012 faces? Please provide an explanation to your answer.

Problem 402. Let S be a 30 element subset of $\{1,2,\ldots,2012\}$ such that every pair of elements in S are relatively prime. Prove that at least half of the elements of S are prime numbers.

Problem 403. On the coordinate plane, 1000 points are randomly chosen. Prove that there exists a way of coloring each of the points either red or blue (but not both) so that on every line parallel to the x-axis or y-axis, the number of red points minus the number of blue points is equal to −1, 0 or 1.

Problem 404. Let I be the incenter of acute ΔABC. Let Γ be a circle with center I that lies inside ΔABC. D, E, F are the intersection points of circle Γ with the perpendicular rays from I to sides BC, CA, AB respectively. Prove that lines AD, BE, CF are concurrent.

Problem 405. Determine all functions f,g: $(0,+\infty) \to (0,+\infty)$ such that for all positive number x, we have

$$f(g(x))=\frac{x}{xf(x)-2} \ \text{ and } \ g(f(x))=\frac{x}{xg(x)-2}.$$

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Solutions

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Problem 396. Determine (with proof) all functions f: $\mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y, we have

$$f(x^2+xy+f(y))=(f(x))^2+xf(y)+y.$$

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **CHEUNG Wai Lam** (Queen Elizabeth School), **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **Kevin LAU** (St. Paul's Co-educational College, S.4), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Mohammad Reza SATOURI** (Persian Gulf University, Bushehr, Iran) and **Maksim STOKIĆ** (Mathematical Grammar School, Belgrade, Serbia).

Call the required equation (*). For x=0, we get $f(f(y))=y+f(0)^2$ for all y. Call this (**). The right side may be any real number, hence f is surjective. By (**), $y = f(f(y)) - f(0)^2$. If $f(y) = f(y')$, then the last equation implies $y=y'$, i.e. f is injective.

Putting $x=-y$ in (*), we get $f(f(y)) = (f(-y))^2-yf(y)+y$ for all y. Call this (***).

Now f surjective implies there exists z such that $f(z)=0$. Let $x=y=z$, then (*) yields $f(2z^2)=z$. Putting $(x,y)=(0,2z^2)$ in (*), we get $0=2z^2+f(0)^2$. Then z=0 and $f(0)$=0. So (**) reduces to $f(f(y))=y$ for all y. Putting y = 0 in (*), since $f(0)$ = 0, we get $f(x^2) = (f(x))^2$. The last two sentences reduce (***) to $y = (f(y))^2-yf(y)+y$. This simplifies to $f(y)$ = 0 or $f(y) = y$ for every y. Since f is injective and $f(0)$ = 0, we get $f(y) = y$ for all y. Conversely, a quick check shows $f(y) = y$ for all y satisfies (*).

Other commended solvers: **Tobi MOEKTIJONO** (National University of Singapore).

Problem 397. Suppose in some set of 133 distinct positive integers, there are at least 799 pairs of relatively prime integers. Prove that there exist a,b,c,d in the set such that $\gcd(a,b) = \gcd(b,c) = \gcd(c,d) = \gcd(d,a) = 1$.

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **Kevin LAU** (St. Paul's Co-educational College, S.4), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (Princeton University), **Tobi MOEKTIJONO** (National University of Singapore) and **Maksim STOKIĆ** (Mathematical Grammar School, Belgrade, Serbia).

Let $S=\{n_1, n_2,\ldots,n_{133}\}$ be the set of these 133 positive integers. From i=1 to 133, let X_i be the set of all n_k in S such that $k\neq i$ and $\gcd(n_i,n_k)$=1. Denote by $|X|$ the number of elements in set X. For $k\neq i$, $\gcd(n_i,n_k)$=1 implies $n_i\in X_k$ and $n_k\in X_i$. Then $N = |X_1| + |X_2| + \cdots + |X_{133}| \geq 2\times799 = 1598$.

Define $f(x) = x(x-1)/2$. In a set X with j elements, there are exactly $j(j-1)/2 = f(|X|)$ pairs of distinct elements. Since $f(x)$ is concave on $\mathbb{R}$, by Jensen's inequality,

$$\sum_{i=1}^{133} f(|X_i|)\geq 133f\left(\frac{N}{133}\right)\geq 133f\left(\frac{1598}{133}\right)$$

$$> 133f(12)=f(133)=f(|S|).$$

Since every pair of distinct element in X_i is also a pair of distinct element in S, the inequality above implies in counting pairs of distinct elements in the X_i's, there are repetitions, i.e. there are X_i, X_k with $i\neq k$ sharing a common pair of distinct elements a,c. Let $b=n_i$ and $d=n_k$. Then a,b,c,d satisfy $\gcd(a,b) = \gcd(b,c) = \gcd(c,d) = \gcd(d,a) = 1$.

Problem 398. Let k be positive integer and m an odd integer. Show that there exists a positive integer n for which the number n^n-m is divisible by 2^k.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (Princeton University), **Tobi MOEKTIJONO** (National University of Singapore) and **Maksim STOKIĆ** (Mathematical Grammar School, Belgrade, Serbia).

For k=1, let n=1. Suppose it is true for case k (i.e. there exists n such that $2^k \mid n^n-m$). Now m odd implies n odd. For case k+1, if $2^{k+1} \mid n^n-m$, then the same n works for k+1. Otherwise, $n^n-m=2^kl$ for some odd integer l. Let $v=2^k$. By binomial theorem,

$$\begin{aligned}(n+v)^{n+v} &= n^{n+v}+(n+v)n^{n+v-1}v+v^2x\\ &= n^{n+v}+vn^{n+v}+v^2y\end{aligned}$$

for some integers x,y. By Euler's theorem, since n is odd and $\varphi(2^{k+1})=2^k$,

$$n^v=n^{2^k}\equiv 1 \pmod{2^{k+1}}.$$

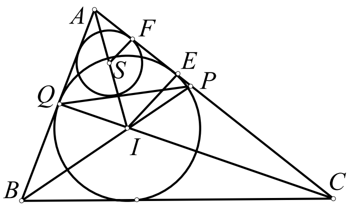
Since $l+n^n$ is even, we have

$$\begin{aligned}(n+v)^{n+v} &= n^{n+v}+vn^{n+v}+v^2y\\ &\equiv n^n+2^kn^n=m+2^k(l+n^n)\\ &\equiv m \pmod{2^{k+1}}.\end{aligned}$$

So $n+v$ works for k+1.

Problem 399. Let ABC be a triangle for which $\angle BAC$=60°. Let P be the point of intersection of the bisector of $\angle ABC$ and the side AC. Let Q be the point of intersection of the bisector of $\angle ACB$ and the side AB. Let r_1 and r_2 be the radii of the incircles of triangles ABC and APQ respectively. Find the radius of the circumcircle of triangle APQ in terms of r_1 and r_2 with proof.

Solution. **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **Kevin LAU** (St. Paul's Co-educational College, S.4), **Andy LOO** (Princeton University), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **Tobi MOEKTIJONO** (National University of Singapore) and **Maksim STOKIC** (Mathematical Grammar School, Belgrade, Serbia).



Let I and S be the incenters of $\triangle ABC$ and $\triangle APQ$ respectively. (Note A,S,I are on the bisector of $\angle BAC$.) Now $\angle PIQ=\angle CIB=180^\circ-(\angle CBI+\angle BCI)$ $=180^\circ-\frac{1}{2}(\angle CBA+\angle BCA)=120^\circ$ using $\angle BAC=60^\circ$. So $APIQ$ is cyclic.

Applying sine law to ΔAPI, we get $IP/(\sin\angle IAP)=2R$. So $R=IP$. By a well-known property of incenter, we have $IP=IS$ (see *vol.11*, *no.2*, *p.1* of Math Excal.). Let the incircles of ΔABC and ΔAPQ touch AC at E and F respectively. Then $R=IP=IS=AI-AS$ $=IE/\sin 30^\circ-SF/\sin 30^\circ=2r_1-2r_2$.

Other commended solvers: **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramure, Romania), **Simon LEE** (Carmel Alison Lam Foundation Sec. School) and **Mihai STOENESCU** (Bischwiller, France).

Problem 400. Determine (with proof) all the polynomials $P(x)$ with real coefficients such that for every rational number r, the equation $P(x)=r$ has a rational solution.

Solution. **Tobi MOEKTIJONO** (National University of Singapore), **Maksim STOKIC** (Mathematical Grammar School, Belgrade, Serbia) and **TAM Ka Yu** (MIT).

We will show $P(x)$ satisfies the desired condition if and only if $P(x)=ax+b$, where $a,b\in\mathbb{Q}$ and $a\neq 0$. For the *if*-part, $P(x)=r\in\mathbb{Q}$ implies $x=(r-b)/a\in\mathbb{Q}$.

Conversely, let $P(x)$ satisfy the desired condition and let $n=\deg P$. For each $r=0,1,\ldots,n$, let $P(x_r)=r$ for some $x_r\in\mathbb{Q}$. By the Lagrange interpolation formula,

$$P(x)=\sum_{r=0}^{n}\left(r\prod_{0\le r\le n,r\ne i}\frac{x-x_i}{x_r-x_i}\right).$$

Expanding the right side, we see $P(x)$ has rational coefficients.

Letting M be the product of the denominators, we see $Q(x)=MP(x)$ has integer coefficients. Let k be the leading coefficient of $Q(x)$ and c be the constant term of $P(x)$. Let p_1, p_2, p_3, … be the sequence of prime numbers. Let $P(x)=c+p_i/M$ has solution $t_i\in\mathbb{Q}$. Then $Q(x)-(cM+p_i)$ has k as the leading coefficient and $-p_i$ as constant term. Now $Q(t_i)=0$, which implies $t_i=1/d_i$ or p_i/d_i for some (not necessarily positive) divisor of k. Since $P(t_i)$'s are distinct, so the t_i's are distinct. Hence, $t_i=1/d_i$ for at most as many time as the number of divisors of k. So there must exist a divisor d of k such that there are infinitely many times $t_i=p_i/d$. This imply that $P(x)-(c+dx/M)=0$ has infinitely many solutions. So the left side is the zero polynomial. Then $P(x)=ax+b$ with $a=d/M\neq 0$ and $b=c$ rational.

Other commended solvers: **Simon LEE** (Carmel Alison Lam Foundation Secondary School).

IMO 2012 (Member Perspective)

(continued from page 2)

The arrival of Dr. Leung stirred up much happiness after the contest. We reported on how we did. Albert and Jimmy shone on Day 2, solving Problems 4 and 5. Kit was also comfortable with Problem 4 while Kevin had some technical troubles in one particular case. Nobody achieved anything substantial on Problem 6.

We celebrated that evening at a Chinese restaurant. It was especially memorable that our deputy leader raised a couplet (對聯), which he regarded as an open puzzle for millenniums (千古絕對):

望江樓，望江流
望江樓上望江流
江樓千古，江流千古

It took me nearly an hour to come up with a so-so solution:

觀雨亭，觀雨停
觀雨亭下觀雨停
雨亭四方，雨停四方

<u>**July 12**</u> It was the contestants' turn to have fun and the leaders' turn to work hard. At night, Dr. Leung briefed us on the progress of the first day of coordination. In addition to our previous expectations, Albert pocketed one point for proving the necessary condition on Problem 6. Regretfully, Kit lost one point on Problem 4 for not having verified the feasibility of the functions obtained. Dr. Leung had refused to sign Alice's and Kevin's scores on Problem 4 in order to bargain later.

<u>**July 13**</u> The marking scheme stipulated that any solution of Problem 5 with coordinate geometry would score a 0 if not a 7. Despite our leaders' relentless effort, the coordinators were able to detect a fatal error of mine. So my Problem 5 was destined to be a 0.

On another note, Dr. Leung succeeded in getting 1 point for Alice on Problem 4, which in his words was "an achievement". Kevin's Problem 4 was finalized with a score of 4.

<u>**July 14**</u> We got up early in the morning to enjoy the sunrise scene at the seaside. Kevin had a pitiful blunder. His shoes and trousers were wetted by a sudden strike of waves. That morning the last coordination on Problem 2 was done. Albert was awarded 3 marks for his analytic struggle. The uncertainties of our results then shifted from our actual scores to the medal cutting scores.

We went shopping for souvenirs in the afternoon and as soon as we got back to the hotel, I learned from the Chinese leaders that the cutting scores for Gold, Silver and Bronze Medals were 28, 21 and 14 respectively, all being multiples of 7. I breathed a sigh of relief as my Silver Medal was ultimately secure.

<u>**July 15**</u> In the afternoon we had the Closing Ceremony followed by a chain of photo-taking. We won three Silver Medals (Albert, Jimmy and me), one Bronze Medal (Alice) and two Honorable Mentions (Kit and Kevin).

<u>**July 16, 17 and 18**</u> The six-hour bus journey from Mar del Plata to Buenos Aires passed rapidly in our dreams. Then after a long flight, we were finally home in one piece and me with several bonus pimples.

In conclusion I shall stress one point – succinctly but with all the strength that I command – one can never pay sufficient tribute to our IMO trainers, who have so selflessly devoted countless hours of their own time to Mathematical Olympiad over the years. I can find no words to thank them the way they truly deserve.

"Ask not what your country can do for you; ask what you can do for your country." With this John F. Kennedy exclamation I urge you all to support the 2016 Hong Kong IMO by whatever means you can, so that together we can make it an all-time success.

Mathematical Excalibur

Volume 17, Number 3 *November-December 2012*

Olympiad Corner

Below are the problems of the 15th Hong Kong China Math Olympiad.

Problem 1. For any positive integer n, let a_1, a_2, …, a_m be all the positive divisors of n, where $m\geq 1$. If there exist m integers b_1, b_2, …, b_m such that

$$n=\sum_{i=1}^{m}(-1)^{b_i}a_i,$$

then we say that n is a *good number*. Prove that there exists a good number with exactly 2013 distinct prime factors.

Problem 2. Some of the lattice points (x,y), with $1\leq x\leq 101$ and $1\leq y\leq 101$ are marked so that no 4 marked points form the vertices of an isosceles trapezoid with bases parallel to the x-axis or the y-axis (a rectangle is counted as an isosceles trapezoid). Determine the maximum number of marked points. (A lattice point is a point with integral coordinates.)

Problem 3. Prove that for every positive integer n and every group of real numbers a_1, a_2, …, $a_n>0$,

$$\sum_{k=1}^{n}\frac{k}{a_1^{-1}+a_2^{-1}+\ldots+a_k^{-1}}\leq 2\sum_{k=1}^{n}a_k.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 3, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Primes in Arithmetic Progressions

Kin Y. Li

To see there are infinitely many prime numbers, we assume only finitely many of them exist, say $p_1,p_2,\ldots,p_m$. Consider $q=p_1p_2\cdots p_m+1$. Let p be a prime in the prime factorization of q. Then p is one of the p_i's. So p divides q and $q-1$. Then p divides $q-(q-1)=1$, contradiction.

Other than 2, the rest of the prime numbers are in the arithmetic progression $2n+1$, where n denotes a positive integer. It is natural to ask how many prime numbers are in the other arithmetic progressions $an+b$, where a and b are given integers with $a>0$. Certainly, if $\gcd(a,b)>1$, then no primes will be in the sequence $an+b$.

In case $(a,b)=(4,-1)$ we can see the answer is infinitely many by modifying the proof above. Assume $p_1,p_2,\ldots,p_m$ are all the primes of the form $4n-1$. Then let $q=4p_1p_2\cdots p_m-1$. Now $q\equiv -1 \pmod 4$. Assume q is a product of primes in the sequence $4n+1$. Then $q\equiv 1 \pmod 4$, contradiction. So q must have at least one prime divisor p in the sequence $4n-1$. Then p is one of the p_i's. So p divides q and $q+1$. Then p divides $(q+1)-q=1$, contradiction.

In case $(a,b)=(p,1)$, where p is a prime, we will need facts from number theory.

Fact 1 (Bezout's Theorem). For all positive integers a and b, there exist integers r and s such that $ar+bs=\gcd(a,b)$.

Fact 2 (Euler's Theorem). For positive integer n, let $\varphi(n)$ be the number of integers among $1,2,\ldots,n$ that is relatively prime to n. If $\gcd(a,n)=1$, then $a^{\varphi(n)}\equiv 1 \pmod n$. In case n is a prime, we have $\varphi(n)=n-1$ and $a^{n-1}\equiv 1 \pmod n$. This case is *Fermat's Little Theorem*.

Example 1 (*2004 Korean Mathematical Olympiad*). Let p be a prime and $f_p(x)=x^{p-1}+x^{p-2}+\cdots+x+1$.

(1) For each integer m divisible by p, is there an integer q such that q divides $f_p(m)$ and $\gcd(q,m(m-1))=1$?

(2) Prove that there are infinitely many integers n such that $pn+1$ is prime.

Solution. (1) Yes. Let q be a prime divisor of $f_p(m)$. As $f_p(m)\equiv 1 \pmod m$, we see q does not divide m. Hence $\gcd(m,q)=1$. Assume $m\equiv 1 \pmod q$. Then $0\equiv f_p(m)\equiv p \pmod q$, which implies $p=q$. Since p divides m, we get $1\equiv f_p(m)\equiv p \pmod p$, contradiction. Hence q does not divide $m-1$. Then $\gcd(q,m(m-1))=1$.

(2) Assume $p_1,p_2,\ldots,p_k$ are all the primes of the form $pn+1$. Let $m=p_1p_2\cdots p_k\,p$ and q be a prime divisor of $f_p(m)$. By (1), $m\not\equiv 0$ or $1 \pmod q$, which implies $\gcd(m,q)=1$. By Fermat's little theorem, $m^{q-1}\equiv 1 \pmod q$. Now $m^p-1=(m-1)\,f_p(m)$ implies $m^p\equiv 1 \pmod q$.

Assume $\gcd(q-1,p)=1$. By Bezout's theorem, there are integers r and s such that $(q-1)r+ps=1$. Then $m=m^{(q-1)r}m^{ps}\equiv 1 \pmod q$, contradicting the last underlined expression. Then $\gcd(q-1,p)=p$, i.e. q is of the form $pn+1$. As q divides $f_p(m)$ and $f_p(m)\equiv 1 \pmod{p_i}$, we see $q\neq p_1,p_2,\ldots,p_k$.

In the general case $\gcd(a,b)>1$, we have

Dirichlet's Theorem. If a and b are given integers with $a>0$ and $\gcd(a,b)>1$, then there are infinitely many primes in the arithmetic progression $an+b$.

All known proof of this theorem is beyond the scope of secondary school curriculum. Below we will look at some examples. First we need more facts.

Fact 3 (Chinese Remainder Theorem). If k_1, k_2, …, k_n are pairwise relatively prime positive integers and c_1, c_2, …, c_n are integers, then there exist a unique integer x in the interval $[1, k_1k_2\cdots k_n]$ such that $x\equiv c_i \pmod{k_i}$ for $i=1,2,\ldots,n$.

Fact 4 (Wilson's Theorem). If p is a prime, then $(p-1)!\equiv -1 \pmod p$.

At the end of the article, we will give explanations for facts 1 to 4.

Example 2 (*1996 St Petersburg Math Olympiad*) Prove that there are no positive integers a and b such that for each pair p, q of distinct primes greater than 1000, the number $ap+bq$ is also prime.

Solution. Assume such a and b exist. Let r be a prime number with $\gcd(r,a)=\gcd(r,b)=1$. By Dirichlet's theorem, there exist positive integers x and y such that $p=rx+b$ and $q=ry-a$ are prime numbers greater than 1000. Then $ap+bq=(ax+by)r$ is not prime, contradiction.

Example 3 (*1997 British Mathematical Olympiad*) Let $S = \{1/r : r = 1,2,3,\ldots\}$. For all integer $k > 1$, prove that there is a k-term arithmetic progression in S such that no addition term in S can be added to it to form a $(k+1)$-term arithmetic progression.

Solution. By Dirichlet's theorem, there exists a positive integer n such that $kn+1$ is prime. Let $a_1=1/(kn)!$ and $d=n/(kn)!$. For $i=2,\ldots,k$, $a_i=a_1+(i-1)d =(1+(i-1)n)/(kn)!$ are in S. However, the term $a_{k+1}=a_1+kd=(kn+1)/(kn)!$ is not in S since $kn+1$ is a prime. So a_1, a_2, …, a_k is such an example.

Example 4 Prove that for every positive integer s, a, b with $\gcd(a,b)=1$, there are infinitely many integers n such that $an+b$ is a product of s pairwise distinct prime numbers.

Solution. The case $s=1$ is Dirichlet's theorem. Suppose the case s is true. Then there exists an integer N such that $aN+b= q_1q_2\cdots q_s$, where q_1, q_2, …, q_s. are pairwise distinct primes. Next, by Dirichlet's theorem, there exist infinitely many positive integers n such that $an+1$ is a prime greater than all of q_1, q_2, …, q_s. Let $t_n= q_1q_2\cdots q_sn+N$. Then $at_n+b = aq_1q_2\cdots q_sn+aN+b = q_1q_2\cdots q_s(an+1)$ is a product of $s+1$ pairwise distinct prime numbers. This completes the induction.

Example 5 (*2011 Mongolian Math Olympiad Team Selection Test*) Let m be a positive odd integer. Prove that there exist infinitely many positive integer n such that $(2^n-1)/(mn+1)$ is an integer.

Solution. By Dirichlet's theorem, there exist infinitely many primes $p > m$ and $p= \varphi(m)k+1$ for some positive integer k. By Euler's theorem, $2^{\varphi(m)} \equiv 1 \pmod m$. Then

$$2^p = 2^{\varphi(m)k+1} \equiv 2 \pmod m.$$

This leads to $n=(2^p-2)/m$ is an integer. By Fermat's little theorem, p divides $2^p - 2$. Since $p>m$, we see p divides n. Then $mn+1=2^p -1$ divides $2^n -1$. Therefore, $(2^n-1)/(mn+1)$ is an integer.

Example 6 (*American Math Monthly 4772*) Let p_k be the k-th prime number. For every integer N, prove that there exists a positive integer k such that both p_{k-1} and p_{k+1} are not in the interval $[p_k-N, p_k+N]$.

Solution. Let q be a prime number greater than $N+2$. Observe that $a=q!$ and $b=(q-1)!-1$ are relatively prime because the prime divisors of $q!$ are the primes less than or equal to q, however $(q-1)!-1$ is not divisible by any prime number less than q and $(q-1)!-1 \equiv -2 \pmod q$ by Wilson's theorem.

By Dirichlet's theorem, there is a prime $p_k \equiv (q-1)! -1 \pmod{q!}$. Then $p_k+1\equiv 0 \pmod{(q-1)!}$. Also, by Wilson's theorem, $p_k+2 \equiv (q-1)!+1\equiv 0 \pmod q$. These showed p_k+1 and p_k+2 are not primes. For $j=2,\ldots, q-1$, we have

$$p_k+1\pm j \equiv p_k+1 \equiv (q-1)! \equiv 0 \pmod j.$$

So integers in $[p_k-q+2, p_k+q]$ except p_k are not primes. Since $q>N+2$, both p_{k-1} and p_{k+1} cannot be in the $[p_k-N, p_k+N]$.

Example 7 (*American Math Monthly E1632*) Prove that if $f(x)$ is a polynomial with rational coefficients such that $f(p)$ is a prime number for every prime number p, then either $f(x)=x$ for all x or $f(x)$ is the same prime constant for all x.

Solution. Assume the conclusion is false. Let k be the least common multiple of the denominators of the coefficients of $f(x)$ and let $g(x)=kf(x)$. Then $g(x)$ has integer coefficients. Now there must be a prime p such that p and $g(p)$ are relatively prime (otherwise, for the infinitely many primes p that are relatively prime to k, we have $\gcd(p,g(p))=p$, so p divides $g(p)=kf(p)$, hence both primes $f(p)$ and p are equal, which forces $f(x)=x$).

By Dirichlet's theorem, there are infinitely many integers n_i such that $m_i=g(p)n_i+p$ is prime. Now $g(m_i) \equiv g(p) \equiv 0 \pmod{g(p)}$ for all i. Then $kf(p)$ divides $kf(m_i)$. Hence $f(p)$ divides $f(m_i)$. Since $f(p)$ and $f(m_i)$ are primes, we get $f(m_i)=f(p)$ for infinitely many i. This leads to $f(x)$ being the constant polynomial $f(p)$, contradiction.

Example 8 (*American Math Monthly 4524*) Prove that for every pair of positive integers n and N, there are consecutive positive integers k, $k+1$, …, $k+N$ such that $\varphi(k)$, $\varphi(k+1)$, …, $\varphi(k+N)$ are all divisible by n, where $\varphi(n)$ is as defined in Euler's theorem.

Solution. We need the *fact* that if integer $w=ab$, where $a=p^m$, p is prime and $\gcd(b,p)=1$, then $\varphi(w)$ is divisible by $p -1$.

Granting the *fact*, by Dirichlet's theorem, there are distinct primes p_0, p_1, …, $p_N \equiv 1 \pmod n$. By the Chinese remainder theorem, there is an integer k such that $k \equiv 0 \pmod{p_0}$, $k \equiv -1 \pmod{p_1}$,…, $k\equiv -N \pmod{p_N}$. So for $j=0,1$, …, N, the number $k+j$ is divisible by the prime p_j. Then $\varphi(k+j)$ is divisible by $p_j -1$ by the fact, which is a multiple of n.

For the *fact*, note $\gcd(a,b)=1$. Then $\varphi(ab)= \varphi(a)\varphi(b)$. (This follows from the Chinese remainder theorem, since for every k in $[1,ab]$ with $\gcd(k,ab)=1$, let r and s be the remainders when k is divided by a and b respectively. Now $\gcd(k,ab)=1$ if and only if $\gcd(r,a) = 1 = \gcd(s,b)$. The Chinese remainder theorem asserts that $x \equiv k \pmod{ab}$ if and only if $x \equiv r \pmod a$ and $x \equiv s \pmod b$. Thus $x\leftrightarrow(r,s)$ is bijective.) For x in $[1,p^m]$, $\gcd(x,p^m)>1$ if and only if x is a multiple of p. So $\varphi(a)=\varphi(p^m) = p^m - p^{m-1} = p^{m-1}(p-1)$. Then $\varphi(w)= \varphi(a)\varphi(b)$ is divisible by $p -1$.

Example 9 Prove that there are infinitely many positive integers n such that the equation $x^n+y^n=z^n$ has no solution (x,y,z) in integers with $xyz\neq0$ and $\gcd(n,xyz)=1$. (*These n's may even be chosen to be pairwise relatively prime.*)

(*Remark* Barry Powell published this result in the *American Mathematical Monthly* on November 1978.)

Solution. The case $n=4$ is well-known. Next, suppose n_1, n_2, …, n_k are such n's. By Dirichlet's theorem, there is a prime p such that $p \equiv -1 \pmod{4n_1n_2\cdots n_k}$. We define a new $n = p(p-1)/2$. Note $n \equiv 1 \pmod 4$. Since $(p-1)/2$, $(p+1)/2$ are consecutive integers and $p > (p+1)/2$, so $\gcd(p(p-1)/2, (p+1)/2) = 1$. Hence, $\gcd(n, 4n_1n_2\cdots n_k)=1$. (*In particular, n is relatively prime to every one of n_1, n_2, ..., n_k.*)

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 3, 2013.***

Problem 406. For every integer $m>2$, let P be the product of all those positive integers that are less than m and relatively prime to m, prove that P^2-1 is divisible by m.

Problem 407. Three circles S, S_1, S_2 are given in a plane. S_1 and S_2 touch each other externally, and both of them touch S internally at A_1 and at A_2 respectively. Let P be one of the two points where the common internal tangent to S_1 and S_2 meets S. Let B_i be the intersection points of PA_i and S_i (i=1,2). Prove that line B_1B_2 is a common tangent to S_1 and S_2.

Problem 408. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f:\mathbb{Q}\to\{0,1\}$ be a function such that for all x,y in $\mathbb{Q}$ with $f(x)=f(y)$, we have $f((x+y)/2)=f(x)$. If $f(0)=0$ and $f(1)=1$, then prove that $f(x)=1$ for every rational $x>1$.

Problem 409. The population of a city is one million. Every two citizens there know another common citizen (here knowing is mutual). Prove that it is possible to choose 5000 citizens from the city such that each of the remaining citizens will know at least one of the chosen citizens.

Problem 410. (*Due to Titu ZVONARU and Neculai STANCIU, Romania*) Prove that for all positive real x,y,z,

$$\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)} \ge 4(xy+yz+zx)$$

$$+\frac{xy+yz+zx}{3(x^2+y^2+z^2)}((x-y)^2+(y-z)^2+(z-x)^2).$$

Here $\sum_{cyc} f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y)$.

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Solutions

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Problem 401. Suppose all faces of a convex polyhedron are parallelograms. Can it have exactly 2012 faces? Please provide an explanation to your answer.

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)) and **F5 Group** (Carmel Alison Lam Foundation Secondary School).

The answer is negative. Let us call a series of faces F_1, F_2,…, F_k a <u>*loop*</u> if the pairs (F_1, F_2), (F_2, F_3),…, (F_{k-1}, F_k), (F_k, F_1) each have a common edge and all these common edges are parallel. Clearly any two loops have exactly two common faces and conversely each face belongs to exactly two loops. Therefore, if there are n loops, the total number of faces must be $2\ {}_nC_2=n(n-1)$. However, $n(n-1)=2002$ has no solution in integer.

Problem 402. Let S be a 30 element subset of {1,2,…,2012} such that every pair of elements in S are relatively prime. Prove that at least half of the elements of S are prime numbers.

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **F5 Group** (Carmel Alison Lam Foundation Secondary School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia) and **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Assume there are more than 15 elements in S are not prime. Excluding 1, there are at least 15 of them are composite numbers. Each composite number in S has a prime divisor at most $[2012^{1/2}] = 46$. There are 14 prime numbers less than 46. By the pigeonhole principle, two of the 15 composite numbers above will share a common prime divisor, contradiction.

Problem 403. On the coordinate plane, 1000 points are randomly chosen. Prove that there exists a way of coloring each of the points either red or blue (but not both) so that on every line parallel to the x-axis or y-axis, the number of red points minus the number of blue points is equal to −1, 0 or 1.

Solution. **J. S. GLIMMS** (Vancouver, Canada) and **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France).

Replace 1000 by n. We prove by induction on n. The case n=1 is clear. Suppose the case $n=k$ is true. For the case $n=k+1$, we have two cases.

<u>*Case A*</u> (one of the lines L parallel to the x-axis or the y-axis contains an odd number of the points). Ignore one of the points P on L. By inductive step, there is a desired coloring for the remaining k points. Since there is an even number of point on L now, the number of red and blue points must be the same. Then look at the coloring on the line $L^\perp$ through P perpendicular to L. Color P red if $L^\perp$ is a −1 or 0 case and blue if it is a 1 case.

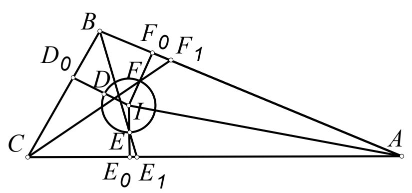
<u>*Case B*</u> (all lines parallel to the x-axis or y-axis contain an even number of the points). Ignore one of the points P on one of the lines L parallel to the x-axis. By inductive step, there is a desired coloring for the remaining k points. Let $L^\perp$ be the line through P parallel to the y-axis.

Since other than L, the lines parallel to x-axis all contain an even number of the points, they must all be 0 case lines. Ignoring P, if L is a case 1 line, then in the whole plane there is exactly one more red point than blue point. Also, other than $L^\perp$, the lines parallel to y-axis all contain an even number of the points, they must all be 0 case lines. Then $L^\perp$ must also be a case 1 line. We then color P blue so both L and $L^\perp$ become case 0 lines. Similarly, ignoring P, both lines may be 0 cases, then color P red or blue. Otherwise both lines are −1 cases, then color P red.

Other commended solvers: **F5 Group** (Carmel Alison Lam Foundation Secondary School).

Problem 404. Let I be the incenter of acute ΔABC. Let Γ be a circle with center I that lies inside ΔABC. D, E, F are the intersection points of circle Γ with the perpendicular rays from I to sides BC, CA, AB respectively. Prove that lines AD, BE, CF are concurrent.

Solution. **F5 Group** (Carmel Alison Lam Foundation Secondary School) and **J. S. GLIMMS** (Vancouver, Canada).



(Below $P=\alpha\cap\beta$ will mean lines α and β meet at point P, $d(P,\alpha)$ will denote the distance from point P to line α and $[XYZ]$ will denote the area of ΔXYZ.)

Let $D_0 = ID\cap BC$, $E_0 = IE\cap CA$, $F_0 = IF\cap AB$. Since AI bisects $\angle CAB$, IE_0 and IF_0 are symmetric respect to AI. Now $IE=IF$ implies E and F are symmetric respect to AI. Hence, $d(E,AB)=d(F,AC)$. Then

$$\frac{[CFA]}{[AEB]}=\frac{CA\cdot d(F,AC)/2}{AB\cdot d(E,AB)/2}=\frac{CA}{AB}.$$

Similarly,

$$\frac{[BEC]}{[CDA]}=\frac{BC}{CA}\quad and\quad \frac{[ADB]}{[BFC]}=\frac{AB}{BC}.$$

Let $D_1 = AD\cap BC$, $E_1 = BE\cap CA$, $F_1 = CF\cap AB$. We have

$$\frac{AF_1}{F_1B}=\frac{[CF_1A]}{[BF_1C]}=\frac{d(A,CF)}{d(B,CF)}=\frac{[CFA]}{[BFC]}.$$

Similarly,

$$\frac{BD_1}{D_1C}=\frac{[ADB]}{[CDA]}\quad and\quad \frac{CE_1}{E_1A}=\frac{[BEC]}{[AEB]}.$$

From the equations above, we get

$$\frac{AF_1}{F_1B}\frac{BD_1}{D_1C}\frac{CE_1}{E_1A}=\frac{CA}{BC}\frac{AB}{CA}\frac{BC}{AB}=1.$$

By Ceva's theorem, lines AD, BE, CF are concurrent.

Other commended solvers: **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece).

Comment: **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania) mentioned that the problem was well-known and the point of concurrency is called the Kariya point.

Problem 405. Determine all functions f,g: $(0,+\infty) \to (0,+\infty)$ such that for all positive number x, we have

$$f(g(x))=\frac{x}{xf(x)-2}\quad and\quad g(f(x))=\frac{x}{xg(x)-2}.$$

Solution. **F5 Group** (Carmel Alison Lam Foundation Secondary School) and **J. S. GLIMMS** (Vancouver, Canada).

Let $F(x)=xf(x)$ and $G(x)=xg(x)$. For all $x > 0$, $f(g(x)) > 0$ and $g(f(x)) > 0$ imply $F(x)>2$ and $G(x)>2$. Define $a_1=2$. Now

$$\frac{G(x)}{F(x)-2}=g(x)f(g(x))=F(g(x))>a_1.$$

Then $\quad G(x) > a_1F(x) - 2a_1. \qquad (1)$

Similarly, $F(x) > a_1G(x) - 2a_1. \qquad (2)$

Doing $(1)\times a_1+(2)$ and simplifying, we get

$$F(x)<\frac{2a_1}{a_1-1}=4.$$

Define $b_1=4$. Similarly we get $G(x) < b_1$. Repeating the above steps, but reversing all the inequality signs, we can get

$$F(x)>\frac{2b_1}{b_1-1}=a_2,\quad G(x)>a_2,$$

$$F(x)<\frac{2a_2}{a_2-1}=b_2\quad and\quad G(x)<b_2.$$

This suggest defining

$$a_{n+1}=\frac{2b_n}{b_n-1}\quad and\quad b_n=\frac{2a_n}{a_n-1}$$

for n=1,2,3,…. Replacing a_1, b_1, a_2, b_2 by a_n, b_n, a_{n+1}, b_{n+1} and repeating the steps above, we can prove $a_n< F(x)$, $G(x) < b_n$ for n=1,2,3,… by induction on n. Next we will show a_n, b_n have same limit. Now

$$a_{n+1}=\frac{2b_n}{b_n-1}=\frac{4a_n/(a_n-1)}{(a_n+1)/(a_n-1)}=\frac{4a_n}{a_n+1}.$$

Taking reciprocal, we get

$$\frac{1}{a_{n+1}}=\frac{1}{4}+\frac{1}{4}\frac{1}{a_n}.$$

Defining $c_n=1/a_n$, we get $c_{n+1}=(1+c_n)/4$. Subtracting 1/3 from both sides, we get $c_{n+1}-1/3 = (c_n-1/3)/4$. Using this, we get

$$c_{n+1}-\frac{1}{3}=\frac{1}{4^n}(c_1-\frac{1}{3})=\frac{1}{6\cdot 4^n}.$$

From this, letting n tends to infinity, we can see c_n has limit 1/3. Then a_n has limit 3. Similarly b_n has limit 3. Thus, for all $x>0$, $F(x)=3=G(x)$, i.e. $f(x)=3/x=g(x)$. Plugging these into the given equations, we see indeed they are solutions.

Olympiad Corner

(continued from page 1)

Problem 3. (*Cont.*) Can "2" immediately to the right of the inequality be replaced by a smaller positive number?

Problem 4. In ΔABC, $AB > AC$, M is the midpoint of BC of its circumcircle containing A. Its incircle with incentre I is tangent to BC at D. The line passing through D and parallel to AI intersects the incircle again at P. Prove that the lines AP and IM intersect at a point on the circumcircle of ΔABC.

Primes in Arithmetic Progressions

(continued from page 2)

Assume there are integers x, y, z satisfying $x^n+y^n=z^n$ with $xyz\neq 0$ and $\gcd(n,xyz)=1$. Then $\gcd(p,x) = \gcd(p,y) = \gcd(p,z) = 1$. Let $w = x^{(p-1)/2}$. By Euler's theorem, $w^2=x^{p-1}\equiv 1 \pmod p$. Then p divides $w-1$ or $w+1$. Hence $x^{(p-1)/2}= w\equiv \pm 1 \pmod p$. Then $x^n \equiv \pm 1 \pmod p$. Similarly, $y^n, z^n \equiv \pm 1 \pmod p$. But then $x^n + y^n \equiv 0$ or $\pm 2 \pmod p$, contradicting $x^n+y^n=z^n$.

<u>Explanations for Facts 1 to 4.</u>

For fact 1, let $n=\min\{a,b\}$. For n=1,we may assume $a\geq b=1$ and take $(r,s)=(0,1)$. Suppose cases n=1 to k are true. For case $n=k+1$, say $a\geq b=k+1$. Dividing a by b, we can write $a=qb+c$, where $q=[a/b]$ and $0\leq c<b$. If c=0, then take $(r,s) = (1,q-1)$ to get $ar+bs = b = \gcd(a,b)$. If $c\geq 1$, then since $k+1=b>c\geq 1$ and $\gcd(b,c) = \gcd(b,a-qb) = \gcd(b,a)$, we can apply inductive step to get r', s' so that $\gcd(b,c)=br'+cs'$. Then $\gcd(a,b) = br'+(a-qb)s'= as'+b(r'-qs')$.

<u>Remark</u>: In case $\gcd(a,b)=1$, fact 1 gives $ar\equiv 1 \pmod b$. We denote this r by a^{-1} in (mod b). Hence we can cancel a in $ax\equiv ay \pmod b$ to get $x\equiv y \pmod b$ by multiplying both sides by a^{-1}.

For fact 2, let $k= \varphi(n)$ and let $r_1, r_2, \ldots, r_k$ be the integers in $[1,n]$ relatively prime to n. If $\gcd(a,n)=1$, then $ar_i\equiv ar_j \pmod n$ implies $r_i=r_j$ by the remark above. Then $ar_1,ar_2, \ldots,ar_k$ is just a permutation of $r_1, r_2, \ldots, r_k \pmod n$. So $(ar_1)(ar_2)\cdots(ar_k)\equiv r_1r_2\cdots r_k \pmod n$. As $\gcd(r_1r_2\cdots r_k,n)=1$, by the remark above we may cancel $r_1r_2\cdots r_k$ to get $a^k\equiv 1 \pmod n$, which is Euler's theorem.

For fact 3, let $K= k_1k_2\cdots k_n$ and $M_i =K/k_i$. Then $\gcd(M_i,k_i) =1$ and for $j\neq i$, $M_j\equiv 0 \pmod{k_i}$. Let x be the integer in the interval $[1,K]$ such that

$$x\equiv c_1M_1^{\varphi(k_1)}+\cdots+c_nM_n^{\varphi(k_n)}\pmod K.$$

Using Euler's theorem, $x\equiv c_i \pmod{k_i}$. If x' in $[1,K]$ is another solution, then $x-x'\equiv c_i-c_i=0 \pmod{k_i}$ for $i=1,2,\cdots,n$. This leads to $x-x'\equiv 0 \pmod K$. As x, x' are both in $[1,K]$, we get $x=x'$.

For fact 4, p=2 or 3 cases are clear. For $p>3$, let a be in $[1, p-1]$. If $a\equiv a^{-1} \pmod p$, then $a^2\equiv 1 \pmod p$. So p divides $(a-1)(a+1)$. Hence a=1 or $p-1$. For a in $[2,p-2]$, we can form $(p-3)/2$ pairs a and a^{-1}. Then $(p-1)! \equiv 1(aa^{-1})^{(p-3)/2}(-1) = -1 \pmod p$.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 28th Italian Math Olympiad.

Problem 1. Let ABC be a triangle with right angle at A. Choose points D, E, F on sides BA, CA, AB respectively so that $AFDE$ is a square. Denote by x the side-length of this square. Prove that

$$\frac{1}{x}=\frac{1}{AB}+\frac{1}{AC}.$$

Problem 2. Determine all positive integers that are 300 times the sum of their digits.

Problem 3. Let n be an integer greater than or equal to 2. There are n persons in a line, and each of these persons is either a villain (and this means that he/she always lies) or a knight (and this means he/she always tells the truth). Apart from the first person in the line, every person indicates one of those before him and declares either "this person is a villain" or "this person is a knight". It is known that the number of villains is greater than the number of knights. Prove that, watching the declarations, it is possible to determine, for each of the n persons, whether he/she is a villain or a knight.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 10, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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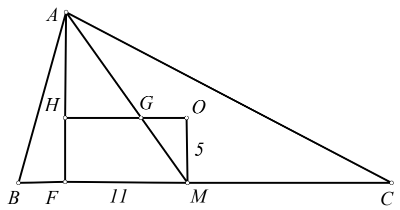
Putnam Exam

Kin Y. Li

The title of our article is an abbreviated name for the famous *William Lowell Putnam Mathematical Competition*. It started in the year 1938. Thousands of students in many US and Canadian universities participate in this competition annually. The top five scorers each year are designated as Putnam Fellows. These Putnam Fellows include the Physics Nobel Laureates Richard Feynman, Kenneth Wilson, the Fields' Medalists John Milnor, David Mumford, Dan Quillen and many other famous celebrities.

Although it is a math competition for undergraduate students, some of the problems may be solved by secondary school students interested in math olympiads. Below we will provide some examples.

Example 1 (*1997 Putnam Exam*) A rectangle $HOMF$ has sides HO=11 and OM=5. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC and F the foot of the altitude from A. What is the length of BC?



Solution. Recall the centroid G of ΔABC is on the Euler line OH (see *Math Excalibur*, *vol.* 3, *no.* 1, *p.* 1) and AG/GM = 2. As FH, $MO \perp OH$ and $\angle AGH = \angle MGO$, so $\Delta AHG \sim \Delta MOG$. Hence $AH = 2OM = 10$. Then $OC^2 = OA^2 = AH^2 + OH^2 = 221$ and $BC = 2MC = 2(OC^2 - OM^2)^{1/2} = 28$.

Example 2 (*1991 Putnam Exam*) Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 2^p+1 \pmod{p^2}.$$

Solution. Let W be the left side of the equation. Since $\binom{p+j}{j}=\binom{p+j}{p}$, W is the coefficient of x^p in the polynomial

$$\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+j}\binom{p+j}{k}x^k$$
$$=\sum_{j=0}^{p}\binom{p}{j}(1+x)^{p+j}$$
$$=(1+x)^p\sum_{j=0}^{p}\binom{p}{j}(1+x)^j$$
$$=(1+x)^p(2+x)^p.$$

Expanding $(1+x)^p(2+x)^p$, we see

$$W=\sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k}2^k.$$

For $0 < k < p$, p divides $p!$, but not $k!(p-k)!$. So p divides $\binom{p}{k}=\binom{p}{p-k}$.

In (mod p^2) of W, we may ignore the terms with $0 < k < p$ to get

$$W\equiv\binom{p}{0}\binom{p}{p}2^0+\binom{p}{p}\binom{p}{0}2^p=1+2^p \pmod{p^2}.$$

Example 3 (*2000 Putnam Exam*) Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form (± 1, ± 1, …, ± 1) in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Solution. Let S be the set of all points $(x_1, x_2, \ldots, x_n)$ with all $x_i = \pm 1$. For each P in B, let S_P be the set of all points in S which differ from P in exactly one coordinate. Each S_P contains n points. So the union of all S_P's over all P in B (counting points repeated as many times as they appeared in the union) must contain more than 2^{n+1} points. Since this is more than twice 2^n, by the pigeonhole principle, there must exist a point T appeared in at least three of the sets S_P, S_Q, S_R, where P, Q, R are distinct points in B. Then any two of P, Q, R have exactly two different coordinates. Then ΔPQR is equilateral with sides $2^{3/2}$.

Example 4 (*1947 Putnam Exam*) Given $P(z) = z^2 + az + b$, a quadratic polynomial for the complex variable z with complex coefficients a and b. Suppose that $|P(z)| = 1$ for every z such that $|z| = 1$. Prove that $a = b = 0$.

Solution. Let $\omega \neq 1$ be a cube root of unity. Let $\alpha = P(1)$, $\beta = \omega P(\omega)$ and $\gamma = \omega^2 P(\omega^2)$. We have $|\alpha| = |\beta| = |\gamma| = 1$ and $\alpha+\beta+\gamma = 3+a(1+\omega^2+\omega^4)+b(1+\omega+\omega^2) = 3$. Hence, $|\alpha+\beta+\gamma| = |\alpha| + |\beta| + |\gamma|$. By the equality case of the triangle inequality, we get $\alpha = \beta = \gamma = 1$. Then $P(1) = 1$, $P(\omega) = \omega^2$ and $P(\omega^2) = \omega = \omega^4$. Since P is of degree 2 and $P(z)-z^2 = 0$ has three distinct roots 1, ω and ω^2, we get $P(z) = z^2$ for all complex number z.

Example 5 (*1981 Putnam Exam*) Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of n^2+3, then p is also a divisor of k^2+3 for some integer k with $k^2 < n$.

Solution. First we look at the sequence m^2+3 with $m \geq 0$. The terms are 3, 4, 7, 12, 28, 39, 52, 67, 84, …. We can observe that 3×4, 4×7, 7×12,… are also in the sequence. This suggests multiplying $(m^2+3)[(m+1)^2+3]$. By completing square of the result, we see

$$(m^2+3)[(m+1)^2+3] = (m^2+m+3)^2+3.$$

Let $n = (m^2+m+2)(m^2+m+3)+3$. Using the identity above twice, we see $n^2+3 = (m^2+3)[(m+1)^2+3][(m^2+m+2)^2+1]$. So if p is a prime divisor of n^2+3, then p is also a divisor of either m^2+3 or $(m+1)^2+3$ or $(m^2+m+2)^2+1$ and m^2, $(m+1)^2$, $(m^2+m+2)^2 < n$. Letting $m = 1,2,3,\ldots$, we get infinitely many such n.

Example 6 (*1980 Putnam Exam*) Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > ½|X| \geq 5$ for $1 \leq i \leq 1066$. Prove there exists ten elements $x_1, x_2, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, x_2, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)

Solution. Let $X = \{x_1, x_2, \ldots, x_m\}$ with $m = |X|$ and n_k be the number of i such that x_k is in A_i. We may arrange the x_k's so that n_k is decreasing. For $1 \leq i \leq 1066$ and $1 \leq k \leq m$, let $f(i,k)=1$ if x_k is in A_i and $f(i,k)=0$ otherwise. Then

$$n_1|X| \geq \sum_{k=1}^{m} n_k = \sum_{k=1}^{m}\sum_{i=1}^{1066} f(i,k) = \sum_{j=1}^{1066}\sum_{k=1}^{m} f(i,k)$$
$$= \sum_{i=1}^{1066}|A_i| \geq \sum_{i=1}^{1066}\frac{1}{2}m = 533|X|.$$

Then n_1 is greater than 533, i.e. x_1 is in more than 533 A_i's.

Next let $B_1, B_2, \ldots, B_r$ be those A_i's not containing x_1 and $Y = \{x_2, x_3, \ldots, x_m\}$. Then $r = 1066-n_1 \leq 532$ and each $|B_i| > ½|X| > ½|Y|$. Repeating the reasoning above, we will get $n_2 > r/2$. Let $C_1, C_2, \ldots, C_s$ be those A_i's not containing x_1, x_2 and $Z = \{x_3, x_4, \ldots, x_m\}$. Then $s = r-n_2 < r/2$, i.e. $s \leq 265$. After 532 and 265, repeating the reasoning, we will get 132, 65, 32, 15, 7, 3, 1. Then at most 1 set is left not containing $x_1, x_2, \ldots, x_9$. Finally, we may need to use x_{10} to take care of the last possible set.

Example 7 (*1970 Putnam Exam*) A quadrilateral which can be inscribed in a circle is said to be *inscribable* or *cyclic*. A quadrilateral which can be circumscribed to a circle is said to be *circumscribable*. If a circumscribable quadrilateral of sides a, b, c, d has area $A = \sqrt{abcd}$, then prove that it is also inscribable.

Solution. Since the two tangent segments from a point (outside a circle) to the circle are equal and the quadrilateral is circumscribable, we have $a+c=b+d$. Let k be the length of a diagonal and α, β be opposite angles of the quadrilateral so that

$$a^2+b^2-2ab\cos\alpha = k^2 = c^2+d^2-2cd\cos\beta.$$

Subtracting $(a-b)^2 = (c-d)^2$, we get

$$2ab(1-\cos\alpha) = 2cd(1-\cos\beta). \quad (*)$$

Now $2\sqrt{abcd} = 2A = ab\sin\alpha + cd\sin\beta$. Squaring and using (*) twice, we get

$$\begin{aligned} 4abcd &= a^2b^2(1-\cos^2\alpha)+2abcd\sin\alpha\sin\beta \\ &\quad +c^2d^2(1-\cos^2\beta) \\ &= abcd(1+\cos\alpha)(1-\cos\beta) \\ &\quad +2abcd\sin\alpha\sin\beta \\ &\quad +abcd(1+\cos\beta)(1-\cos\alpha). \end{aligned}$$

Simplifying this, we get $4=2-2\cos(\alpha+\beta)$, i.e. $\alpha+\beta=180°$. Therefore the quadrilateral is cyclic.

Example 8 (*1964 Putnam Exam*) Show that the unit disk in the plane cannot be partitioned into two disjoint congruent subsets.

Solution. Let D be the unit disk, O be its center and $d(X,Y)$ denote the distance between X and Y in D. Assume D can be partitioned into two disjoint congruent subsets A and B. Without loss of generality, suppose O is in A. For each X in A, let X^* be the corresponding point in B. Then O^* is in B. For all X, Y in A, $d(X,Y) = d(X^*,Y^*)$.

Since $d(O,X) \leq 1$ for all X in A and the set $B = \{X^* : X \text{ in } A\}$, so $d(O^*,Z) \leq 1$ for all Z in B. Let R and S be the endpoints of the diameter perpendicular to line OO^*. Then $d(O^*,R) = d(O^*,S) > 1$. Hence, R and S are in A. Now $d(R^*,S^*) = d(R,S) = 2$, so R^*S^* is a diameter. Since O is the midpoint of diameter RS in A, O^* must be the midpoint of the diameter R^*S^*. Then $O^*=O$, which contradicts A, B are disjoint.

Example 9 (*1950 Putnam Exam*) In each of N houses on a straight street are one or more boys. At what point should all the boys meet so that the sum of the distances that they walk is as small as possible?

Solution. Think of the street is the real axis. Suppose the i-th boy's house is at x_i so that $x_1 \leq x_2 \leq \cdots \leq x_n$. Suppose they meet at x, the first and the n-th boy together must walk a distance of $x_n - x_1$ if x is in $[x_1, x_n]$ and more if x is outside $[x_1, x_n]$. This is similar for the second boy and the $(n-1)$-st boy, etc.

If n is even, say $n=2k$, then the least distance all n boys have to walk is

$$(x_n-x_1)+(x_{n-1}-x_2)+\cdots+(x_{k+1}-x_k)$$

with equality if x is in $[x_k, x_{k+1}]$. If n is odd, say $n = 2k - 1$, then the least distance they have to walk is

$$(x_n-x_1)+(x_{n-1}-x_2)+\cdots+(x_{k+1}-x_k)+0$$

with equality if $y = x_k$.

Example 10 (*1956 Putnam Exam*) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. Prove that $P(z) \equiv Q(z)$.

Solution. Observe that if $P(z)$ has c as a zero with multiplicities $k > 0$, then the derivative $P'(z)$ has c as a zero with multiplicities $k-1$, which follows from differentiating $P(z) = (z-c)^k R(z)$ on both sides.

Now suppose $P(z)$ has degree m and $Q(z)$ has degree n. By symmetry, we may assume $m \geq n$. Let the distinct zeros of $P(z)$ be $a_1, a_2, \ldots, a_s$ and let the distinct zeros of $P(z)+1$ be $b_1, b_2, \ldots, b_t$. Clearly, $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t$ are all distinct.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***March 10, 2013.***

Problem 411. A and B play a game on a square board divided into 100×100 squares. Each of A and B has a checker. Initially A's checker is in the lower left corner square and B's checker is in the lower right corner square. They take turn to make moves. The rule is that each of them has to move his checker one square up, down, left or right within the board and A goes first. Prove that no matter how B plays, A can always move his checker to meet B's checker eventually.

Problem 412. $\triangle ABC$ is equilateral and points D, E, F are on sides BC, CA, AB respectively. If

$$\angle BAD + \angle CBE + \angle ACF = 120°,$$

then prove that $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$ cover $\triangle ABC$.

Problem 413. Determine (with proof) all integers $n\ge3$ such that there exists a positive integer M_n satisfying the condition for all n positive numbers a_1, a_2, …, a_n, we have

$$\frac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}} \le M_n\left(\frac{a_2}{a_1}+\frac{a_3}{a_2}+\cdots+\frac{a_n}{a_{n-1}}+\frac{a_1}{a_n}\right).$$

Problem 414. Let p be an odd prime number and $a_1, a_2, \ldots, a_{p-1}$ be positive integers not divisible by p. Prove that there exist integers $b_1, b_2, \ldots, b_{p-1}$, each equals 1 or -1 such that

$$a_1b_1+a_2b_2+\cdots+a_{p-1}b_{p-1}$$

is divisible by p.

Problem 415. (*Due to MANOLOUDIS Apostolos, Piraeus, Greece*) Given a triangle ABC such that $\angle BAC=103°$ and $\angle ABC=51°$. Let M be a point inside $\triangle ABC$ such that $\angle MAC=30°$ and $\angle MCA=13°$. Find $\angle MBC$ with proof.

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Solutions

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Problem 406. For every integer $m>2$, let P be the product of all those positive integers that are less than m and relatively prime to m, prove that P^2-1 is divisible by m.

Solution. **Jon GLIMMS** (Vancouver, Canada), **Corneliu MĂNESCU- AVRAM** (Technological Transportation High School, Ploieşti, Romania), **WONG Ka Fai** and **YUNG Fai**.

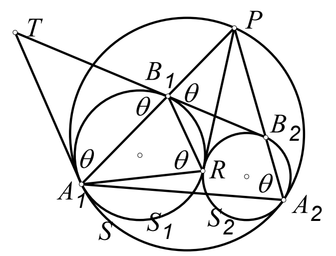
Let a in interval $[1,m)$ be relatively prime to m. By Bezout's theorem, there exists a unique a^{-1} in $[1,m)$ such that $aa^{-1}\equiv1$ (mod m). Then $\gcd(a^{-1},m) = 1$. Since $a^{-1}a \equiv 1$ (mod m), by uniqueness, $(a^{-1})^{-1} = a$.

For those factor a in the product P satisfying $a\ne a^{-1}$, a will be cancelled by a^{-1} (mod m). Thus, P is congruent modulo m to the product of those remaining factor a satisfying $a= a^{-1}$. Now $a = a^{-1}$ implies $a^2 = aa^{-1}\equiv1$ (mod m). It follows $P^2 \equiv 1$ (mod m) and we are done.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 407. Three circles S, S_1, S_2 are given in a plane. S_1 and S_2 touch each other externally, and both of them touch S internally at A_1 and at A_2 respectively. Let P be one of the two points where the common internal tangent to S_1 and S_2 meets S. Let B_i be the intersection points of PA_i and S_i $(i=1,2)$. Prove that line B_1B_2 is a common tangent to S_1 and S_2.

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School), **William FUNG and Jacob HA** and **NGUYEN Van Thien** (Luong The Vinh High School, Dongnai Province, Vietnam)**.**



Let the tangent at A_1 to S (and S_1) and line B_1B_2 meet at T. Let R be the tangent point of S_1 and S_2. By the intersecting chord theorem, we have

$$PB_1\times PA_1 = PR^2 = PB_2\times PA_2.$$

So A_1, B_1, B_2, A_2 are concyclic. Using (1) line TA_1 is tangent to S_1, (2) line TA_1 is tangent to S, (3) A_1, B_1, B_2, A_2 concyclic and (4) vertical angles are congruent in that order, we get

$$\angle B_1RA_1 = \angle B_1A_1T = \angle PA_2A_1 = \angle PB_1B_2 = \angle TB_1A_2.$$

Then line TB_1=B_1B_2 is tangent to S_1 at B_1. Similarly, line B_1B_2 is tangent to S_2 at B_2. Therefore, line B_1B_2 is a common tangent to S_1 and S_2.

Other commended solvers: **Dusan DROBNJAK**(Mathematical Grammar School, Belgrade, Serbia), **Jon GLIMMS** (Vancouver, Canada), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece) and **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India).

Problem 408. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f:\mathbb{Q}\to\{0,1\}$ be a function such that for all x,y in $\mathbb{Q}$ with $f(x)=f(y)$, we have $f((x+y)/2)=f(x)$. If $f(0)=0$ and $f(1)=1$, then prove that $f(x)=1$ for every rational $x > 1$.
(*Source*: 2000 *Indian Math Olympiad*)

Solution. **Ioan Viorel CODREANU,** (Secondary School Satulung, Maramures, Romania) and **Dusan DROBNJAK**(Mathematical Grammar School, Belgrade, Serbia).

We *claim* that if a,b are rational numbers and $f(a)\ne f(b)$, then for all positive integer n, we have $f(a+n(b-a))=f(b)$.

We will prove this by induction on n. The case n=1 is clear. Suppose the case $n=k$ is true. Then we have $f(a+k(b-a))=f(b)$. *Assume* $f(a+(k+1)(b-a)) \ne f(b)$. Since $f(r)$=0 or 1 for all r in $\mathbb{Q}$ and $f(a)\ne f(b)$, we get $f(a+(k+1)(b-a)) = f(a)$. Let $x = a$, $y = a+(k+1)(b-a)$, $x' = b$, $y' = a+k(b-a)$. From above, we have $f(x) = f(y)$ and $f(x') = f(y')$. By the given property of f, since

$$\frac{x+y}{2}=\frac{(k+1)b-(k-1)a}{2}=\frac{x'+y'}{2},$$

we get $f(a)=f(x)=f(x')=f(b)$, contradiction. Hence the case $n=k+1$ is true and we complete the induction.

Now by the claim, since $f(0)=0\ne1=f(1)$, for all positive integer n, we get $f(n) = f(1) =1$. For a rational $r > 1$, let $r-1=p/q$, where p, q are positive integers. *Assume* $f(r)\ne1$. Using the claim with $a = 1$, $b = r$, and $n = q$, we get $f(1+q(r-1)) = f(r)$. But $f(1+q(r-1)) = f(1+p) = 1$, contradiction. So, for all rational $r>1$, $f(r) = 1$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 409. The population of a city is one million. Every two citizens there know another common citizen (here knowing is mutual). Prove that it is possible to choose 5000 citizens from the city such that each of the remaining citizens will know at least one of the chosen citizens.
(*Source*: 63*rd* *St. Petersburg Math Olympiad*)

Solution. **Jon GLIMMS** (Vancouver, Canada).

Let $m=10^6$ and $x_1, x_2, \ldots, x_m$ be all the citizens in the city. Let $F(x_i)$ be all the citizens (not including x_i) that x_i knows and $|F(x_i)|$ denote the number of such citizens.

If there exists a x_i with $|F(x_i)|\le 5000$, then let us choose any 5000 citizens including all members of $F(x_i)$. For any x_j not among the chosen 5000 citizens, by the given assumption, x_i and x_j know a common citizen in $F(x_i)$, who is in the chosen 5000 citizens.

Otherwise, we may assume for every x_i, $|F(x_i)| > 5000$. Now there are m^{5000} ordered 5000-tuples $(C_1, C_2, \ldots, C_{5000})$, where each C_k may be any one of the m citizens. For each x_i, let

$$S(x_i)=\{(C_1,C_2,\ldots,C_{5000})\text{: all } C_k\notin F(x_i)\}$$

Now $S(x_i)$ has less than $(m-5000)^{5000}$ members since $|F(x_i)|>5000$. Let S be the union of $S(x_1), S(x_2),\ldots,S(x_m)$. We <u>claim</u> that $m(m-5000)^{5000} < m^{5000}$. The claim means there exists $(C_1, C_2, \ldots, C_{5000})$ not in every $S(x_i)$. That means by choosing $C_1, C_2, \ldots, C_{5000}$, every x_i will know at least one C_k and we are done.

For the claim, using $(1+x)^n \ge 1+nx$ from the binomial theorem, we have the equivalent inequality

$$\left(\frac{m}{m-5000}\right)^{5000}=\left(1+\frac{5000}{m-5000}\right)^{5000}$$
$$>\left(1+\frac{1}{200}\right)^{200\times 25}>2^{25}>(10^3)^{2.5}>m.$$

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 410. (*Due to Titu ZVONARU and Neculai STANCIU, Romania*) Prove that for all positive real x,y,z,

$$\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)}\ge 4(xy+yz+zx)$$
$$+\frac{xy+yz+zx}{3(x^2+y^2+z^2)}((x-y)^2+(y-z)^2+(z-x)^2).$$

Here $\sum_{cyc} f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y).$

Solution of Proposers.

Observe that $4(xy+yz+zx)$ is the cyclic sum of $x(y+z)+y(x+z)$. Now

$$\begin{aligned}
&(x+y)\sqrt{(x+z)(y+z)}-x(y+z)-y(x+z)\\
&=x\sqrt{y+z}(\sqrt{x+z}-\sqrt{y+z})\\
&\quad+y\sqrt{x+z}(\sqrt{y+z}-\sqrt{x+z})\\
&=(x\sqrt{y+z}-y\sqrt{x+z})(\sqrt{x+z}-\sqrt{y+z})\\
&=\frac{(x^2(y+z)-y^2(x+z))}{(x\sqrt{y+z}+y\sqrt{x+z})}\frac{((x+z)-(y+z))}{(\sqrt{x+z}+\sqrt{y+z})}\\
&=\frac{(xy+yz+zx)(x-y)^2}{(x\sqrt{y+z}+y\sqrt{x+z})(\sqrt{x+z}+\sqrt{y+z})}.
\end{aligned}$$

By the *AM-GM* inequality, we have $(x+y)^2 \le 2(x^2+y^2)$ and $xy+yz+zx \le x^2+y^2+z^2$. Using these, we get

$$\begin{aligned}
&(x\sqrt{y+z}+y\sqrt{x+z})(\sqrt{x+z}+\sqrt{y+z})\\
&=(x+y)\sqrt{(x+z)(y+z)}+x(y+z)+y(x+z)\\
&\le(x+y)\frac{x+y+2z}{2}+2xy+yz+zx\\
&=\frac{(x+y)^2}{2}+2(xy+yz+zx)\\
&\le x^2+y^2+2(x^2+y^2+z^2)\\
&\le 3(x^2+y^2+z^2).
\end{aligned}$$

So it follows that

$$\begin{aligned}
&(x+y)\sqrt{(x+z)(y+z)}-2xy-yz-zx\\
&\ge\frac{xy+yz+zx}{3(x^2+y^2+z^2)}(x-y)^2.
\end{aligned}$$

Rotating x,y,z to y,z,x to z,x,y, we get two other similar inequalities. Adding the three inequalities, we will get the desired inequality. Equality holds if and only if $x=y=z$.

Other commended solvers: **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Comment: The proposers mention that this is a refinement of problem 2 of the 2012 Balkan Math Olympiad.

Olympiad Corner

(continued from page 1)

Problem 4. Let x_1, x_2, x_3, … be the sequence defined by the following recurrence: $x_1=4$ and for $n \ge 1$,

$$x_{n+1} = x_1x_2x_3\cdots x_n + 5.$$

(The first few terms of the sequence are then $x_1=4$, $x_2=4+5=9$, $x_3=4\cdot 9+5=41$, …) Find all pairs $\{a,b\}$ of positive integers such that x_ax_b is a perfect square.

Problem 5. Let $ABCD$ be a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB+\angle CPD=180^\circ.$$

Problem 6. Determine all pairs $\{a,b\}$ of positive integers with the following property: for any possible coloring of the set of all positive integers with two colors A and B, there exist either two positive integers colored by A with difference a or two positive integers colored by B with difference b.

Putnam Exam

(continued from page 2)

Now let r_i's be the multiplicities of the a_i's as zeros of $P(z)$, then the sum of the r_i's is m. By the observation above, the multiplicity of a_i as zeros of $P'(z)$ is r_i-1 and these multiplicities sum to $m-s$. Similarly, the sum of the multiplicities of the b_i's as zeros of $P'(z)= (P+1)'(z)$ is $m-t$. So

$$(m-s)+(m-t)\le \deg P'(z) < m.$$

Hence $s+t > m$. However, $a_1, a_2,\ldots,a_s$, $b_1, b_2, \ldots, b_t$ are zeros of $P(z)-Q(z)$ with degree at most m. So, $P(z)\equiv Q(z)$.

The interested readers are highly encouraged to browse the following books for more problems of the Putnam Exam.

A. M. Gleason, R. E. Greenwood and L. M. Kelly, <u>The William Lowell Putnam Mathematical Competition Problems and Solutions: 1938-1964</u>, MAA, USA, 1980.

G. L. Alexanderson, L. F. Klosinski and L. C. Larson, <u>The William Lowell Putnam Mathematical Competition Problems and Solutions: 1965-1984</u>, MAA, USA, 1985.

K. S. Kedlaya, B. Poonen and R. Vakil, <u>The William Lowell Putnam Mathematical Competition 1985-2000 Problems, Solutions and Commentary</u>, MAA, USA, 2002.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Final Selection Test for the 2012 Croatian IMO Team.

Problem 1. Determine all functions f: $\mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y holds

$$f(x^2 + f(y)) = (f(x) + y^2)^2.$$

(*Tonći Kokan*)

Problem 2. Along the coast of an island there are 20 villages. Each village has 20 fighters. Every fighter fights all the fighters from all the other villages. No two fighters have equal strength and the stronger fighter wins the fight.

We say that the village A is *stronger* than the village B if in at least k fights among the fighters from A and B a fighter from the village A wins. It turned out that every village is stronger than its neighbour (in the clockwise direction).

Show that the maximal possible k is 290.

(*Moscow Olympiad 2003, modified*)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 10, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Inequality of A. Oppenheim

Prof. Marcel Chirita, Bucharest, Romania

In this note we establish conditions solving the problems of A. Oppenheim and O. Bothema, then we solve some problems. Below, we let a,b,c,S,s,R,r denote the sides BC, CA, AB, area, semiperimeter, circumradius, inradius of a triangle ABC respectively. In [1], two problems are stated as follow:

Problem 1. (*O. Bothema*) For $\triangle ABC$, give conditions on real numbers x,y,z so

$$yza^2+zxb^2+xyc^2 \le R^2(x+y+z)^2 \qquad (1)$$

with equality if and only if

$$\frac{x}{\sin 2A} = \frac{y}{\sin 2B} = \frac{z}{\sin 2C}. \qquad (2)$$

Problem 2. (*A.Oppenheim*) For $\triangle ABC$, give conditions on real numbers x,y,z so

$$xa^2 + yb^2 + zc^2 \ge 4S\sqrt{xy+yz+zx} \qquad (3)$$

with equality if and only if

$$\frac{x}{-a^2+b^2+c^2} = \frac{y}{a^2-b^2+c^2} = \frac{z}{a^2+b^2-c^2}. \qquad (4)$$

The author will solve problem 2, then use it to solve problem 1. It is easy to see these problems are false for some x,y,z. For example, if one of x,y,z is negative, problems 1 and 2 may be false.

Theorem. For $\triangle ABC$, if $x+y>0$, $y+z>0$, $z+x>0$ and $xy+yz+zx>0$, then (3) and (4) hold.

Proof. Let $k = 4\sqrt{xy+yz+zx}$. Using $c^2 = a^2+b^2-2ab\cos C$ and $S=\frac{1}{2}ab\sin C$, we can rewrite (3) as

$$2(x+z)a^2+2(y+z)b^2 \ge (4z\cos C+k\sin C)ab.$$

By the *AM-GM* inequality, the left side is greater than or equal to

$$4\sqrt{(x+z)(y+z)}ab = \sqrt{16z^2+k^2}ab,$$

which is greater than or equal to the right side by the Cauchy-Schwarz inequality. So (3) is true. Equality holds (from *AM-GM* and Cauchy-Schwarz) if and only if

$$\frac{a^2}{y+z} = \frac{b^2}{z+x} = \frac{c^2}{x+y}.$$

Let t be this ratio. Then $a^2 = t(y+z)$, $b^2 = t(z+x)$, $c^2 = t(x+y)$. So $-a^2+b^2+c^2 = 2tx$, $a^2-b^2+c^2 = 2ty$ and $a^2+b^2-c^2 = 2tz$. This gives (4) and steps can be reversed. Using the cosine law, we can see (4) is equivalent to

$$\frac{xa}{\cos A} = \frac{yb}{\cos B} = \frac{zc}{\cos C}.$$

From (4), we see x,y,z can be all positive or one negative and two positive.

To solve problem 1, in place of x,y,z, we use x/a^2, y/b^2, z/c^2, which also satisfy the conditions of the theorem. Then (3) is

$$x+y+z \ge 4S\sqrt{\frac{xy}{a^2b^2}+\frac{yz}{b^2c^2}+\frac{zx}{c^2a^2}}.$$

Using the formula $S=abc/(4R)$ (which is from $S=\frac{1}{2}ab\sin C$ and $c/(\sin C)=2R$), the last inequality becomes

$$x+y+z \ge \frac{1}{R}\sqrt{xyc^2+yza^2+zxb^2},$$

which is equivalent to (1). For equality case, observe that using the cosine law and $a/(\sin A)=2R$,

$$\frac{x/a^2}{-a^2+b^2+c^2} = \frac{x}{2a^2bc\cos A} = t\frac{x}{\sin 2A},$$

where $t = 1/(2Rabc)$. This gives (2).

Next we give many applications of these inequalities.

<u>Example 1</u> If we take $x=y=z$ in (3), then we get $a^2+b^2+c^2 \ge 4S\sqrt{3}$, which dated back to Ionescu (1897), later to Weitzenböck (1919) and Carlitz (1961).

<u>Example 2</u> If we take $x=a^2$, $y=b^2$ and $z=c^2$ in (3), then we get

$$a^4+b^4+c^4 \ge 4S\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

Since Heron's formula gives

$$2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) = 16S^2$$

(continued on page 2)

and expanding $(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 \ge 0$ leads to

$$a^2b^2+b^2c^2+c^2a^2 \le a^4+b^4+c^4,$$

it follows immediately that

$$a^2b^2+b^2c^2+c^2a^2 \ge 16S^2$$

and hence $a^4+b^4+c^4 \ge 16S^2$.

Example 3 (*a*) If $x = 9$, $y = 5$ and $z = -3$ in (3), then we get $9a^2+5b^2-3c^2 \ge 4S\sqrt{3}$.

(b) If x=27, y=27 and z=−13 in (3), then we get $27a^2+27b^2-13c^2 \ge 12S\sqrt{3}$.

(c) If x=3, y=−1 and z=15 in (3), then we get $3a^2-b^2+15c^2 \ge 12S\sqrt{3}$.

These were exercises proposed in [6] and [9].

Example 4 If we consider $x=bc/a$, $y=ca/b$ and $z=ab/c$ in (3), then we have

$$3abc \ge 4\sqrt{a^2+b^2+c^2}\,S.$$

Taking into account that $4RS=abc$ and $ab+bc+ca \le a^2+b^2+c^2$, we have $ab+bc+ca \le a^2+b^2+c^2 \le 9R^2$.

Example 5 If we consider $x=bc$, $y=ca$ and $z=ab$ in (3), then we have

$$abc(a+b+c) \ge 4S\sqrt{abc(a+b+c)},$$

which implies $abc(a+b+c) \ge 16S^2$. Using $S = ½(a+b+c)r = sr$, we get $abc \ge 8sr^2$. Using abc=4RS=4Rsr, we have $R \ge 2r$.

Example 6 Let $x > 0$. If we consider $2x-1, \dfrac{2}{x}-1$ and 1, then we can easily check that they satisfy the conditions in the theorem. So (3) yields

$$(2x-1)a^2+\left(\frac{2}{x}-1\right)b^2+c^2 \ge 4S\sqrt{3}.$$

This was a proposed exercise of B. Suceavă in [9].

Example 7 If we consider

$$x=\frac{s-a}{a^2}, y=\frac{s-b}{b^2}, z=\frac{s-c}{c^2}$$

in (3), then we get

$$s \ge 4S\sqrt{\frac{(s-a)(s-b)}{a^2b^2}+\frac{(s-b)(s-c)}{b^2c^2}+\frac{(s-c)(s-a)}{c^2a^2}}.$$

Squaring both sides and applying the *AM-GM* inequality on the right side, we get

$$s^2 \ge 48S^2\sqrt[3]{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^4b^4c^4}},$$

which is equivalent to

$$a^4b^4c^4s^6 \ge 48^3S^6(s-a)^2(s-b)^2(s-c)^2.$$

Using abc=4RS=4Rsr on the left and Heron's formula on the right, we can simplify this to $sR^2 \ge 12\sqrt{3}r^3$.

Example 8 Consider

$$x=\frac{s-a}{a}, y=\frac{s-b}{b}, z=\frac{s-c}{c}.$$

Then

$$xa^2+yb^2+zc^2 = a(s-a)+b(s-b)+c(s-c)$$

$$=\frac{2ab+2bc+2ca-(a^2+b^2+c^2)}{2}.$$

From [3], we have $ab+bc+ca=s^2+r^2+4Rr$ and $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$. Putting these into the above equation, we get

$$xa^2+yb^2+zc^2 = 2r^2+8Rr.$$

Recall by cosine law

$$\frac{(s-a)(s-b)}{ab}=\frac{c^2-a^2-b^2+2ab}{4ab}$$

$$=\frac{1-\cos C}{2}$$

$$=\sin^2\frac{C}{2}.$$

Using this and similar equations, we have

$$4S\sqrt{xy+yz+zx}$$

$$=4S\sqrt{\frac{(s-a)(s-b)}{ab}+\frac{(s-b)(s-c)}{bc}+\frac{(s-c)(s-a)}{ca}}$$

$$=4S\sqrt{\sin^2\frac{C}{2}+\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}$$

$$\ge 4S\sqrt{\frac{3}{4}} = 2S\sqrt{3},$$

where the last inequality follows by applying Jensen's inequality to $f(x) = \sin^2(x/2)$ on $[0,\pi/2]$. Thus, (3) yields

$$r^2+4Rr \ge S\sqrt{3}.$$

Example 9 If instead of x, y, z, we replace them by $\dfrac{yz}{a^2}, \dfrac{xy}{b^2}, \dfrac{zx}{c^2}$ in (3), then we get after calculations that

$$xa^2+yb^2+zc^2 \le R^2\frac{(xy+yz+zx)^2}{xyz}.$$

Example 10 If instead of x, y, z, we consider yz, zx, xy, then (1) and (3) yield the following inequality

$$4S\sqrt{xyz(x+y+z)} \le a^2yz+b^2zx+c^2xy$$

$$\le (x+y+z)^2R^2,$$

which is the subject of the article "On an inequality in a triangle" from GM 8 in 1984 by Prof. Virgil Nicula.

Example 11 If instead of x,y,z, we consider

$$\frac{p}{q+r},\ \frac{q}{r+p},\ \frac{r}{p+q},$$

where $p,q,r > 0$, then (3) yields

$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2 \ge 2S\sqrt{3}.$$

This is problem *E3150* proposed by G. Tsintsifas in the American Math. Monthly in 1988.

Example 12 If instead of x,y,z, we consider

$$\frac{b}{a}m,\ \frac{c}{b}n,\ \frac{a}{c}p,$$

where $m, n, p > 0$, then (3) yields

$$mab+nbc+pca \ge 4S\sqrt{\frac{c}{a}mn+\frac{a}{b}np+\frac{b}{c}mp}.$$

By the AM-GM inequality, we have

$$\frac{c}{a}mn+\frac{a}{b}np+\frac{b}{c}mp \ge 3\sqrt[3]{m^2n^2p^2}.$$

Combining the last two inequalities, we get

$$mab+nbc+pca \ge 4S\sqrt{3\sqrt[3]{m^2n^2p^2}}.$$

If we take $m = n = p = 1$, then we get

$$ab+bc+ca \ge 4S\sqrt{3},$$

which is due to V. E. Olhov, see [7] and [8] in the bibliography on page 4.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 10, 2013.***

Problem 416. If $x_1 = y_1 = 1$ and for $n > 1$,

$$x_n = -3x_{n-1} - 4y_{n-1} + n$$

and $$y_n = x_{n-1} + y_{n-1} - 2,$$

then find x_n and y_n in terms of n only.

Problem 417. Prove that there does not exist a sequence $p_0, p_1, p_2, \ldots$ of prime numbers such that for all positive integer k, p_k is either $2p_{k-1}+1$ or $2p_{k-1}-1$.

Problem 418. Point M is the midpoint of side AB of acute $\triangle ABC$. Points P and Q are the feet of perpendicular from A to side BC and from B to side AC respectively. Line AC is tangent to the circumcircle of $\triangle BMP$. Prove that line BC is tangent to the circumcircle of $\triangle AMQ$.

Problem 419. Let $n \ge 4$. M is a subset of $\{1,2,\ldots,2n-1\}$ with n elements. Prove that M has a nonempty subset, the sum of all its elements is divisible by $2n$.

Problem 420. Find (with proof) all positive integers x and y such that $2x^2y+xy^2+8x$ is divisible by xy^2+2y.

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Solutions

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Problem 411. A and B play a game on a square board divided into 100×100 squares. Each of A and B has a checker. Initially A's checker is in the lower left corner square and B's checker is in the lower right corner square. They take turn to make moves. The rule is that each of them has to move his checker one square up, down, left or right within the board and A goes first. Prove that no matter how B plays, A can always move his checker to meet B's checker eventually.

Solution. **Jon GLIMMS** (Vancouver, Canada) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Suppose the squares are unit length. A can apply the following strategy. After B made the n-th move, let $R(n)$ denote the rectangle bounded by the squares in the same row or same column as one of the two squares containing the checkers. Let $a(n)$ be the length (i.e. long side) and $b(n)$ be the width (i.e. short side) of $R(n)$. As $R(0)$ is consisted of the lowest row squares, $a(0)=100$ and $b(0)=1$. Following the rules, A can always make a move to decrease the length of $R(n)$. After B made $n+1$ moves, $a(n+1)+b(n+1)$ will either be $a(n)+b(n)$ or $a(n)+b(n)-2$. In particular, $a(n)+b(n)$ is always odd, non-increasing and $a(n) > b(n)$. Since the side of the board is finite, eventually $a(n)+b(n)$ must decrease to 3 and A can move his checker to meet B's checker in the next move.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)) and **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 412. $\triangle ABC$ is equilateral and points D, E, F are on sides BC, CA, AB respectively. If

$$\angle BAD + \angle CBE + \angle ACF = 120^\circ,$$

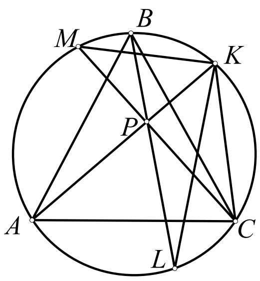
then prove that $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$ cover $\triangle ABC$.
(*Source*: 2006 *Indian Math Olympiad Team Selection Test*)

Solution. **Jon GLIMMS** (Vancouver, Canada) and **William PENG**.

Assume P is in $\triangle ABC$ not covered by $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$. Then $\angle BAD < \angle BAP$, $\angle CBE < \angle CBP$ and $\angle ACF < \angle ACP$. Adding these, we have

$$120^\circ < \angle BAP + \angle CBP + \angle ACP.$$

Now P cannot be the circumcenter of $\triangle ABC$ (otherwise $\angle BAP + \angle CBP + \angle ACP = 90^\circ$ would contradict the inequality above). So PA, PB, PC are not all equal. Suppose $PA > PB$. Let rays AP, BP, CP intersect the circumcircle of $\triangle ABC$ at points K, L, M respectively.



Since $\angle BAP = \angle KLP$ and $\angle ABP = \angle LKP$, $\triangle ABP$ and $\triangle LKP$ are similar. Then $PA > PB$ implies $PL > PK$ and so $\angle BAP = \angle KLP < \angle LKP$. We get

$$\begin{aligned} &\angle BAP + \angle CBP + \angle ACP \\ &= \angle KLP + \angle CKL + \angle AKM \\ &< \angle LKP + \angle CKL + \angle AKM \\ &< \angle BKC = 120^\circ, \end{aligned}$$

which contradicts the inequality above.

Other commended solvers: **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)) and **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France).

Problem 413. Determine (with proof) all integers $n \ge 3$ such that there exists a positive integer M_n satisfying the condition for all n positive numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} \le M_n \left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}} + \frac{a_1}{a_n} \right).$$

(*Source*: 2005 *Chinese Taipei Math Olympiad Team Selection Test*)

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

For $n=3$, let $a_1, a_2, a_3 > 0$ and

$$x = \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_1}{a_3}.$$

Suppose $a_3 \ge a_1, a_2$. Then $x > a_2/a_1, a_3/a_2, a_1/a_3$. So $a_2 > a_3/x$ and $a_1 > a_2/x > a_3/x^2$. Hence,

$$\frac{a_1 + a_2 + a_3}{\sqrt[3]{a_1 a_2 a_3}} \le \frac{3a_3}{\sqrt[3]{\frac{a_3}{x^2}\frac{a_3}{x}a_3}} = 3x.$$

So we can take $M_3=3$. For $n>3$, assume there is such M_n. Let $a_1 = c, a_2 = c^2, \ldots, a_n = c^n$. Then

$$M_n \ge \frac{c + c^2 + \cdots + c^n}{\sqrt[n]{c^{n(n+1)/2}}} \left((n-1)c + \frac{1}{c^{n-1}} \right)^{-1}$$

$$\ge \frac{c^n}{c^{(n+1)/2}} \frac{1}{c((n-1)+c^{-n})} = \frac{c^{(n-3)/2}}{n-1+c^{-n}}.$$

As $c \to \infty$, $c^{(n-3)/2}/(n-1+c^{-n}) \to \infty$. Then M_n cannot be finite, contradiction.

Problem 414. Let p be an odd prime number and $a_1, a_2, \ldots, a_{p-1}$ be positive integers not divisible by p. Prove that there exist integers $b_1, b_2, \ldots, b_{p-1}$, each equals 1 or -1 such that

$$a_1b_1+a_2b_2+\cdots+a_{p-1}b_{p-1}$$

is divisible by p.

Solution. **Jon GLIMMS** (Vancouver, Canada).

For $k = 1, 2, \ldots, p - 1$, we will prove the numbers of the form $a_1c_1 + a_2c_2 + \cdots + a_kc_k$ (where each c_i is 0 or 1) when divided by p will yield at least $k + 1$ different remainders. For $k = 1$, we are given that $a_1 \not\equiv 0 \pmod p$.

Suppose a case $k < p-1$ is true. For the case $k+1$, if the numbers $a_1c_1+a_2c_2+\cdots+a_kc_k$ when divided by p yield at least $k+2$ different remainders, then the case $k+1$ is also true. Otherwise, there are numbers $m_1, m_2, \ldots, m_{k+1}$ of the form $a_1c_1+a_2c_2+\cdots+a_kc_k$ when divided by p yield exactly $k+1$ different remainders. Considering (mod p), we see m_1+a_{k+1}, $m_2+a_{k+1},\ldots, m_{k+1}+a_{k+1}$ also have $k+1$ different remainders.

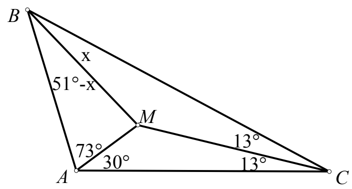
Assume these two groups of $k+1$ remainders are the same. Then we get $m_1+m_2+\cdots+m_{k+1} \equiv (m_1+a_{k+1}) + (m_2+a_{k+1}) + \cdots +(m_{k+1}+a_{k+1}) \pmod p$. This implies $(k+1)a_{k+1}\equiv 0 \pmod p$, which is not possible as $k+1<p$ and a_{k+1} is not divisible by p. Hence, there must be at least $k+2$ different remainders among the two groups. So the case $k+1$ is true.

Let $S=a_1+a_2+\cdots+a_{p-1}$. Since $\gcd(2,p) = 1$, there is an integer r such that $2r \equiv S \pmod p$. From the case $k = p - 1$ above, we see there is $a_1c_1 + a_2c_2 + \cdots + a_{p-1}c_{p-1} \equiv r \pmod p$. Let $b_i = 1 - 2c_i$, then $b_i = \pm 1$ and $a_1b_1 + a_2b_2 + \cdots + a_{p-1}b_{p-1} \equiv S-2r \equiv 0 \pmod p$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 415. (*Due to MANOLOUDIS Apostolos, Piraeus, Greece*) Given a triangle ABC such that $\angle BAC = 103^\circ$ and $\angle ABC = 51^\circ$. Let M be a point inside $\triangle ABC$ such that $\angle MAC = 30^\circ$ and $\angle MCA = 13^\circ$. Find $\angle MBC$ with proof.

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Adrian Iain LAM** (St. Paul's College), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



Let $x = \angle MBC$. By the trigonometric form of Ceva's theorem, we have

$$\frac{\sin 13^\circ}{\sin 13^\circ}\,\frac{\sin 73^\circ}{\sin 30^\circ}\,\frac{\sin x}{\sin(51^\circ - x)} = 1.$$

Then $2\sin 73^\circ = \dfrac{\sin 51^\circ \cos x - \cos 51^\circ \sin x}{\sin x}$

$= \sin 51^\circ \cot x - \cos 51^\circ$.

Using $\sin 73^\circ = \cos 17^\circ$, we get

$$\cot x = (2\cos 17^\circ+\cos 51^\circ)/\sin 51^\circ. \quad (*)$$

Since cot is strictly decreasing on $(0^\circ,51^\circ)$, there is at most one such x. Now we have

$$\begin{aligned}2\sin y \cos y &= \sin 2y\\ &= \sin(3y-y)\\ &= \sin 3y \cos y - \cos 3y \sin y.\end{aligned}$$

Dividing by $\sin y$ leads to

$$2\cos y = \sin 3y \cot y - \cos 3y.$$

Solving for $\cot y$ and setting $y=17^\circ$, we get

$$\cot 17^\circ = (2\cos 17^\circ+\cos 51^\circ)/\sin 51^\circ.$$

Therefore, $x = 17^\circ$.

Other commended solvers: **Christian Pratama BUNAIDI** (University of Tarumanagara, Jakarta, Indonesia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Prithwijit DE** (HBCSE, Mumbai, India), **Uma GIRISH** (Vidya Mandir Senior Secondary School, Chennai, India), **KWOK Man Yi** (S2, Baptist Lui Ming Choi Secondary School), **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France), **Mihai STOENESCU** (Bischwiller, France) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Olympiad Corner

(continued from page 1)

Problem 3. Trapzoid $ABCD$ with a longer base AB is inscribed in the circle k. Let A_0, B_0 be respectively the midpoints of segments BC, CA. Let N be the foot of the altitude from the point C to AB, and G the centroid of the trangle ABC. Circle k_1 goes through A_0 and B_0 and touches the circle k in the point X, different than C. Prove that the points D, G, N and X are collinear.

(*IMO Shortlist 2011, modified*)

Problem 4. For a given positive integer k let $S(k)$ denote the sum of all numbers from the set $\{1,2\ldots,k\}$ relatively prime to k. Let m be a positive integer and n an odd positive integer. Prove that there exist positive integers x and y such that m divides x and $2S(x) = y^n$.

(*Columbia 2008*)

The Inequality of A. Oppenheim

(continued from page 2)

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Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2013 International Mathematical Olympiad.

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers m_1, m_2, …, m_k (not necessarily different) such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

• no line passes through any point of the configuration;
• no region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 8, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2013 – Leader Report (I)

Leung Tat-Wing

The 54th International Mathematical Olympiad (IMO) was held in Santa Marta, Colombia from July 18th to July 28th, 2013. It took me 40 hours of flight and waiting time to travel from Hong Kong to Amsterdam, then to Panama City, and then to Barranquilla, Colombia (where the leaders stayed before they met the contestants in Santa Marta after two days of 4½-hour contests held on the mornings of 23rd and 24th of July). Tired and exhausted, I were picked up in the airport of Barranquilla and delivered to Hotel El Prada. We managed to settle down and be prepared for the next two days' Jury meetings. Our team arrived at Santa Marta, three days later, safe and intact, luckily. The next day they still had to travel two hours from Santa Marta to Barranquilla, participating in another opening ceremony, then another two hours back to Santa Marta. It was tough for them. Accommodation was fine though. Contestants stayed in a nice seaside resort hotel (Iratoma), while leaders stayed in a hotel in Barranquilla. They would join the contestants after the two day contests.

Jury meetings were chaired by Maria Losada, a long time veteran of IMO activities. She was very experienced and chaired the meetings well. Interesting to note, she kept on reminding us (leaders) that we should try to form the best possible paper, a paper that can provide intellectual challenge to contestants, that has some aesthetic sense and that allows every contestant to achieve the most. We were also supposed to work out as many possible solutions as possible. We should be able to tell whether a problem is easy, medium and/or hard. Really sometimes I did not know how the goals may be attained or even verified. She also reminded us ethically we should keep the problems with strict security, not to disclose any information to any contestant beforehand, etc. Indeed the Jury meetings were very educational.

After the two days' contests students enjoyed a break. Leaders and deputy leaders had to check the solutions of the contestants, discussed or argued with coordinators and sorted out how many points should be award to contestants. (This process is called coordination). Luckily this year many coordinators were again very experienced. Many of them are old time leaders from Europe and are experienced problem solvers. They were able to discern mistakes made by the contestants (trivial, small or big) and were able to award points accurately. Personally I recognized many of them and I think I have known many of them for at least more than 10 years. That is why little trouble was observed during the coordination process.

The awards (closing) ceremony was held near a historical site, 45 minute drive from the hotel. We were delivered to the site around 7:00 pm. Then the ceremony lasted for more than two hours. Participants were than sent back to the resort for the banquet. That night was surely hectic. The next day we started our trip home. When we arrived at Bogota, we found that the flight from Bogota to Paris was overbooked. Eventually two of us (deputy leader and a member of the team) had to take another flight from Bogota to Frankfurt, then back to Hong Kong, about 10 hours late. Air France is famous (notorious) in terms of scheduling, here is another example. All in all, we did not get delayed too much and we eventually returned home safely. Lucky! Lucky!

Talking about organization of the event, personally I have no problem with the Jury meeting and/or coordination. Accommodation was very nice. However anything concerned a coach (transportation) was simply not good enough. Say, what is the point of waiting for several hours for a bus, then

(continued on page 2)

visit an old town or take a short walk for less than an hour, and then heading back? ? I do not mean to blame the host country. Indeed I want only to illustrate the point that it is such a gigantic and complicated task to host an IMO!

Our team brought home 1 silver and 5 bronze medals. Among 97 teams, we ranked 31. I cannot say that our team did badly. Indeed all our team members managed to get medals, indicating they achieved certain standard. However in these few years, we trailed behind teams like Singapore, Canada, Australia and other teams, not to say the even stronger teams such as China, USA, Korea and Russia, etc. Do we want to do better? Can we recruit better team members? Can we afford time and energy to do that? We have to think about these problems. I can identify some weak points for our team. For example, our team members simply don't like to do geometry and/or combinatorics problems. Our team members usually get stuck in harder problems, presentations and other things. Or perhaps our team members are too much occupied also by other contests? I know for sure IMO team members of teams such as USA, Australia and Canada would not be allowed to compete in other contests such as IOI or IPhO in the same year. Another suggestion is that we do not train our team enough, we have no intensive camp before IMO (compared with China, USA or UK), and perhaps we should start an intensive camp that will also used as a selection criterion of our team. This idea comes from none other than our old team members! We should pause to think about all these for a while, I suppose.

On the other hand, in this IMO, we confirmed that we will host IMO2016, so in 2016, IMO will be held in Hong Kong. Now we just have to do it, and do it right.

I shall discuss the problems of this IMO. First let us see how they were selected. Indeed the host country (Problem Selection Committee) shortlisted about 30 problems from hundred or so problems submitted by various countries. In the last few years, the Jury first chose an easy pair (problem 1 and 4), then a hard pair (problem 3 and 6), then a medium pair (problem 2 and 5). The 6 selected problems will be then juggled to form the papers. However this year, it was proposed (and accepted) 4 easy problems in algebra, combinatorics, geometry and number theory were selected. Likewise 4 medium problems again from the different topics were selected. Then two easy problems were selected from the 4 easy problems, say problems of algebra and conbinatorics were selected. The medium problems of other topics (geometry and number theory) were automatically selected as the medium pair. The idea is to guarantee problems of all topics be selected either as an easy problem or a medium problem. After that it doesn't matter what problems were selected as the hard pair. However, perhaps the end result was not as ideal as we wanted. Eventually in this IMO there are two synthetic geometry problems (Problem 3 and 4). Problem 2, which was supposed to be a combinatorics problem, is actually a problem of combinatorial geometry. Problem 6, which is a combinatorics problem, also has some geometry favor. Problem 1, which was supposed to be a number theory problem, is more like an algebra problem (no prime numbers, no factorization of integers, merely algebraic manipulation and some induction). And finally of course problem 5 is a problem of functional inequalities. So this paper is very much skewed to geometry and with no number theory. Can we say it is balanced? Really at the very beginning, the problems selected were not quite balanced. The problem selection committee suggested there were no easy combinatorics problems and no hard geometry problems! In short, Jury members tended to select problems that demand "ad hoc" considerations, no need to resort to more advanced techniques and/or theorems.

(For the statement of the problems, please see the Olympiad Corner on page 1-*Ed.*)

Problem 1: Problem 1 and 4 (easy pair) turned out to be too easy. Many strong teams get full score in these two problems. For $k = 1$, we have

$$1+\frac{2^1-1}{n}=1+\frac{1}{n},$$

and it is already of the required form. Hence it is natural to solve the problem using some kind of induction procedure. Essentially all of us did the problem using iterations. One of our team members did the problem as follows. Denote the statement that $1+(2^k-1)/n$ is of the form

$$\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\ldots\left(1+\frac{1}{m_k}\right)$$

by $S(n,k)$. Note that

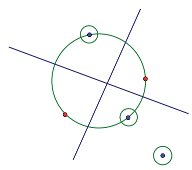
$$1+\frac{2^{k+1}-1}{2n}=\left(1+\frac{1}{2n+2^{k+1}-2}\right)\left(1+\frac{2^k-1}{n}\right),$$

hence if $S(n,k)$ is valid, so is $S(2n,k+1)$. Likewise

$$1+\frac{2^{k+1}-1}{2n-1}=\left(1+\frac{1}{2n-1}\right)\left(1+\frac{2^k-1}{n}\right).$$

Hence if $S(n,k)$ is valid, so is $S(2n–1,k+1)$. Clearly the cases $S(n,1)$ or $S(1,k)$ are valid. Hence by reducing the cases $S(2n,k)$ to $S(n,k–1)$, or $S(2n–1,k)$ to $S(n,k–1)$, (odd or even cases), one can always obtain the cases $S(p,1)$ or $S(1,q)$, and we are done.

Problem 2: All our members guessed the correct answer. The trouble is how to present a proof that is complete (no missing cases). Jury members also worried students didn't realize the minimum value of k should work for all possible configurations. Thus they defined the term "Colombian". (Another definition is the "beautiful" labeling in problem 6. In my opinion it was quite unnecessary.) First we show $k \geq 2013$. Indeed we mark 2013 red points and 2013 blue points alternately on a circle, (and another blue point elsewhere), then there are 4026 arcs formed. All these arcs have two endpoints of different colors and there must be a line passing through an arc to separate the two points, also each line passing through an arc will meet another arc only once, so we see at least 4026/2=2013 lines are needed.



A case of 2 red points and 3 blue points

Now we have to show $k = 2013$ is indeed enough. The official solution goes as follows. First if there are two points of the same color, say A and B, then one can draw two lines parallel to AB, and are sufficiently close and there are only two points between these lines, namely A and B. This statement is intuitively clear. Draw the convex hull P of the points, and there are two cases.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 8, 2013.***

Problem 426. Real numbers a, b, x, y satisfy the property that for all positive integers n, $ax^n+by^n=1+2^{n+1}$. Determine (with proof) the value of x^a+y^b.

Problem 427. Determine all (m,n,k), where m, n, k are integers greater than 1, such that $1!+2!+\cdots+m!=n^k$.

Problem 428. Let $A_1A_2A_3A_4$ be a convex quadrilateral. Prove that the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$ and $\Delta A_4A_1A_2$ pass through a common point.

Problem 429. Inside ΔABC, there is a point P such that $\angle APB=\angle BPC=\angle CPA$. Let $PA=u$, $PB=v$, $PC=w$, $BC=a$, $CA=b$ and $AB=c$. Prove that

$$(u+v+w)^2 \le ab+bc+ca - \left(\sqrt{a(b+c-a)}-\sqrt{b(c+a-b)}\right)^2.$$

Problem 430. Prove that among any $2n+2$ people, there exist two of them, say A and B, such that there exist n of the remaining $2n$ people, each either knows both A and B or does not know A nor B. Here, *x knows y* does not necessarily imply *y knows x*.

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Solutions

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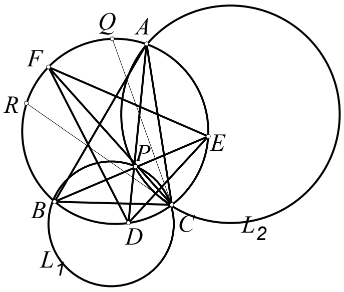
Problem 421. For every acute triangle ABC, prove that there exists a point P inside the circumcircle ω of ΔABC such that if rays AP, BP, CP intersect ω at D, E, F, then $DE: EF: FD = 4:5:6$.

Solution. **Jon GLIMMS** (Vancouver, Canada), **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **William PENG**.

For such a point P, let us apply the exterior angle theorem to ΔABP and ΔACP. Then we have

$$\angle BPC=\angle BAC+\angle ABE+\angle ACF$$
$$=\angle BAC+\angle FDE.$$

Similarly, $\angle CPA=\angle CBA+\angle DEF$.



To get such a point P, we first draw ΔXYZ with $XY=4$, $YZ=5$ and $ZX=6$. Let $\alpha=\angle ZXY$ and $\beta=\angle XYZ$. Next we consider the locus L_1 of point P such that $\angle BPC=\angle BAC+\alpha$, which is a circle through B and C. Also, let L_2 be the locus of point P such that $\angle CPA=\angle CBA+\beta$, which is a circle through C and A.

Let the tangents to L_1 and L_2 at C intersect ω at Q and R. Then

$$\angle QCB+\angle RCA$$
$$=180^\circ-(\angle BAC+\alpha)+180^\circ-(\angle CBA+\beta)$$
$$=\angle ACB+\angle YZX>\angle ACB.$$

This implies L_1 and L_2 intersect at a point P inside ω. Define D, E, F as in the statement of the problem. From the last two paragraphs, we get $\angle ZXY=\alpha=\angle FDE$ and $\angle XYZ=\beta=\angle DEF$. These imply ΔDEF and ΔXYZ are similar. Therefore, $DE: EF: FD = 4:5:6$.

Problem 422. Real numbers $a_1, a_2, a_3, \ldots$ satisfy the relations

$$a_{n+1}a_n+3a_{n+1}+a_n+4=0$$

and $a_{2013}\le a_n$ for all positive integer n. Determine (with proof) all the possible values of a_1.

Solution. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 4), **Jon GLIMMS** (Vancouver, Canada), **William PENG** and **TAM Pok Man** (Sing Yin Secondary School, Form 6).

The recurrence relation can be written as $(a_{n+1}+2)(a_n+2)=(a_n+2)-(a_{n+1}+2)$. If $a_i=-2$ for some i, then all $a_n=-2$ by induction. So $a_1=-2$ is a possible value. Suppose no $a_i=-2$. Then

$$\frac{1}{a_{n+1}+2}=1+\frac{1}{a_n+2}.$$

Letting $b_n=1/(a_n+2)$, we easily get $b_n=n-1+b_1\neq 0$ for all positive integer n. Then $b_1\neq 0,-1,-2,\ldots$ and $a_n=-2+1/(n-1+b_1)$. Now for positive integer n, a_n is least when $n-1+b_1<0$ and nearest 0, i.e.

$$n-1+b_1<0<n+b_1.$$

Setting $n=2013$ and $b_1=1/(a_1+2)$, we can solve the inequality to get

$$-\frac{4025}{2012}<a_1<-\frac{4027}{2013}.$$

Other commended solvers: **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 423. Determine (with proof) the largest positive integer m such that a $m\times m$ square can be divided into seven rectangles with no two having any common interior point and the lengths and widths of these rectangles form the sequence 1,2,3,4,5,6,7,8,9,10, 11,12,13,14.

Solution. **Jon GLIMMS** (Vancouver, Canada), **William PENG** and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Let $a_1, a_2, a_3, a_4, \ldots, a_{2n-1}, a_{2n}$ be a permutation of $1, 2, 3, 4, \ldots, 2n-1, 2n$. We claim the maximum of $a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}$ is $S_n=1\times 2+3\times 4+\cdots+(2n-1)\times 2n$. The cases $n=1$ or 2 can be checked. Suppose cases 1 to n are true. For the case $n+1$, if $(2n+1)(2n+2)$ is one of the term, then we can switch it with the last term and apply the case n to get

$$a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}+(2n+1)(2n+2)$$
$$\le S_n+(2n+1)\times(2n+2)=S_{n+1}.$$

Otherwise, $2n+1$ and $2n+2$ are in different terms. We can switch terms so that $a_{2n-1}=2n+1$ and $a_{2n+1}=2n+2$. If we try switching $(2n+1)a_{2n}+(2n+2)a_{2n+2}$ to $a_{2n}a_{2n+2}+(2n+1)(2n+2)$, then since a_{2n} and a_{2n+2} are at most $2n$, we have

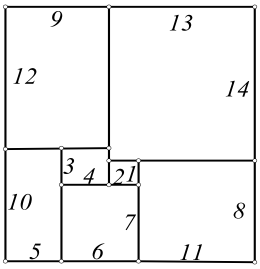
$$[(2n+2)-a_{2n}][(2n+1)-a_{2n+2}]>0.$$

Expanding, we see

$$a_{2n}a_{2n+2}+(2n+1)(2n+2)$$
$$>(2n+1)a_{2n}+(2n+2)a_{2n+2}$$
$$=a_{2n-1}a_{2n}+a_{2n+1}a_{2n+2}.$$

Adding $a_1a_2+a_3a_4+\cdots+a_{2n-3}a_{2n-2}$ and using case $n-1$, we see S_{n+1} again is the maximum.

For the problem, the claim implies $m^2\le S_7=1\times 2+3\times 4+\cdots+13\times 14=504$. Then $m\le 22$. To finish, we show a 22×22 square which can be so divided.



Other commended solvers: **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 424. (*Due to Prof. Marcel Chirita, Bucuresti, Romania*) In ΔABC, let $a=BC$, $b=CA$, $c=AB$ and R be the circumradius of ΔABC. Prove that

$$\max(a^2+bc, b^2+ca, c^2+ab) \ge \frac{2\sqrt{3}abc}{3R}.$$

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **TAM Pok Man** (Sing Yin Secondary School, Form 6), **Alex TUNG Kam Chuen** (La Salle College), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

By the extended sine law, $c/\sin C = 2R$. Let $[ABC]$ denote the area of ΔABC. Then $[ABC] = ½ab \sin C = abc/(4R)$. So $ab = 2[ABC]/\sin C$. Using these below, we have

$$\begin{aligned}
&3\max(a^2+bc, b^2+ca, c^2+ab)\\
&\ge a^2+bc+b^2+ca+c^2+ab\\
&\ge 2(ab+bc+ca)\\
&= 4[ABC]\left(\frac{1}{\sin C}+\frac{1}{\sin A}+\frac{1}{\sin B}\right)\\
&\ge 4[ABC]\frac{3}{\sin((A+B+C)/3)}\\
&= \frac{2\sqrt{3}abc}{R},
\end{aligned}$$

where the second inequality is by expanding $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$ and the third inequality is by applying Jensen's inequality to $f(x)=1/\sin x$.

Other commended solvers: **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, Form 2).

Problem 425. Let p be a prime number greater than 10. Prove that there exist distinct positive integers $a_1, a_2, \ldots, a_n$ such that $n \le (p+1)/4$ and

$$\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

is a positive integral power of 2.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Alex TUNG Kam Chuen** (La Salle College) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

More generally, we prove this is true for all odd integers $p \ge 3$. Let

$$X = \frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

If $p \equiv 1 \pmod 4$, then let $n=(p-1)/4$ and for $i=1,2,\ldots,n$, let $a_i = 2i-1$. We have

$$\begin{aligned}
X &= \frac{4n(4n-2)\cdots(2n+2)}{1\cdot 3\cdots(2n-1)}\\
&= \frac{4n(4n-2)\cdots(2n+2)}{1\cdot 3\cdots(2n-1)} \times \frac{2\cdot 4\cdots(2n)}{2\cdot 4\cdots(2n)}\\
&= 2^{2n}.
\end{aligned}$$

If $p \equiv 3 \pmod 4$, then let $n=(p+1)/4$ and for $i=1,2,\ldots,n$, let $a_i = 2i-1$. We have

$$\begin{aligned}
X &= \frac{(4n-2)(4n-4)\cdots(2n)}{1\cdot 3\cdots(2n-1)}\\
&= \frac{(4n-2)(4n-4)\cdots(2n)}{1\cdot 3\cdots(2n-1)} \times \frac{2\cdot 4\cdots(2n-2)}{2\cdot 4\cdots(2n-2)}\\
&= 2^{2n-1}.
\end{aligned}$$

Olympiad Corner

(continued from page 1)

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Problem 4. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y, H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let f: $\mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x,y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \ge f(xy)$;
(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \ge f(x)+f(y)$;
(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 6. Let $n \ge 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0,1,\ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a+d = b+c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x,y) of positive integers such that $x+y \le n$ and $gcd(x,y)=1$. Prove that $M = N+1$.

IMO 2013–Leader Report (I)

(continued from page 2)

<u>Case 1.</u> If there is a red point A on the convex hull P, we can draw a line separating A draw all other points. Then we pair up the remaining 2012 red points into 1006 pairs, and as remarked, draw 1006 pairs of parallel lines (2012 lines), separating each pair of red points from all other points. Thus 2012+1=2013 lines are needed.

<u>Case 2.</u> All vertices of the convex hull P are blue. Take any pair of consecutive blue points A and B, separating them from all other points by a line (one line) parallel to AB. Then pair up the remaining 2012 blue points into 1006 pairs as before, separating each pair from all other points by 1006 pairs of parallel lines (2012 lines). Thus again 2013 lines are used.

(To be continued)

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the North Korean Team Selection Test for IMO 2013.

Problem 1. The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at A_4 ($\neq A_2$). Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 21, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Sequences

Kin Y. Li

Sequence problems occur often in math competitions. Below we will look at some of these problems involving limits in their solutions.

Example 1. (1980 *British Math Olympiad*) Find all real a_0 such that the sequence defined by $a_{n+1}=2^n-3a_n$ for n=0,1,2,… satisfies $a_0<a_1<a_2<\cdots$.

Solution. We have

$$\begin{aligned}a_{n+1} &= 2^n-3a_n = 2^n-3\times 2^{n-1}+3^2a_{n-1}\\ &=\cdots = 2^n-\sum_{j=1}^{n}(-3)^j2^{n-j}+(-3)^{n+1}a_0\\ &=\frac{2^{n+1}}{5}+\left(a_0-\frac{1}{5}\right)(-3)^{n+1}.\end{aligned}$$

If a_0=1/5, then it is good. If $a_0\neq 1/5$, then since $(2/3)^n$ goes to 0 as $n\to\infty$, so $a_n/3^n$ will have the same sign as $(a_0-1/5)(-1)^n$ when n is large. Hence, $a_n<a_{n+1}$ will not hold, contradiction.

Example 2. (1971-1972 *Polish Math Olympiad*) Prove that when n tends to infinity, the sum of the digits of 1972^n in base 10 will go to infinity.

Solution. Let a_i be the i-th digit of 1972^n from right to left in base 10. For $1\le k\le n/4$, we claim that among $a_{k+1}, a_{k+2},\ldots, a_{4k}$, at least one of them is nonzero.

Assume not. Then let

$$C = a_1+a_2\times 10+\cdots+a_k\times 10^{k-1}.$$

We have 1972^n-C divisible by 10^{4k}. Since $4k\le n$, so C is divisible by $2^{4k}=16^k>10^k>C$, contradiction.

From the claim, we get at least one digit in each of the following m+1 groups of digits will not be zero

$$\begin{aligned}&a_2,a_3,a_4,\\ &a_5,a_6,a_7,\ldots,a_{16},\\ &\ldots\\ &a_{j+1,}a_{j+2},a_{j+3},\ldots,a_{4j},\end{aligned}$$

where $n/16<j=4^m\le n/4$. The digit sum of 1972^n is at least $m+1>(\log_4 n)-1$. So, the digit sum of 1972^n goes to infinity.

Example 3. Let a_1, a_2, a_3, … be a sequence of positive numbers. Prove that there exists infinitely many n such that $1+a_n>2^{1/n}a_{n-1}$.

Solution. Assume not. Then there is a M such that for all $n>M$, we have $1+a_n\le 2^{1/n}a_{n-1}$. Since $(1+1/n)^n\ge 2$, we have

$$a_n\le 2^{1/n}a_{n-1}-1\le((n+1)/n)a_{n-1}-1.\ (*)$$

We claim that for $k\ge M$,

$$a_k\le(k+1)\left(\frac{a_M+1}{M+1}-\sum_{j=M+1}^{k+1}\frac{1}{j}\right).$$

The case $k=M$ is true as the right side is a_M. Suppose case k is true. By (*),

$$\begin{aligned}a_{k+1}&\le\frac{k+2}{k+1}a_k-1=\frac{k+2}{k+1}a_k-\frac{k+2}{k+2}\\ &\le(k+2)\left(\frac{a_M+1}{M+1}-\sum_{j=M+1}^{k+2}\frac{1}{j}\right).\end{aligned}$$

This concludes the induction. As $k\to\infty$, the above sum of $1/j$ goes to infinity, hence some $a_{k+1}<0$, contradiction.

Example 4. (2007 *Chinese Math Olympiad*) Let $\{a_n\}_{n\ge 1}$ be a bounded sequence satisfying

$$a_n<\sum_{k=n}^{2n+2006}\frac{a_k}{k+1}+\frac{1}{2n+2007},\quad n=1,2,3,\ldots.$$

Prove that $a_n<1/n$ for $n=1,2,3,\ldots$.

Solution. Let $b_n=a_n-1/n$. Then for $n\ge 1$,

$$b_n<\sum_{k=n}^{2n+2006}\frac{b_k}{k+1}. \qquad (*)$$

It suffices to show $b_n<0$. Since a_n is bounded, so there is a constant M such that $b_n<M$. For $n>100{,}000$, we have

$$\begin{aligned}b_n&<\sum_{k=n}^{2n+2006}\frac{b_k}{k+1}<M\sum_{k=n}^{2n+2006}\frac{1}{k+1}\\ &=M\sum_{k=n}^{[3n/2]}\frac{1}{k+1}+M\sum_{k=[3n/2]+1}^{2n+2006}\frac{1}{k+1}\\ &<\frac{M}{2}+M\frac{2006+n/2}{1+3n/2}<\frac{6}{7}M.\end{aligned}$$

Repeating this m times, if $n>100{,}000$, then $b_n<(6/7)^mM$. Letting $m\to\infty$, we get $b_n\le 0$ for $n>100{,}000$. Using (*), we see if for $n\ge N+1$, we have $b_n<0$, then $b_N<0$. This gives $b_n<0$ for $n\ge 1$.

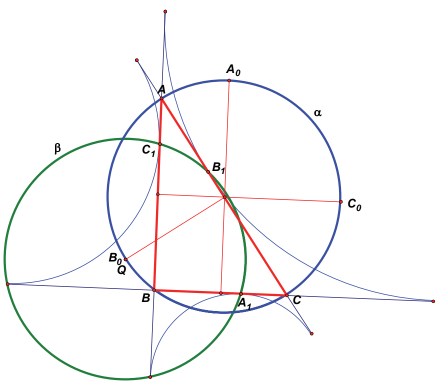
IMO 2013 - Leader Report(II)

Leung Tat-Wing

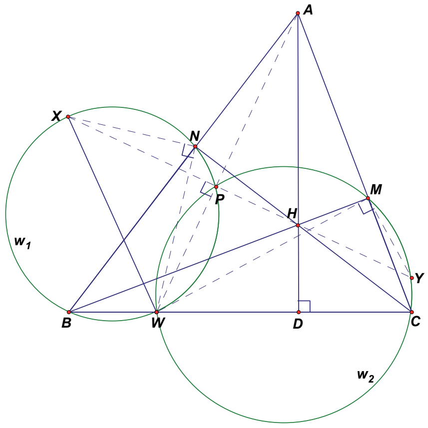
We will continue with our discussion on the IMO 2013 problems, which can be found in the Olympiad Corner of the last issue of *Math Excalibur*.

Problem 3: The problem was selected in the very last minute of the Jury meetings. Indeed another geometry problem concerning properties of hexagons was initially selected as a member of the hard pair. It was however discovered the problem was similar to an USAMO problem. I myself also recalled several similar problems. So the problem was rejected and replaced by this problem 3. After the selection process, it was announced both problem 3 and 6 come from Russia, indeed a problem similar to problem 4 was also found in a Russian geometry problem book. Truly the Russians are masters of posing problems!

Despite being a difficult problem (solved by 40 contestants), problem 3 is indeed a pure geometry problem and can be solved by pure synthetic geometry method. Indeed denote the circumcircles of ABC and $A_1B_1C_1$ by α and β respectively and let Q be the centre of the circumcircle of $A_1B_1C_1$. Let A_0 be the midpoint of arc BC containing A, and define B_0 and C_0 respectively. Then one can check $A_0B_1=A_0C_1$ and A,A_0,B_1,C_1 concyclic. (Likewise $B_0C_1=B_0C_1$ and B,B_0,C_1,A_1 concyclic; $C_0A_1=C_0B_1$ and C,C_0,B_1,A_1 concyclic.) One then consider the largest angle of $A_1B_1C_1$, say B_1, and if Q is on α, then Q must coincide with B_0, and hence $\angle B$=90°, not easy though!



Problem 4: There are more than 19 different solutions and surely there are more. It is possible to solve the problem using complicated angle chasings and/or coordinate geometry. But of course the basic or most natural approach is to look at the radical axis of the two circles. The following proof is given by Lau Chun Ting, a team member of ours.



Suppose ω_1 and ω_2 meet at another point P ($\neq W$). Since $\angle WPX = \angle WPY$ = 90°, so X, P and Y are collinear. To show H lies on XY, (X, Y, H collinear), it suffices to show $\angle HPW$ = 90°. Suppose now AH meets BC at D. Now B, N, M, C are concyclic (since $\angle BNC = \angle BMC$ = 90°), we have $AN\times AB=AM\times AC$. So the powers of the point A with respect to the circles ω_1 and ω_2 are the same, that means A lies on the radical axis WP, or A, P, W collinear (radical axis theorem). Now note that H, M, C, D are also concyclic, hence $AH\times AD=AM\times AC$ (quite a few concyclic conditions). As before

$$AM\times AC=AN\times AB=AP\times AW,$$

we get $AP\times AW=AH\times AD$. Therefore, W, P, H, D are concyclic and we get $\angle HPW$ = 90°, as required.

Using coordinate attack, we may let $A=(a_1,a_2)$, $B=(-b,0)$, $C=(c,0)$ and $W=(0,0)$. By computing slopes and equations of lines, (complicated but still manageable), one eventually gets the coordinates of X, H and Y. Hence can verify X, H and Y collinear by calculating slopes of XH and HY.

Problem 5: For problem of this kind, one can try many things to obtain partial results. But the essential (crucial) part of this problem is actually how to make use of condition 3. Indeed if this condition is released, then the function $f(x) = bx^2$, with $b \geq 1$, will satisfy the first and second condition. Now see what we can get by putting different values of x and y into the equations. For examples, put $x = a$ and $y = 1$, one gets $a\,f(1) = f(a)\,f(1) \geq f(a) = a$, hence $f(1) \geq 1$. We let $f(1) = c \geq 1$. By induction, one can then show $f(n) \geq nc$, for all natural numbers n. So in particular $f(n)$ is positive. Now we show $f(x)$ is strictly increasing. Indeed if $f(x+\Delta x) \leq f(x)$ for some positive rational numbers x and Δx, then

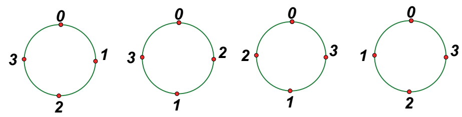
$$f(x) \geq f(x+\Delta x) \geq f(x) + f(\Delta x),$$

therefore $f(\Delta x) \leq 0$. However, we also have $f(n)\,f(\Delta x) \geq f(n\Delta x)$. Now since $f(\Delta x) \leq 0$, so we must also have $f(n\Delta x) \leq 0$ for all n, however surely we can find n so that $n\Delta x$ is a natural number and $f(n\Delta x)$ is positive, a contradiction. Using the same argument, we can show $f(x) > 0$ for all positive rational numbers. One then proceeds to show $f(1) = 1$. Hence $f(x) = x$ for all positive rationals. I am not going to produce all the details here. Suffices to say, we often need to expand a positive rational number in terms of a, say for a rational number $b < a$, it is of the form

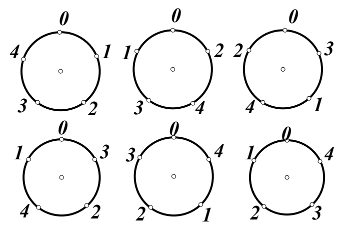
$$k_0 + \frac{k_1}{a} + \frac{k_2}{a^2} + \cdots \quad \text{(finite sum)},$$

some kind of a-adic expansion!

Problem 6: Problem 6 is even harder than problem 3, only 7 contestants solved it. A nice point of the problem is that it links a geometric fact (intersecting chords) to a certain number property, and the relation is an exact relation ($M=N+1$). For n=3, the beautiful labellings are given below (we always label 0 at the top).



The pairs of positive integers satisfying the stated property are (1,1), (1,2) and (2,1). For n=4, to complete the list of integers with the stated property, we just have to consider those x and y satisfying $x+y$=4. Indeed we get two more pairs (1,3) and (3,1). Indeed the six beautiful labellings are



(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 21, 2013.***

Problem 431. There are 100 people, composed of 2 people from 50 distinct nations. They are seated in a round table. Two people sitting next to each other are <u>*neighbors*</u>.

Prove that it is possible to divide the 100 people in two groups of 50 people so that no 2 people from the same nation are in the same group and each person in a group has at most one neighbor in the group.

Problem 432. Determine all prime numbers p such that there exist integers a,b,c satisfying $a^2+b^2+c^2=p$ and $a^4+b^4+c^4$ is divisible by p.

Problem 433. Let P_1, P_2 be two points inside ΔABC. Let $BC=a$, $CA=b$ and $AB=c$. For $i=1,2$, let $P_iA=a_i$, $P_iB=b_i$ and $P_iC=c_i$. Prove that

$$aa_1a_2+bb_1b_2+cc_1c_2\ge abc.$$

Problem 434. Let O and H be the circumcenter and orthocenter of ΔABC respectively. Let D be the foot of perpendicular from C to side AB. Let E be a point on line BC such that $ED\perp OD$. If the circumcircle of ΔBCH intersects side AB at F, then prove that points E, F, H are collinear.

Problem 435. Let $n>1$ be an integer that is not a power of 2. Prove that there exists a permutation $a_1, a_2,\ldots, a_n$ of $1,2,\ldots, n$ such that

$$\sum_{k=1}^{n} a_k\cos\frac{2k\pi}{n}=0.$$

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Solutions

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Problem 426. Real numbers a, b, x, y satisfy the property that for all positive integers n, $ax^n+by^n=1+2^{n+1}$. Determine (with proof) the value of x^a+y^b.

Solution. **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain).

Considering the generating functions of the left and right sides of $ax^n+by^n=1+2^{n+1}$, we have

$$\sum_{n=1}^{\infty}ax^nz^{n-1}+\sum_{n=1}^{\infty}by^nz^{n-1}=\sum_{n=1}^{\infty}z^{n-1}+\sum_{n=1}^{\infty}2^{n+1}z^{n-1}.$$

For $|z|<\min\{1/2,1/|x|,1/|y|\}$, using the geometric series formula, we have

$$\frac{ax}{1-xz}+\frac{by}{1-yz}=\frac{1}{1-z}+\frac{4}{1-2z}.$$

The right side is a rational function of z. By the uniqueness of the partial fraction decomposition, either $ax=1$, $x=1$, $by=4$, $y=2$ or $ax=4$, $x=2$, $by=1$, $y=1$. In both cases, $x^a+y^b=1^1+2^2=5$.

Other commended solvers: **CHAN Long Tin** (Cambridge University, Year 1), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, Form 2), **LO Wang Kin**(Wah Yan College, Kowloon), **Math Group** (Carmel Alison Lam Foundation Secondary School), **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 427. Determine all (m,n,k), where m, n, k are integers greater than 1, such that $1!+2!+\cdots+m!=n^k$.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **LO Wang Kin** (Wah Yan College, Kowloon), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania School, Ploieşti, Romania), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG.**

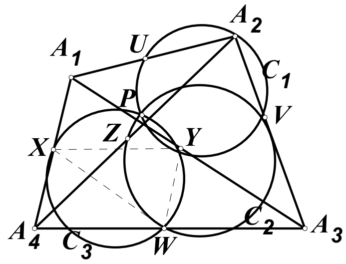
Let $S(m)=1!+2!+\cdots+m!$. Then $S(2)=3$, $S(3)=9=3^2$, $S(4)=33=3\times 11$, $S(5)=153=3^2\times 17$, $S(6)=873=3^2\times 97$, $S(7)=5913=3^4\times 73$, $S(8)=46233=3^2\times 11\times 467$.

For $m>8$, since $9!\equiv 0 \pmod{3^3}$, so $S(m)\equiv S(8)\equiv 0 \pmod{3^2}$ and $S(m)\equiv S(8)\not\equiv 0 \pmod{3^3}$. These imply that if $S(m)=n^k$ and $k>1$, then $k=2$.

Since $S(4)=33\equiv 3 \pmod 5$, $S(m)\equiv 3 \pmod 5$. Now $n^2\equiv 0, 1, 4 \pmod 5$. So $S(m)\ne n^2$. We have the only solution is $(m,n,k)=(3,3,2)$.

Problem 428. Let $A_1A_2A_3A_4$ be a convex quadrilateral. Prove that the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$ and $\Delta A_4A_1A_2$ pass through a common point.

Solution. **HOANG Nguyen Viet** (Hanoi, Vietnam), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania School, Ploieşti, Romania), **Apostolis MANOLOUDIS, Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6).



Let C_1, C_2, C_3, C_4 be the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$, $\Delta A_4A_1A_2$ respectively. Let U, V, W, X, Y, Z be the midpoints of A_1A_2, A_2A_3, A_3A_4, A_4A_1, A_1A_3, A_2A_4 respectively. Let C_1 and C_3 intersect at Y and P (in case C_1, C_3 are tangent, P will be the same as Y). We claim P is on C_2. For that it suffices to show P,V,W,Z are concyclic.

By the midpoint theorem, $XY=\frac{1}{2}A_4A_3=WA_3$ and $XW=\frac{1}{2}A_1A_3=YA_3$. So we have (1) $WXYA_3$ is a parallelogram. Similarly, (2) $YUVA_3$ and (3) $WZVA_3$ are also parallelograms. Now (4) P,U,V,Y are on C_1 and (5) P,X,W,Y are on C_3. We have

$$\begin{aligned}\angle VPW &= \angle YPV+\angle YPW\\ &=\angle YUV+\angle YXW \text{ by (4), (5)}\\ &=\angle YA_3V+\angle YA_3W \text{ by (2), (1)}\\ &=\angle VA_3W\\ &=\angle VZW \text{ by (3).}\end{aligned}$$

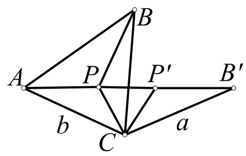
So P is on C_2. Similarly, P is on C_4.

Other commended solvers: **William FUNG, Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 429. Inside ΔABC, there is a point P such that $\angle APB = \angle BPC = \angle CPA$. Let $PA = u$, $PB = v$, $PC = w$, $BC = a$, $CA = b$ and $AB = c$. Prove that

$$(u+v+w)^2 \le ab+bc+ca - \left(\sqrt{a(b+c-a)} - \sqrt{b(c+a-b)}\right)^2.$$

Solution. **LO Wang Kin** (Wah Yan College, Kowloon).



Rotate ΔABC about C by 60° away from A. Let the images of B, P be B', P' respectively. As $\angle PCP' = 60° = \angle BCB'$, so $\Delta PCP'$ and $\Delta BCB'$ are equilateral. As $\angle B'PC = \angle CPA = 120°$, A, P, P', B' are collinear. So $AB' = AP+PP'+P'B' = u+w+v$. By the cosine law, $AB'^2 = a^2+b^2-2ab\cos(C+60°)$.

After expansion and cancellation, the right side of the desired inequality becomes

$$a^2+b^2-ab+2\sqrt{ab(b+c-a)(c+a-b)}.$$

Now

$$\begin{aligned}&\sqrt{ab(b+c-a)(c+a-b)}\\ &= ab\sqrt{2\left(1-\frac{a^2+b^2-c^2}{2ab}\right)}\\ &= ab\sqrt{2(1-\cos C)}.\end{aligned}$$

Using these, the right side minus the left side of the desired inequality is

$$\begin{aligned}&ab(-1+2\sqrt{2(1-\cos C)}+2\cos(C+60°))\\ &= ab(-1+2\sqrt{2(1-\cos C)}+\cos C-\sqrt{3}\sin C)\\ &= 2ab\sqrt{1-\cos C}(\sqrt{2}-\sqrt{1+\cos(C-60°)}) \ge 0\end{aligned}$$

and we are done.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **T. W. LEE** (Alumni of New Method College), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6).

Problem 430. Prove that among any $2n+2$ people, there exist two of them, say A and B, such that there exist n of the remaining $2n$ people, each either knows both A and B or does not know A nor B. Here, *x knows y* does not necessarily imply *y knows x*.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **Math Group** (Carmel Alison Lam Foundation Secondary School).

Take a person P out of the $2n+2$ people. Suppose among the remaining $2n+1$ people, he knows k of them and does not know $2n+1-k$ of them. Among these $2n+1$ people, there are ${}_{2n+1}C_2 = n(2n+1)$ pairs. Call a pair <u>*good*</u> if P knows both of them or does not know both of them, <u>*bad*</u> if P knows one, but not both. By the AM-GM inequality, there are at most $\lceil k(2n+1-k)\rceil \le \lceil (n+½)^2\rceil = n^2+n$ bad pairs. Adding up all the bad pairs for all $2n+2$ people, the number is at most $(2n+2)(n^2+n) = 2n(n+1)^2$. There are ${}_{2n+2}C_2 = (n+1)(2n+1)$ pairs altogether. Since the average

$$\frac{2n(n+1)^2}{(n+1)(2n+1)} = \frac{2n(n+1)}{2n+1} < n+1,$$

some pair $\{A, B\}$ will be a bad pair for at most n of the remaining $2n$ people. Then at least n other people will call $\{A,B\}$ a good pair and we are done.

Olympiad Corner

(*Continued from page 1*)

Problem 2. Let $a_1, a_2, \ldots, a_k$ be numbers such that $a_i \in \{0,1,2,3\}$, $i=1$ to k and $z = (x_k, x_{k-1}, \ldots, x_1)_4$ be a base 4 expansion of $z \in \{0, 1, 2, \ldots, 4^k-1\}$. Define A as follows:

$A = \{z \mid p(z) = z, z = 0,1,2,\ldots,4^k-1\}$, where

$$p(z) = \sum_{i=1}^{k} a_i x_i 4^{i-1}.$$

Prove that $|A|$ is a power of 2. ($|X|$ denotes the number of elements in X).

Problem 3. Find all $a,b,c \in \mathbb{Z}$, $c \ge 0$ such that $(a^n+2^n) \mid b^n+c$ for all positive integers n, where $2ab$ is non-square.

Problem 4. Positive integers 1 to 9 are written in each square of a 3×3 table. Let us define an operation as follows: Take an arbitrary row or column and replace these numbers a, b, c with either non-negative numbers $a-x$, $b-x$, $c+x$ or $a+x$, $b-x$, $c-x$, where x is a positive number and can vary in each operation.

1) Does there exist a series of operations such that all 9 numbers turn out to be equal from the following initial arrangement a) ?, b) ?

1	2	3
4	5	6
7	8	9

a)

2	8	5
9	3	4
6	7	1

b)

2) Determine the maximum value which all 9 numbers turn out to be equal to after some steps.

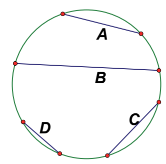
Problem 5. The incircle ω of a quadrilateral $ABCD$ touches AB, BC, CD, DA at E, F, G, H, respectively. Choose an arbitrary point X on the segment AC inside ω. The segments XB, XD meet ω at I, J respectively. Prove that FJ, IG, AC are concurrent.

Problem 6. Show that $x^3+x+a^2=y^2$ has at least one pair of positive integer solution (x,y) for each positive integer a.

IMO 2013–Leader Report (II)

(continued from page 2)

The problem is how to connect the geometry and the number theory information. In general, how to get started? I can only describe it roughly from the official solution. Call three chords *aligned* if one of them separates the other two. For more than three chords, they are aligned if any three of them aligned.



In the figure the chords A, B and C are aligned (the line formed by B separated the two chords A and C; while B, C and D are not aligned (none of the lines formed by B, C or D separates the other two chords). Now call a chord a *k-chord* if the sum of its two endpoints is k (the chord may be degenerated into a point of value k). The crucial observation is: in a beautiful labeling, the k-chords are aligned for any k. To prove this claim, one proceeds by induction. Indeed the only case is when there are three chords not aligned and such that one of the chords has endpoints 0 and n. After the claim is proved, one proceeds again using delicate induction arguments to show $M=N+1$. Indeed the beautiful labellings are eventually divided into classes. Elements of the first class are as before in the induction step. Elements of the second class correspond precisely with the pairs of positive integers satisfying $x+y=n$ and $\gcd(x,y)=1$, (which correspond exactly to the elements $\{x \mid 1 \le x \le n, \gcd(x,y) = 1\}$ with size $\varphi(n)$. Tough!

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Dutch Team Selection Test for IMO 2013.

Problem 1. Show that

$$\sum_{n=0}^{2013}\frac{4026!}{\left(n!(2013-n)!\right)^2}$$

is the square of an integer.

Problem 2. Let P be the intersection of the diagonals of a convex quadrilateral $ABCD$. Let X, Y and Z be points on the interior of AB, BC and CD respectively such that

$$\frac{AX}{XB}=\frac{BY}{YC}=\frac{CZ}{ZD}=2.$$

Suppose moreover that XY is tangent to the circumcircle of ΔBXY. Show that $\angle APD=\angle XYZ$.

Problem 3. Fix a sequence a_1, a_2, a_3, … of integers satisfying the following condition: for all prime numbers p and all positive integers k, we have

$$a_{pk+1}=pa_k-3a_p+13.$$

Determine all possible values of a_{2013}.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 25, 2014***.

For individual subscription for the next five issues for the 13-14 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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PUMaC 2013

Andy Loo (Princeton University)

In the United States there are several annual math competitions organized by undergraduate students at different universities for high school enthusiasts, including the Harvard-MIT Math Tournament (HMMT), the Stanford Math Tournament (SMT), and, last but not least, the Princeton University Mathematics Competition (PUMaC). Started in 2006, PUMaC has grown into an international event in which high schoolers across America are joined by teams from as far away as Bulgaria and China on Princeton campus each year.

PUMaC 2013 was held on November 16, engaging over 600 participants, and I was honored to serve as Problem Tsar (academic coordinator who heads the problem writing team). The responsibility of creating, grading and defending the problems and solutions of a competition of such scale and repute gave me an inspiring learning experience.

The competition is split into Division A (more challenging) and Division B (for less experienced contestants). Each team consists of eight students. In the morning, each contestant takes two out of four one-hour answer-only individual tests (Algebra, Geometry, Combinatorics and Number Theory, eight problems each) of his/her choice, followed by the one-hour Team Round, where members of the same team may discuss and work together (each team enjoying a separate room!).

The top 10 performers on each individual test (possibly with nonempty intersection) qualify for the Individual Finals, a one-hour proof-based test with three problems. I personally feel that an average Individual Finals problem lies somewhere near an IMO problem 1 or 4 in terms of difficulty. Remarkably, in PUMaC 2013, two contestants got a perfect score on the Division A Individual Finals despite the time pressure! Also worth mentioning is the Power Round, which is a relatively long series of problems revolving around a central theme – knot theory in 2013 – released one week before the competition day for the teams to work on and turn in on competition day. (Teams may also enroll on a Power Round-only basis.) It usually takes frantic grading to determine the individual and team rankings in time for the award ceremony in the late afternoon, while mini-events such as Math (quiz) Bowl and Rubik's cube as well as a lecture by a Princeton professor keep the participants entertained.

I would like to discuss a few problems in PUMaC 2013, not necessarily because they are the hardest, but mostly because they bring out certain lessons of problem solving we can learn.

Individual Finals B1.

Let $a_1 = 2013$ and $a_{n+1} = 2013^{a_n}$ for all positive integers n. Let $b_1 = 1$ and $b_{n+1} = 2013^{2012b_n}$ for all positive integers n. Prove that $a_n > b_n$ for all positive integers n.

At first sight, one natural reaction to this problem would be to do induction. However, we would quickly realize that the assumption $a_n > b_n$ does not imply $a_{n+1} > b_{n+1}$, as it does not imply $2013^{a_n} > 2013^{2012b_n}$. Many contestants performed pages of tedious calculations in vain. Are we doomed? It turns out that a clever little tweak to the induction idea would lead us to a crisp and compact solution:

Instead of $a_n > b_n$, we shall prove $a_n \geq 2013b_n$ for all positive integers n. This is clearly true for $n = 1$. If $a_k \geq 2013b_k$ for

(continued on page 2)

some positive integer k, then

$$\begin{aligned} a_{k+1} &= 2013^{a_k} \\ &\geq 2013^{2013b_k} \\ &= 2013^{b_k} \cdot 2013^{2012b_k} \\ &\geq 2013b_{k+1}. \end{aligned}$$

There is something intriguing about this seemingly easy proof: if we cannot even prove just the original result, how come we can miraculously prove a stronger result? The answer to this paradox lies in the nature of mathematical induction: when we use induction, our task is essentially to prove the original statement about an arbitrary positive integer but equipped with an additional tool – the assumption that the statement is true for the preceding positive integer(s). If the statement is strengthened, what we need to prove becomes more demanding but the inductive hypothesis that we can use also gets more powerful. In the case of this problem, since the recurrence relations are exponential, the upgrade of the inductive hypothesis outweighs the increase in difficulty of the desired result.

Individual Finals A1.

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}$$

$$\leq \frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

for any positive real numbers a, b and c satisfying $a^2 + b^2 + c^2 = 1$.

The usual first step in proving such a symmetric inequality is to use the given condition to *homogenize* the inequality, i.e. to make the terms carry equal degrees. Afterwards, various inequality theorems can be applied. Here we first write the left-hand side as

$$\frac{1}{3a^2+2b^2+2c^2}+\frac{1}{3b^2+2c^2+2a^2}+\frac{1}{3c^2+2a^2+2b^2}$$

and note that by the AM-GM inequality, $3a^2 + 3b^2 \geq 6ab$ and analogous inequalities hold. So

$$\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

$$\geq \frac{1}{3a^2+3b^2+c^2}+\frac{1}{3b^2+3c^2+a^2}+\frac{1}{3c^2+3a^2+b^2}.$$

It suffices to prove the following inequality

$$\frac{1}{3x+2y+2z}+\frac{1}{3y+2z+2x}+\frac{1}{3z+2x+2y}$$

$$\leq \frac{1}{3x+3y+z}+\frac{1}{3y+3z+x}+\frac{1}{3z+3x+y}$$

where x, y and z are positive real numbers.

At this stage, one may resort to passionate expansion and then apply Muirhead's inequality and/or Schur's inequality, or, alternatively, factorization and completing the square.

But I wish to share a solution using the *majorization inequality* (see Math Excalibur, vol. 5, no. 5, p.2): Without loss of generality we may assume $x \geq y \geq z$. Then

$$(3x + 3y + z,\ 3y + 3z + x,\ 3z + 3x + y)$$

majorizes

$$(3x + 2y + 2z,\ 3y + 2z + 2x,\ 3z + 2x + 2y).$$

Due to the convexity of the function $f(t) = 1/t$, the desired inequality follows by the majorization inequality.

Readers may also be interested in an alternative solution involving calculus: First, by Muirhead's inequality (see *Mathematical Excalibur*, vol. 11, no. 1), we have

$$u^3v^3w + v^3w^3u + w^3u^3v$$

$$\geq u^3v^2w^2 + v^3w^2u^2 + w^3u^2v^2$$

for any positive u, v, w. Letting

$$u = t^{x-1/7},\ v = t^{y-1/7} \text{ and } w = t^{z-1/7}$$

where $0 < t < 1$, we get

$$t^{3x+3y+z-1} + t^{3y+3z+x-1} + t^{3z+3x+y-1}$$

$$\geq t^{3x+2y+2z-1} + t^{3y+2z+2x-1} + t^{3z+2x+2y-1}.$$

Now, integrating both sides with respect to t from 0 to 1, we obtain nothing but the desired inequality!

Lastly I encourage all readers to try out the following problem which only one out of the 123 contestants attempting Combinatorics A got right. This is really my favorite problem in PUMaC 2013 because I love eating sushi and find the setting very interesting:

Combinatorics A8.

Eight different pieces of sushi are placed evenly around a round table which can rotate about its center. Eight people sit evenly around the table. Each person has one favorite piece of sushi among the eight, and their favorites are all distinct. Sadly, they find that no matter how they rotate the table, there are never more than three people who have their favorite sushi in front of them simultaneously.

How many possible arrangements of the eight pieces of sushi are there? (Two arrangements that differ by a rotation are considered the same.)

In 1908, a classic Chinese newspaper article famously raised three questions for the country: When can China first send an individual athlete to the Olympic Games? When can China first send a delegation to the Olympic Games? When can China first host the Olympic Games?

In closing, I would also like to ask three questions: When can Hong Kong first take part in the Power Round of PUmaC? When can Hong Kong first send a team to Princeton to join the main competition of PUMaC? When can a university in Hong Kong first host a math competition run by undergraduates for secondary school students?

As Dr. Kin Li (editor of *Math Excalibur*) observes, Hong Kong students need more opportunities to participate in different competitions and broaden their horizons. They will also be able to experience a beautiful university, make friends with some of the most brilliant brains from around the world, and learn team spirit especially through the Power Round and Team Round. With optimism, I hope my three questions will find answers before long.

For further information and past papers, please visit PUMaC's website http://www.pumac.princeton.edu/

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 25, 2014.***

Problem 436. Prove that for every positive integer n, there exists a positive integer $p(n)$ such that the interval $[1, p(n)]$ can be divided into n pairwise disjoint intervals with each contains at least one integer and the sum of the integers in each of these intervals is the square of some integer.

Problem 437. Determine all real numbers x satisfying the condition that $\cos x, \cos 2x, \cos 4x, \ldots, \cos 2^n x, \ldots$ are all negative.

Problem 438. Suppose $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is divisible by at least one of the positive integers $a_1, a_2, \ldots, a_m$. Prove that there exists one of the a_i such that for all integer n, $P(n)$ is divisible by that a_i.

Problem 439. In acute triangle ABC, T is a point on the altitude AD (with D on side BC). Lines BT and AC intersect at E, lines CT and AB intersect at F, lines EF and AD intersect at G. A line ℓ passing through G intersects side AB, side AC, line BT, line CT at M, N, P, Q respectively.

Prove that $\angle MDQ = \angle NDP$.

Problem 440. There are n schools in a city. The i-th school will send C_i students to watch a performance at a field. It is <u>known</u> that $0 \le C_i \le 39$ for i=1, 2, …, n and $C_1+C_2+\cdots+C_n=1990$. The seats will be put in a rectangle arrangement with each row having 199 seats. Determine the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions above.

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Solutions

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Problem 431. There are 100 people, composed of 2 people from 50 distinct nations, are seated in a round table. Two people sitting next to each other are <u>*neighbors*</u>.

Prove that it is possible to divide the 100 people in two groups of 50 people so that no 2 people from the same nation are in the same group and each person in a group has at most one neighbor in the group.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Suppose these 100 people $V_1, V_2, \ldots, V_{100}$ are seated in a round table in clockwise order. For $n = 1,2,\ldots, 50$, call $\{V_{2n-1}, V_{2n}\}$ a <u>*partner pair*</u>. We color V_1 in black and color the person with the same nation as him, say V_r, in white. If V_r's partner is not yet colored, then we color V_r's partner, say V_s, in black (this completes the coloring of the partner pair $\{V_r, V_s\}$) and go on to color the person with the same nation as V_s in white. Repeat this process until we reach a V_r whose partner V_s was colored already, then $V_r=V_2$ and $V_s=V_1$ since the only partner pair not yet completing the coloring is $\{V_1,V_2\}$ with V_1 black and V_2 waiting to be colored. This gives the first cycle. Then we start to form another cycle with a remaining partner pair. Since there are 100 people, we will eventually stop. At the end, there are two groups with 50 black's and 50 white's and the required conditions are satisfied.

Problem 432. Determine all prime numbers p such that there exist integers a,b,c satisfying $a^2 + b^2 + c^2 = p$ and $a^4+b^4+c^4$ is divisible by p.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Math Center** (Carmel Alison Lam Foundation Secondary School) and **O Kin Chit Alex** (G.T. (Ellen Yeung) College).

Without loss of generality, we may assume $a \ge b \ge c \ge 0$. Then

$$\begin{aligned} 0 &\equiv (p - b^2 - c^2)^2 + b^4 + c^4 \\ &\equiv (b^2 + c^2)^2 + b^4 + c^4 = 2(b^4 + b^2c^2 + c^4) \\ &= 2(b^2 - bc + c^2)(b^2 + bc + c^2) \pmod p. \end{aligned}$$

Next,

$$\begin{aligned} 0 \le bc &\le b^2 - bc + c^2 \\ &\le b^2 + bc + c^2 \\ &\le a^2 + b^2 + c^2 = p. \end{aligned}$$

Since $a \ge b \ge c \ge 0$, if $bc=a^2$, then $a=b=c$ and p being prime implies $a = 1$ and $p = 3$. Otherwise $bc < a^2$ leads to $b^2+bc+c^2= 0$ or 1. If $b = 0$, then $a^2 = p$ contradicts p is prime. Then $c = 0$, $b = 1$ and $a^2 + 1 = p$, which leads to

$$0 \equiv a^4+b^4+c^4 = a^4+1 \equiv 2 \pmod p.$$

Then p=2 and a=b=1, c=0. Therefore, the only solutions are $p = 2$ or 3.

Problem 433. Let P_1, P_2 be two points inside ΔABC. Let $BC = a$, $CA = b$ and $AB = c$. For $i = 1,2$, let $P_iA = a_i$, $P_iB = b_i$ and $PC_i = c_i$. Prove that

$$aa_1a_2+bb_1b_2+cc_1c_2 \ge abc.$$

Solution. **Math Group** (Carmel Alison Lam Foundation Secondary School).

Let the complex numbers $\alpha, \beta, \gamma, \mu, \nu$ correspond to the points A, B, C, P_1, P_2 in the complex plane respectively. By expansion, we have

$$\frac{(\mu-\alpha)(\nu-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}+\frac{(\mu-\beta)(\nu-\beta)}{(\alpha-\beta)(\gamma-\beta)}+\frac{(\mu-\gamma)(\nu-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}=1.$$

Then

$$\frac{a_1a_2}{cb}+\frac{b_1b_2}{ca}+\frac{c_1c_2}{ba}$$

$$=\left|\frac{(\mu-\alpha)(\nu-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}\right|+\left|\frac{(\mu-\beta)(\nu-\beta)}{(\alpha-\beta)(\gamma-\beta)}\right|+\left|\frac{(\mu-\gamma)(\nu-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}\right|$$

$$\ge\left|\frac{(\mu-\alpha)(\nu-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}+\frac{(\mu-\beta)(\nu-\beta)}{(\alpha-\beta)(\lambda-\beta)}+\frac{(\mu-\gamma)(\nu-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}\right|$$

$= 1.$

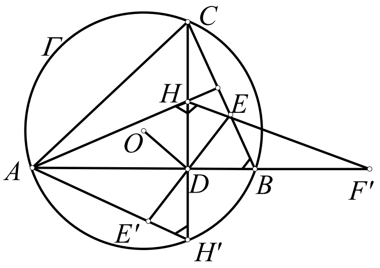
Multiplying both sides by abc, we get the desired result.

Problem 434. Let O and H be the circumcenter and orthocenter of ΔABC respectively. Let D be the foot of perpendicular from C to side AB. Let E be a point on line BC such that $ED \perp OD$. If the circumcircle of ΔBCH intersects line AB at F, then prove that points E, F, H are collinear.

Solution 1. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **T. W. LEE** (Alumni of New Method College).

Let lines HE and AB intersect at F'. Let Γ be the circumcircle of ΔABC. Let H'

be the intersection of line CD and Γ different from C. Let E' be the intersection of lines DE and AH'.

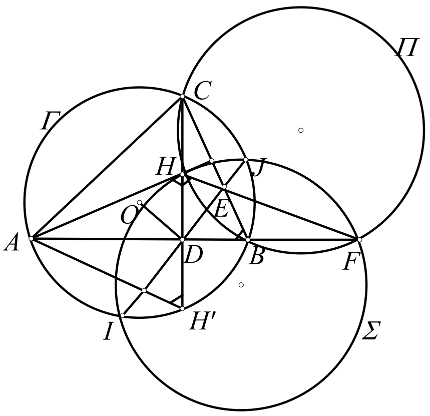


Observe that since $\angle H'DE' = \angle HDE$, $\angle AH'C = \angle ABC = 90^\circ - \angle BAH = \angle AHD$ implies $H'D=HD$ and the butterfly $H'CBAH'$ on Γ gives $E'D=ED$ as $OD\perp DE$, we have $\triangle H'E'D \cong \triangle HED$. Then

$$\angle F'HD = \angle EHD = \angle E'H'D = \angle AH'C = \angle DBC.$$

It follows $\angle CHF' = \angle CBF'$. Then F' is on the line AB and the circumcircle of $\triangle BCH$. Therefore, $F'=F$ and E, F, H are collinear.

Solution 2. **Jerry AUMAN, Georgios BATZOLIS** (*Mandoulides High School, Thessaloniki, Greece*) and **Jon GLIMMS** (Vancouver, Canada).



Let Π be the circle passing through C, H, B, F and let Γ be the circumcircle of $\triangle ABC$. Let line DE meet Γ at I and J. Since $OD\perp DE$, D bisects chord IJ. Next,

$$\angle DCF = \angle DBH = 90^\circ - \angle BAC = \angle DCA$$

implies D bisects AF. Hence $AIFJ$ is a parallelogram. Then $\angle IFJ = \angle IAJ$.

Let H' be the intersection point (different from C) of line CD and Γ. Then D bisects HH' (see solution 1 -- *Ed.*) and $IHJH'$ is a parallelogram. So $\angle IHJ = \angle IH'J$. Then

$$\angle IFJ + \angle IHJ = \angle IAJ + \angle IH'J = 180°.$$

So I,F,J,H lies on a circle Σ.

Finally, the radical axis of Γ and Π is line BC, while the radical axis of Γ and Σ is line IJ. So the radical center of Γ, Π, Σ is the intersection of lines BC and IJ, which is E. Therefore, E is also on the radical axis of Π and Σ, which is line HF.

Comments: One can also solve via coordinate geometry by assigning lines AB and CD as the x-axis and y-axis respectively.

Other commended solvers: **Math Group** (Carmel Alison Lam Foundation Secondary School), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 435. Let $n > 1$ be an integer that is not a power of 2. Prove that there exists a permutation $a_1, a_2, \ldots, a_n$ of $1,2,\ldots, n$ such that

$$\sum_{k=1}^{n} a_k \cos\frac{2k\pi}{n} = 0.$$

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **Math Center** (Carmel Alison Lam Foundation Secondary School).

For integer $n > 1$, let $c_k = \cos(2k\pi/n)$ for $k = 1, 2, \ldots, n$. We have $c_n = 1$, $c_k = c_{n-k}$ and

$$\sum_{k=1}^{n} c_k = \operatorname{Re}\sum_{k=1}^{n}\omega^k = \operatorname{Re}\frac{1-\omega^n}{1-\omega} = 0, \qquad (*)$$

where $\omega = e^{2\pi i/n}$.

Suppose $n = 2m+1$, where $m = 1,2,3,\ldots$. We have $c_1+c_2+\cdots+c_m = -1/2$ (using $c_n = 1$ and $c_k = c_{n-k}$). Hence

$$(2m+2)(c_1+c_2+\cdots+c_m) = -(m+1)c_{2m+1}.$$

Since $c_k = c_{2m+1-k}$, we have

$$(2m+1)c_1+2mc_2+\cdots+(m+2)c_m = (m+2)c_{m+1}+\cdots+2mc_{2m-1}+(2m+1)c_{2m}.$$

Subtracting the two displayed equations above and transposing all terms to the left, we get

$$\sum_{k=1}^{m} kc_k + \sum_{k=m+1}^{2m}(k+1)c_k + (m+1)c_{2m+1} = 0.$$

This solves the cases $n = 3,5,7,\ldots$.

Next, assuming the case n is true, we will show the case $2n$ is also true. Let $d_m = \cos(m\pi/n)$ for $m = 1,2,\ldots,2n$. The case n gives us an equation of the form

$$a_1d_2 + a_2d_4 + \cdots + a_nd_{2n} = 0, \qquad (**)$$

where $a_1, a_2, \ldots, a_{2n}$ is a permutation of $1,2,\ldots,n$.

Using (*), we have

$$d_1 + d_2 + \cdots + d_{2n} = \sum_{k=1}^{2n}\cos\frac{2k\pi}{2n} = 0$$

and

$$d_2 + d_4 + \cdots + d_{2n} = \sum_{k=1}^{n}\cos\frac{2k\pi}{n} = 0.$$

Subtracting these equations, we have $d_1+d_3+\cdots+d_{2n-1}=0$. For $k=1,3,\ldots,2n-1$, we have

$$d_{2n-k} = \cos((2n-k)\pi/n) = \cos(k\pi/n) = d_k.$$

Using this, $d_1 + 3d_3 + \cdots + (2n-1)d_{2n-1} = d_{2n-1} + 3d_{2n-3} + \cdots + (2n-1)d_1$. Adding the left and right sides, we get the equation $2n(d_1+d_3+\cdots+d_{2n-1}) = 0$. So

$$d_1+3d_3+\cdots+(2n-1)d_{2n-1} = 0. \qquad (***)$$

Finally, taking twice the equation in (**) and adding it to the equation in (***), we solve the case $2n$.

Comments: **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania) pointed out that Problem 435 is the same as Problem 26753 in the Romanian Mathematical Gazette (G.M.-B) and a solution was appeared in G.M-B, No. 10, 2013, pp. 468-469.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Determine all positive integers $n \ge 2$ satisfying

$$i + j \equiv \binom{n}{i} + \binom{n}{j} \pmod 2$$

for all i and j such that $0 \le i \le j \le n$.

Problem 5. Let $ABCDEF$ be a cyclic hexagon satisfying $AB\perp BD$ and $BC=EF$. Let P be the intersection of lines BC and AD and let Q be the intersection of lines EF and AD. Assume that P and Q are on the same side of D and that A is on the opposite side. Let S be the midpoint of AD. Let K and L be the centres of the incircles of ΔBPS and ΔEQS respectively. Prove that $\angle KDL = 90°$.

Mathematical Excalibur

Volume 18, Number 5 *February 2014 – March 2014*

Olympiad Corner

Below are the problems of the Fourth Round of the 53rd Ukrainian National Math Olympiad for 10-th Graders.

Problem 1. Suppose that for real x,y,z,t the following equalities hold: $\{x+y+z\} = \{y+z+t\} = \{z+t+x\} = \{t+x+y\} = 1/4$. Find all possible values of $\{x+y+z+t\}$. (Here $\{x\}=x-[x]$.)

Problem 2. Let M be the midpoint of the side BC of $\triangle ABC$. On the side AB and AC the points F and E are chosen. Let K be the point of the intersection of BF and CE and L be chosen in a way that $CL \| AB$ and $BL \| CE$. Let N be the point of intersection of AM and CL. Show that KN is parallel to FL.

Problem 3. It is known that for natural numbers a,b,c,d and n the following inequalities hold: $a+c < n$ and $a/b+c/d < 1$. Prove that $a/b+c/d < 1-1/n^3$.

Problem 4. There are 100 cards with numbers from 1 to 100 on the table. Andriy and Nick took the same number of cards in a way that the following condition holds: if Andriy has a card with a number n then Nick has a card with a number $2n+2$. What is the maximal number of cards could be taken by the two guys?

(continued on page 4)

Editors: 張百康 (CHEUNG Pak-Hong), Munsang College, HK
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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 12, 2014***.

For individual subscription for the next five issues for the 13-14 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Using Tangent Lines to Prove Inequalities (Part II)

Ibragim Ibatulin and Adilsultan Lepes

The Republican Specialized Physics Mathematics Secondary Boarding School

Named after O. Zhautykov, Almaty, Kazakhstan

We offer a continuation of the paper by Kin-Yin Li (cf. *Math Excalibur*, *vol.* 10, *no.* 5) where he considers using tangent lines to prove inequalities.

Example 1. Suppose that a, b, and c are positive real numbers satisfying $a+b+c=3$. Find the minimum of the expression $a^4+2b^4+3c^4$.

Solution. Let $f_k(x)=kx^4$, where $x\in(0,3)$, $k = 1, 2, 3$. As $f_k''(x) = 12kx^2 > 0$, where $x>0$, so functions f_k are convex, which means that their graphs do not fall below their tangents drawn at any point $x_k\in(0,3)$ $(k=1,2,3)$. Points x_1, x_2 and x_3 are chosen such that $f_1'(x_1) = f_2'(x_2) = f_3'(x_3)$ and $x_1+x_2+x_3=3$. That is,

$$4x_1^3=8x_2^3=12x_3^3 \text{ and } x_1+x_2+x_3=3.$$

Hence,

$$x_1=\frac{3\sqrt[3]{6}}{\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{6}},\quad x_2=\frac{x_1}{\sqrt[3]{2}},\quad x_3=\frac{x_1}{\sqrt[3]{3}}$$

and for any $x\in(0,3)$, we have the inequalities ($k = 1,2,3$, see Fig. 1)

$$kx^4 \ge f_k(x_k)+f_k'(x_k)(x-x_k). \qquad (1)$$

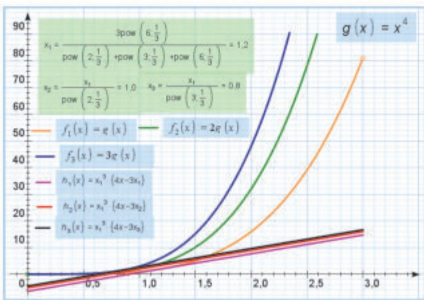

Fig. 1

Adding inequalities (1) for x equals a, b and c, we obtain

$$\begin{aligned} &a^4+2b^4+3c^4\\ &\ge x_1^4\left(1+\frac{1}{2\sqrt[3]{2}}+\frac{1}{3\sqrt[3]{3}}\right)+f_1'(x_1)(3-\sum_{k=1}^{3}x_k)\\ &=\frac{81(6\sqrt[3]{3}+3\sqrt[3]{3}+2\sqrt[3]{2})}{(\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{6})^4}, \end{aligned}$$

which is the minimum (with equality holding at $a=x_1$, $b=x_2$ and $c=x_3$).

Example 2. Let a, b, $c > 0$ be real numbers such that $ab+bc+ca = 1$. Prove the inequality

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{\sqrt{3}}{2}.$$

Solution. Let $S=a+b+c$. Based on the inequality $(a+b+c)^2 \ge 3(ab+bc+ca)$, which is equivalent to $(a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$, we find that $S \ge \sqrt{3}$.

Let $f(x) = x^2/(S-x)$ for $x\in(0,S)$. Let us construct the tangent equation at the point $x_0=S/3$ (see Fig. 2a,b):

$$y=f\left(\frac{S}{3}\right)+f'\left(\frac{S}{3}\right)\left(x-\frac{S}{3}\right)=\frac{S}{6}+\frac{5}{4}\left(x-\frac{S}{3}\right)=\frac{5x-S}{4}.$$

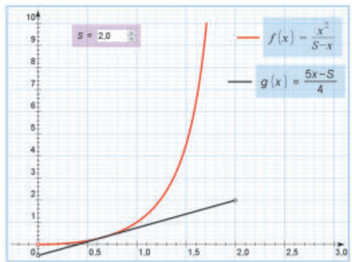

Fig. 2a

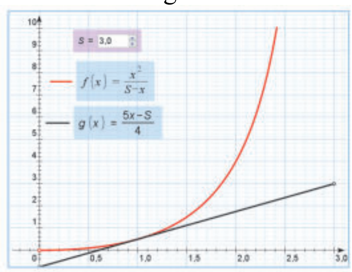

Fig. 2b

Since the inequality $x^2/(S-x) \ge (5x-S)/4$ is equivalent to $(S-3x)^2 \ge 0$ on the interval $(0,S)$, applying it thrice, based on the previously proved inequality $S \ge \sqrt{3}$, we find that

$$\begin{aligned} &\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\\ &=\frac{a^2}{S-a}+\frac{b^2}{S-b}+\frac{c^2}{S-c}\\ &\ge\frac{5(a+b+c)-3S}{4}=\frac{S}{2}\ge\frac{\sqrt{3}}{2}. \end{aligned}$$

(continued on page 2)

Example 3. Let $a, b, c \ge 0$ be real numbers. Prove the inequality

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\ge\sqrt{6(a+b+c)}.$$

Solution. Assume that $S=a+b+c$ and $f(x)=\sqrt{x^2+1}$ for $x\in(0,S)$. We form the tangent equation at the point $x_0=S/3$:

$$\begin{aligned} y&=f\left(\frac{S}{3}\right)+f'\left(\frac{S}{3}\right)\left(x-\frac{S}{3}\right)\\ &=\frac{\sqrt{S^2+9}}{3}+\frac{S}{\sqrt{S^2+9}}\left(x-\frac{S}{3}\right)\\ &=\frac{Sx+3}{\sqrt{S^2+9}}.\end{aligned}$$

Since on the interval $(0,S)$, the inequality

$$\sqrt{x^2+1}\ge\frac{Sx+3}{\sqrt{S^2+9}} \qquad (2)$$

is equivalent to the inequality $(x-S/3)^2 \ge 0$, we find that

$$\begin{aligned}&\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\\ &\ge\sqrt{S^2+9}+\frac{S}{\sqrt{S^2+9}}(a+b+c-S)\\ &=\sqrt{S^2+9}\\ &\ge\sqrt{6S}=\sqrt{6(a+b+c)}.\end{aligned}$$

Example 4. Let a, b and c be positive real numbers such that $a+2b+3c \ge 20$. Prove the inequality

$$a+b+c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c}\ge 13.$$

Solution. Note that if $a=2$, $b=3$, $c=4$, the inequality becomes equality. Let $f(x)=1/x$ for $x>0$. Then f is convex in the interval $(0,+\infty)$. Hence the graph of the function f does not go below the tangent line drawn at any point $x_0 > 0$. Thus, the following inequalities are valid (see Fig. 3):

$$\begin{aligned}\frac{1}{a}&\ge\frac{1}{2}-\frac{1}{4}(a-2)=1-\frac{a}{4},\\ \frac{1}{b}&\ge\frac{1}{3}-\frac{1}{9}(b-3)=\frac{2}{3}-\frac{b}{9},\\ \frac{1}{c}&\ge\frac{1}{4}-\frac{1}{16}(c-4)=\frac{1}{2}-\frac{c}{16}.\end{aligned}$$

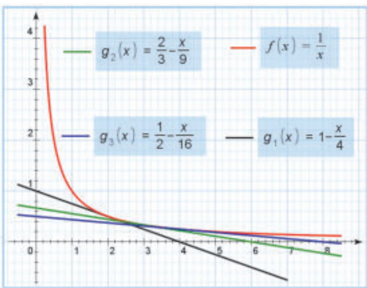

Fig. 3

As given in the statement of the problem, we find that

$$\begin{aligned}&a+b+c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c}\\ &\ge a+b+c+3-\frac{3a}{4}+3-\frac{b}{2}+2-\frac{c}{4}\\ &=8+\frac{a+2b+3c}{4}\ge 8+\frac{20}{4}=13.\end{aligned}$$

Example 5. (Pham Kim Hung) Let a, b and c be positive real numbers such that $a^2+b^2+c^2=3$. Prove the inequality

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}\ge 3.$$

Solution. Note that when $a=b=c=1$, the inequality becomes an equality. Consider $f(x) = 1/(2-x)$ and $g(x) = kx^2+m$, where $x\in(0,\sqrt{3})$. The numbers k and m are to be chosen so that $f(1) = g(1)$ and $f'(1) = g'(1)$. That is, $1=k+m$ and $1=2k$. Hence, $k=m=1/2$ and $g(x)=(x^2+1)/2$. Since the inequality $1/(2-x) \ge (x^2+1)/2$ is equivalent to $x(x-1)^2 \ge 0$, it is true for any $x\in(0,\sqrt{3})$ (see Fig. 4). Hence,

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}\ge\frac{a^2+b^2+c^2+3}{2}=3.$$

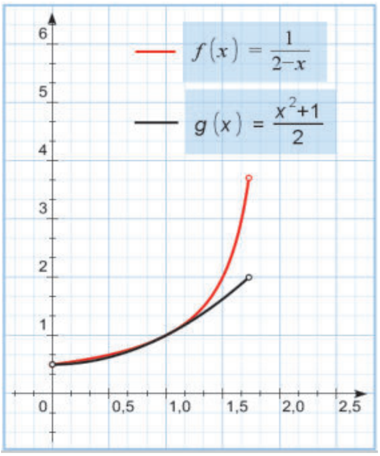

Fig. 4

Example 6. Let a, b and c be positive real numbers. Prove the inequality

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3)\ge(a+b+c)^3.$$

Solution. Note that when $a=b=c=1$, the inequality becomes an equality. Consider $f(x)=x^5-x^2+3$ and $g(x) = kx^3+m$, where $x>0$. The numbers k and m are to be chosen so that $f(1) = g(1)$ and $f'(1) = g'(1)$. That is, $3=k+m$ and $3=3k$. Hence, $k=1$, $m=1/2$ and $g(x)=x^3+2$. The inequality (see Fig. 5)

$$x^5-x^2+3\ge x^3+2 \qquad (3)$$

is true for any $x>0$ as it can be represented in the form $(x-1)^2(x^3+2x^2+2x+1) \ge 0$.

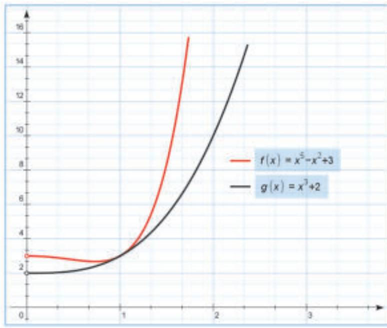

Fig. 5

Example 7. Let a, b, c, d and e be positive real numbers such that

$$\frac{1}{4+a}+\frac{1}{4+b}+\frac{1}{4+c}+\frac{1}{4+d}+\frac{1}{4+e}=1.$$

Prove the inequality

$$\frac{a}{4+a^2}+\frac{b}{4+b^2}+\frac{c}{4+c^2}+\frac{d}{4+d^2}+\frac{e}{4+e^2}\le 1.$$

Solution. Consider $f(x) = x/(4+x^2)$ and $g(x) = m + k/(4+x)$, where $x \ge 0$. The numbers k and m are to be chosen so that $f(1) = g(1)$ and $f'(1) = g'(1)$. Hence $k = -3$ and $m = 4/5$. Since the inequality

$$\frac{x}{4+x^2}\le\frac{4}{5}-\frac{3}{4+x}$$

is equivalent to $(x-1)^2(x+1)\ge 0$, it is true for any $x \ge 0$ (see Fig. 6).

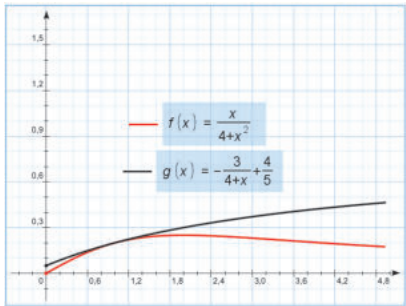

Fig. 6

Applying this inequality, we have

$$\begin{aligned}&\frac{a}{4+a^2}+\frac{b}{4+b^2}+\frac{c}{4+c^2}+\frac{d}{4+d^2}+\frac{e}{4+e^2}\\ &\le 4-3\left(\frac{1}{4+a}+\frac{1}{4+b}+\frac{1}{4+c}+\frac{1}{4+d}+\frac{1}{4+e}\right)\\ &=1.\end{aligned}$$

Finally, we have some exercises for the readers.

Exercise 1. (Gabriel Dospinescu) Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\begin{aligned}&\sqrt{1+a_1^2}+\sqrt{1+a_2^2}+\cdots+\sqrt{1+a_n^2}\\ &\le\sqrt{2}(a_1+a_2+\cdots+a_n).\end{aligned}$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 12, 2014.***

Problem 441. There are six circles on a plane such that the center of each circle lies outside of the five other circles. Prove there is no point on the plane lying inside all six circles.

Problem 442. Prove that if $n > 1$ is an integer, then n^5+n+1 has at least two distinct prime divisors.

Problem 443. Each pair of n ($n\geq 6$) people play a game resulting in either a win or a loss, but no draw. If among every five people, there is one person beating the other four and one losing to the other four, then prove that there exists one of the n people beating all the other n–1 people.

Problem 444. Let D be on side BC of equilateral triangle ABC. Let P and Q be the incenters of ΔABD and ΔACD respectively. Let E be the point so that ΔEPQ is equilateral and D, E are on opposite sides of line PQ. Prove that lines BC and DE are perpendicular.

Problem 445. For each positive integer n, prove there exists a polynomial $p(x)$ of degree n with integer coefficients such that $p(0)$, $p(1)$, …, $p(n)$ are distinct and each is of the form $2\times 2014^k+3$ for some positive integer k.

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Solutions

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Problem 436. Prove that for every positive integer n, there exists a positive integer $p(n)$ such that the interval $[1, p(n)]$ can be divided into n pairwise disjoint intervals with each contains at least one integer and the sum of the integers in each of these intervals is the square of some integer.

Solution. **Jerry AUMAN, Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **Jon GLIMMS** (Vancouver, Canada) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

We look for a pattern. Since $1=1^2$, let $p(1)=1$. Since $2+3+4=3^2$, let $p(2)=1+3=4$ and divide [1,4] into [1,1] and (1,4]. Since

$$5+6+7+8+9+10+11+12+13 = 9^2,$$

let $p(3) = 1+3+9=13$ and divide [1,13] into [1,1], (1,4], (4,13].

This suggests we let $p(n) = 1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n-1)/2$ and divide $[1, p(n)]$ into $[1, p(1)]$, $(p(1), p(2)]$, …, $(p(n-1), p(n)]$. The integers in $(p(k), p(k+1)]$ are from $(3^k+1)/2$ to $(3^{k+1}-1)/2$, which sums to 3^{2k}. So we are done.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India) and **SP47** (Hanoi, Vietnam).

Problem 437. Determine all real numbers x satisfying the condition that $\cos x$, $\cos 2x$, $\cos 4x$, …, $\cos 2^n x$, … are all negative.

***Solution* 1. Jerry AUMAN, T. W. LEE** (Alumni of New Method College) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

For such x, we have $2^n x=2\pi(k_n+\theta_n)$, where $k_n\in\mathbb{Z}$ and $1/4 < \theta_n < 3/4$. In base 2 this is $.01_2< \theta_0=.d_1d_2d_3\ldots_2<.10111\ldots_2$. No d_nd_{n+1} can be 00 or 11, otherwise $\theta_{n-1}=.00\ldots_2$ or $.11\ldots_2$ would not be in (1/4,3/4). So $\theta_0 = .010101\ldots_2=1/3$ or $.101010\ldots_2=2/3$. Then $x=2\pi(k_0+1/3)$ or $2\pi(k_0+2/3)$ and for all $n = 0,1,2,\cdots$, $\cos 2^n x = -1/2$.

Solution 2. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **GLIMMS** (Vancouver, Canada).

Let $t=\cos 2\theta$. Suppose $\cos\theta$, $\cos 2\theta$ and $\cos 4\theta$ are negative. Then $t<0$ and $2t^2-1<0$ imply $-\sqrt{2}/2<t=2\cos^2\theta-1<0$. We get

$$\cos\theta<-\frac{\sqrt{2-\sqrt{2}}}{2}<-\frac{1}{4}.$$

Suppose $s_n = \cos 2^n x < 0$ for $n = 0,1,2,3,\cdots$. Then $s_n \in[-1, -1/4)$. So $|s_n-1/2| > 3/4$. Using this and $s_{n+1}=2s_n^2-1$, we have

$$\left|s_{n+1}+\frac{1}{2}\right|=2\left|s_n^2-\frac{1}{4}\right|=2\left|\left(s_n-\frac{1}{2}\right)\left(s_n+\frac{1}{2}\right)\right|$$
$$\geq\frac{3}{2}\left|s_n+\frac{1}{2}\right|.$$

Repeating this, since $-1\leq s_{n+1}<0$, we get

$$\frac{1}{2}\geq\left|s_{n+1}+\frac{1}{2}\right|\geq\left(\frac{3}{2}\right)^n\left|s_0+\frac{1}{2}\right|.$$

Then $|s_0+1/2| < (2/3)^n$. Taking limit, we see $\cos x = s_0= -1/2$, i.e. $x = \pm 2\pi/3+2k\pi$, where k is integer. Conversely, $s_0= -1/2$ implies $s_n= -1/2$ for $n = 1,2,3,\cdots$.

Other commended solvers: **Henry LEUNG Kai Chung** (Graduate of HKUST Maths).

Problem 438. Suppose $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is divisible by at least one of the positive integers $a_1, a_2,\ldots, a_m$. Prove that there exists one of the a_i such that for all integer n, $P(n)$ is divisible by that a_i.

Solution. **Jerry AUMAN, Jon GLIMMS** (Vancouver, Canada) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Assume the contrary that for each a_i, there exists integer n_i such that $P(n_i)$ is not divisible by a_i. Consider the prime factorizations of a_i and $|P(n_i)|$. Then there exists a prime divisor p_i of a_i such that $d_i = p_i^{e_i}$ is the greatest power of p_i dividing a_i, however d_i does not divide $|P(n_i)|$. If two of the d_i's are powers of the same prime, then eliminate the one with the larger exponent. (In this way, each of $a_1, a_2,\ldots, a_m$ is still divisible by one of the remaining d_i's.)

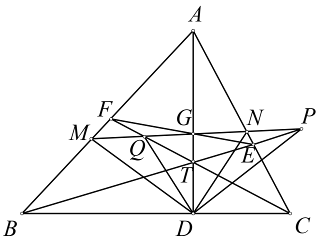
By the Chinese remainder theorem, there exist integers n such that $n \equiv n_i \pmod{d_i}$ for the remaining d_i's. Now $P(n)-P(n_i)$ is divisible by $n-n_i$, which is divisible by d_i. Since $P(n_i)$ is is not divisible by d_i. So $P(n)$ is not divisible by any d_i's, contradicting $P(n)$ is divisible by at least one of the positive integers $a_1, a_2,\ldots, a_m$, hence also divisible by at least one d_i.

Problem 439. In acute triangle ABC, T is a point on the altitude AD (with D on side BC). Lines BT and AC intersect at E, lines CT and AB intersect at F, lines EF and AD intersect at G. A line ℓ passing through G intersects side AB, side AC, line BT, line CT at M, N, P, Q respectively.

Prove that $\angle MDQ = \angle NDP$.

Solution. **William FUNG** and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Set the origin at D and A, B, C at $(0,a)$, $(b,0)$, $(c,0)$ respectively.



Let T be at $(0,1)$. The equations of the lines BT, CT, AB, AC are

$$y = -(x/b) + 1,\quad y = -(x/c) + 1,$$

$$y = -(ax/b) + a,\quad y = -(ax/c) + a$$

respectively. Since $E = BT \cap AC$ and $F = CT \cap AB$, we can solve the equations of the lines to get

$$E = \left(\frac{(a-1)bc}{ab-c}, \frac{a(b-c)}{ab-c}\right)$$

and $$F = \left(\frac{(a-1)cb}{ac-b}, \frac{a(c-b)}{ac-b}\right).$$

From the y-intercept of line EF, we get $G=(0, 2a/(a+1))$. Let the equation of ℓ be $y=mx+2a/(a+1)$. Then $M = \ell \cap AB$ is at

$$\left(\frac{a(a-1)b}{(a+mb)(a+1)}, \frac{a(2a+(a+1)mb)}{(a+mb)(a+1)}\right).$$

Using role symmetry of B and C, we can replace b by c in the coordinates of M to get coordinates of N. Similarly, $P = \ell \cap BT$ is at

$$\left(\frac{-(a-1)b}{(a+mb)(a+1)}, \frac{2a+(a+1)mb}{(a+mb)(a+1)}\right).$$

The coordinates of Q can be found by replacing b by c in the coordinates of P.

Since D is the origin, the slopes of lines DM and DP can be found by taking the y-coordinates of M and P dividing by their respective x-coordinates, which turn out to be the negative of each other! So lines DM and DP are symmetric with respect to the y-axis! Similarly, lines DN and DQ are symmetric with respect to the y-axis. Therefore, $\angle MDQ = \angle NDP$.

Comments: There is a pure geometry solution using a number of equations from applying Menelaus' theorem to different triangles. There is also a solution using harmonic division and cross-ratios from projective geometry.

Other commended solvers: **Georgios BATZOLIS** (Mandoulides High School, Thessaloniki, Greece), **Andrea FANCHINI** (Cantu, Italy), **T. W. LEE** (Alumni of New Method College), **SP47** (Hanoi, Vietnam), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 440. There are n schools in a city. The i-th school will send C_i students to watch a performance at a field. It is known that $0 \le C_i \le 39$ for $i=1, 2, \ldots, n$ and $C_1+C_2+\cdots+C_n=1990$. The seats will be put in a rectangle arrangement with each row having 199 seats. Determine the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions above.

Solution. **Adnan ALI** (9th Grade, Atomic Energy Central School 4 (AECS4), Mumbai, India), **Jerry AUMAN** and **Jon GLIMMS** (Vancouver, Canada).

Let k be the minimal number of rows needed. For $m = 1, 2, \ldots, k$, let there be a_m students in row m. If there are no more than 160 students in some row, then since each school sends at most 39 students, we can put in students from one more school in that row. So we may assume $a_m \ge 161$. Now

$$1990 = a_1 + a_2 + \cdots + a_k \ge 161k,$$

which implies $k \le 12$.

Next, we show 11 rows may not be enough. Suppose there are $n = 80$ schools with $C_i = 25$ for $i = 1, 2, \ldots, 79$ and $C_{80} = 15$. This totals to 1990 students. Then there can only be one row with $25\times7+15 = 190$ students and the other 10 rows with $25\times7=175$ students. This only totals to 1940 students.

So the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions is 12.

Other commended solvers: **T. W. LEE** (Alumni of New Method College) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(*Continued from page 1*)

Problem 5. Find the values of x such that the following inequality holds

$$\min\{\sin x, \cos x\} < \min\{1-\sin x, 1-\cos x\}.$$

Problem 6. Find all pairs of prime numbers p and q that satisfy the following equation

$$3p^q - 2q^{p-1} = 19.$$

Problem 7. Is it possible to choose 24 points in the space, such that no three of them lie on the same line and choose 2013 planes in a way that each plane passes through at least 3 of the chosen points and each triple of points belongs to at least one of the chosen planes?

Problem 8. Let M be the midpoint of the internal bisector AD of $\triangle ABC$. Circle ω_1 with diameter AC intersects BM at E and circle ω_2 with diameter AB intersects CM at F. Show that B, E, F, C belong to the same circle.

Using Tangent Lines …

(*Continued from page 2*)

Exercises 2. Let a, b and c be non-negative real numbers. Prove that

$$\frac{a}{b^2+c^2+d^2} + \frac{b}{c^2+d^2+a^2} + \frac{c}{d^2+a^2+b^2} + \frac{d}{a^2+b^2+c^2} \ge \frac{3\sqrt{3}}{2}\cdot\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}.$$

Exercise 3. Let a, b and c be positive real numbers. Determine the minimal value of

$$\frac{3a}{b+c} + \frac{4b}{c+a} + \frac{5c}{a+b}.$$

Exercise 4. Let a, b and c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$(a^7-a^4+3)(b^5-b^2+3)(c^4-c+3) \ge 27.$$

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Mathematical Excalibur

Volume 19, Number 1 *April 2014 – August 2014*

Olympiad Corner

Below are the problems of the 2014 International Math Olympiad on July 8 and 9, 2014.

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \ldots + a_n}{n} \leq a_{n+1}.$$

Problem 2. Let $n \geq 2$ be an integer. Consider a $n\times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k\times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°$, $\angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 12, 2014***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO2014 and Beyond

Leung Tat-Wing

I write this article with three goals in mind: (1) to report on IMO 2014; (2) to give some idea how we can further train our team members and (3) finally and hopefully provide us some help of how to organize IMO 2016.

Itinerary The 55th International Mathematical Olympiad was held in Cape Town, South Africa from 3 July to 13 July, 2014. It took us 13 hours flying from Hong Kong to Johannesburg, waiting for a couple of hours, then another 2 hours' flight to Cape Town. Surely when compared with Argentina and Colombia, it was a much easier trip. Because we have to host IMO 2016, this year several observers (with leaders or deputy leaders) came with us. We have gathered a lot of information in this trip, which will help us tremendously in our preparation. This IMO was held when world cup matches were going on, and we were lucky that we still managed to watch several games, and at better times (6 pm or 10 pm). We missed only the final game, Germany vs Argentina, when we were exactly in our return flight, and I managed to get the result only when we got off the plane.

Weather in South Africa was nice. It was winter, and usually 20°C during the day time and about 10°C during the night. If it was raining, then it got a bit cooler. We first stayed in a hotel, right below a mountain, which I believe belongs to the Table Mountain range. The view, if I may say, is simply majestic. The city structure looks nice. It looks like a decent English town. The hotel is pretty normal and we stayed there for 6 days. Then we moved to the University of Cape Town (UCT) and stayed with the students. Our students arrived Cape Town three days after us, and they were stationed in dormitories of the University all the time. Though accommodation and food were not as good as in the hotel, I believe I can bear it. Only thing is, every entrance of a dormitory in the University is equipped with heavy iron gate and is watched by a security guard, which I found it a bit scary. This reminds me of the security issue in South Africa. Of course, it is a country with high unemployment rate (25%), high Gini index (6.3), and there are racial problems and other things.

Leaders spent three days to select the 6 problems from a shortlist of 30 problems, then refined the wordings and wrote the English version and other official versions. They discussed the marking schemes proposed by the Problem Group and coordinators, and approved the marking schemes. The students then arrived, and the next day leaders and contestants together participated in the Opening Ceremony, with leaders and contestants still separated so that they could not communicate during the Ceremony. Students then wrote the two 4.5 hour contests on the mornings of the next two days, while leaders had the time to do a bit of sight-seeing and the like. After the two contests, leaders were then moved to the University. After the two contests, students were free, then they had the chances to see further things. I knew my students got the chances to see the Cape of Good Hope, and took a cable-car to the top of the Table Mountain. Because I, as a leader, had to participate in the coordination process, had to miss both events. Coordination is a process in which the leader and deputy leader, plus two coordinators of the host county, come together to decide how many points are to be awarded to a particular problem submission of a student. Given that we had nice and detailed marking schemes, and the coordinators are generally very experienced, we encountered little trouble in deciding points. Then we had a final day excursion and the Closing Ceremony, on the same day. The next day we headed home.

(continued on page 2)

Problem Selection By the end of March 2014, the host country (South Africa) received 141 problem proposals from 43 countries. I don't know when the problem selection group started to work, but surely, it took them more than a month to select 30 shortlisted problems. Furthermore they modified them, supplied alternative solutions and comments, and prepared a booklet for Jury members to consider. Incidentally the problem group was composed of six international members. I was told, they managed to do something before they formally met in South Africa, and also after they left. The selected problems are of course of high quality. However I cannot say I am totally happy with the selection. Indeed I think the problems selected were rather skewed, there were 6 algebra problems, 9 combinatorics problems, 7 geometry problems and 8 number theory problems. Some algebra problems and number theory problems in fact have quite a bit of combinatorics flavor. Moreover, several hard combinatorics problems were simply too hard. The Jury worried very much if one of them was selected, no one would be able to solve it.

When the Jury members met, it was suggested that first we selected 2 out of 4 easy problems, with one problem from each of the topics algebra, combinatorics, geometry and number theory. Again 4 medium problems from the four topics were selected. When the two easy problems were chosen, the two medium problems from the other two categories were automatically selected. Then the hard problems (problem 3 and 6) were chosen arbitrary. The suggestion was adapted. Finally two easy problems of algebra and geometry were selected, and so were two medium problems on combinatorics and number theory. However I am not sure if the easy algebra problem is really an algebra problem, of course it involves some algebraic manipulations, but I think the result very much depends on the discrete structure of integers. It is not an inequality problem nor a functional equation problem anyway. The medium combinatorics problem concerns "holes" within a distribution of rooks in a checker board. The number theory problem again is not really number theory. There is no need for congruence or other number theory things. It basically involves merging or grouping of coins of different values, so it is more like a combinatorics problem. Finally a hard geometry problem and a hard combinatorics problem were selected. It is quite certain in these days two geometry problems are to be selected. Those are the problems contestants cannot easily quote high power theorems or use more specialized techniques. However due to the preference of leaders, in general there is no 3D geometry problems. In this contest, three problems are really of combinatorial flavor. So I think the new method of choosing problems does not guarantee a good distribution of problems. Concerning Problem 6, I have to say I don't like it and I have something more to say, but let's wait.

Coordination The process of coordination was done seriously and rigorously. After the six problems were selected by the Jury (composed of leaders from 101 countries), I believed the chief coordinator then instructed the six problem captains to write up detailed marking schemes, incorporating various solutions supplied by leaders. Each problem captain was responsible for only one specific problem, he knew essentially everything concerning that problem, originality, various solutions, etc. The marking schemes were then formally approved by the Jury. After the two contests, they scanned all the answers scripts of the students. We leaders then got back answer scripts of our students and tried to allocate suitable points for our contestants. A minor mishap was, the scanner could not scan marks of correcting fluid, and thus I was asked several times why were there correcting fluids found on my students' scripts. Luckily of course was, we did not add anything new.

Detailed schedules were given to us, so leaders knew when and where to go. The process of coordination was done formally within two days. I believe because of language issue and other reasons, coordinators were recruited internationally. They were composed of old time leaders, experienced problem solvers etc. Some we met more than 10 years' ago. They were very experienced and were able to spot errors made by students, whether an error is trivial (no point deducted), minor (1 or 2 points deducted) or major (at least 4 to 5 points deducted). I thank my deputy leader, Ching Tak Wing, our old-time trainee and IMO gold medalist, who helped us to go through the many convoluted arguments of our members. We were able to discuss (or argue) with our coordinators, to convince them that our members did do somethings of certain parts of a problem or so, and thus got few extra points. On the whole, I think our papers were fairly marked and the process of coordination was done well.

Results of our Students We got 4 silvers and 2 bronzes, ranked (unofficially) 18 out of 101 countries. Indeed 3 of our 4 silver medalists solved essentially 4 problems and the other silver medalist got 3 problems correct. Also our 2 bronze medalists essentially got 3 problems correct and were real close to silver. I don't think I can blame our students for not trying hard. Indeed they picked up a lot of techniques in these few years, learned (and are still learning) to face a problem fairly and squarely. I observed when they were doing problem 2 and 5 (medium problems), they had generated the habit of gathering data and information, using various grouping and simplification methods, induction and other techniques to solve them, even though their approaches were later found to be a bit clumsy and there were a few gaps (thus few points deducted). Because a lot of time were spent on problems 2 and 5, no one could do problems 3 and 6, thus no one could tackle the hard problems. Four of our six members were old-timers, and they are leaving us for universities. I think we need 2 to 3 years to have another group of members of this caliber.

Think of this issue the other way. If we want to keep our ranking, surely several silver and bronze medals are required. If we want to be ranked within the top 10 countries, for instance, we need two or three gold medals, and some silvers and bronzes. It depends on really what we want. For me, I think it is fine if we can produce a bunch of well-trained students, good and brave to face problems and are ready to pick up necessary skills and other things in the process. Getting a gold medal in an IMO is a process, is part of a training process, but not necessarily is an end, (not like getting a world cup).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 12, 2014.***

Problem 446. If real numbers a and b satisfy $3^a+13^b=17^a$ and $5^a+7^b=11^b$, then prove that $a < b$.

Problem 447. For real numbers x, y, z, find all possible values of $\sin(x+y)+\sin(y+z)+\sin(z+x)$ if

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}.$$

Problem 448. Prove that if s,t,u,v are integers such that $s^2-2t^2+5u^2-3v^2=2tv$, then $s = t = u = v = 0$.

Problem 449. Determine the smallest positive integer k such that no matter how $\{1,2,3,\ldots,k\}$ are partitioned into two sets, one of the two sets must contain two distinct elements m, n such that mn is divisible by $m+n$.

Problem 450. (*Proposed by Michel BATAILLE*) Let $A_1A_2A_3$ be a triangle with no right angle and O be its circumcenter. For $i = 1,2,3$, let the reflection of A_i with respect to O be A_i' and the reflection of O with respect to line $A_{i+1}A_{i+2}$ be O_i (subscripts are to be taken modulo 3). Prove that the circumcenters of the triangles OO_iA_i' ($i = 1,2,3$) are collinear.

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Solutions

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Problem 441. There are six circles on a plane such that the center of each circle lies outside of the five other circles. Prove there is no point on the plane lying inside all six circles.

Solution. **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **William FUNG**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School),

Assume there is a point P inside all six circles C_1, C_2, …, C_6 with centers O_1, O_2, …, O_6 and radii r_1, r_2, …, r_6 respectively. Then $O_iP < r_i$ for $i = 1,2,\ldots,6$. Connecting the six O_i to P, since the six angles about P sum to 360°, there exists $\angle O_mPO_n \le 60°$. Then in ΔO_mPO_n, either $O_mO_n \le O_mP < r_m$ or $O_mO_n \le O_nP < r_n$. This leads to either O_n is inside C_m or O_m is inside C_n, which is a contradiction.

Problem 442. Prove that if $n > 1$ is an integer, then n^5+n+1 has at least two distinct prime divisors.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Luke Minsuk KIM** (Stanford University) and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4).

The case $n = 2$ is true as $n^5+n+1=5\times7$. For $n\ge3$, we have $n^5+n+1=(n^3-n^2+1)(n^2+n+1)$ and $n^3-n^2+1=(n^2+n+1)(n-2)+(n+3)$. Then $n^3-n^2+1 > n^2+n+1 > 1$. Assume n^5+n+1 is a power of some prime p. Then $n^3-n^2+1=p^s$ and $n^2+n+1=p^t$ with $s > t \ge 1$. Now

$$n+3=n^3-n^2+1-(n^2+n+1)(n-2)=p^s-p^t(n-2)$$

is a multiple of $p^t = n^2+n+1$. This leads to $n+3 \ge p^t=n^2+n+1$, i.e. $2 \ge n^2$, contradiction.

Other commended solvers: **Christian Pratama BUNAIDI** (SMA YPK Ketapang I, Indonesia), **CHAN Long Tin** (Cambridge University, Year 2), **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Victorio Takahashi CHU** (Pontifícia Universidade Católica - São Paulo SP, Brazil), **Gabriel Cheuk Hung LOU**, **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Viet Nam), **Milan PAVIC** (Serbia), **Mamedov SHATLYK** (School of Young Physics and Mathematics No. 21, Dashoguz, Turkmenistan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 443. Each pair of n ($n\ge6$) people play a game resulting in either a win or a loss, but no draw. If among every five people, there is one person beating the other four and one losing to the other four, then prove that there exists one of the n people beating all the other $n-1$ people.

Solution. **Jon GLIMMS** (Vancouver, Canada).

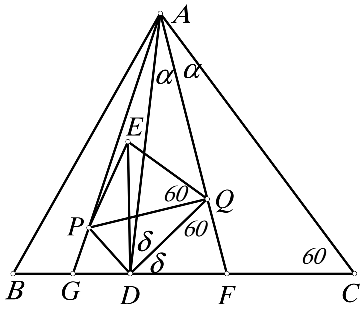
Assume no one beat all other $n-1$ people. Then the number of wins for each of the n people is $0,1,\ldots, n-2$. By the pigeonhole principle, there exist two people, say A and B with the same number of wins. Now, say A beat B. Due to same wins, there exists C such that A beat B, B beat C and C beat A.

Next add two other people to A, B, C. By given condition, one of these five lost to the other four. Observe that this one cannot be A, B, C, say it is D. Since $n\ge6$, ignoring D, we can add two other people to A, B, C. Again, by given condition, one of these five lost to the other four. Observe that this one cannot be A, B, C, D, say it is E. Then none of A,B,C,D,E beat the other four, contradicting the given condition.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 444. Let D be on side BC of equilateral triangle ABC. Let P and Q be the incenters of ΔABD and ΔACD respectively. Let E be the point so that ΔEPQ is equilateral and D, E are on opposite sides of line PQ. Prove that lines BC and DE are perpendicular.

Solution. **Jon GLIMMS** (Vancouver, Canada) and **T. W. LEE** (Alumni of New Method College).



We have $\angle QDP = \angle QDA+\angle PDA = ½(\angle CDA+\angle BDA)= 90°$. Also, $\angle QDA = \angle QDC=90°-\angle PDB$. To show $BC\perp DE$, i.e. $\angle PDE+\angle PDB=90°$, it suffices to show $\angle QDA = \angle PDE$. This is the same as showing lines AD, ED

are symmetric with respect to the angle bisector of $\angle QDP$. For convenience, we refer to this condition by saying lines AD, ED are isogonal with respect to $\angle QDP$.

This will follow from the isogonal conjugacy theorem (see comments below) if we can show that (1) lines AQ, EQ are isogonal with respect to $\angle PQD$ and (2) lines AP, EP are isogonal with respect to $\angle DPQ$. For (1), we have $\angle AQD = 180° - ½(\angle CAD + \angle CDA) = 120°$. Let lines AQ, BC meet at F. Then $\angle FQD = 180° - \angle AQD = 60° = \angle EQP$ implies (1). For (2), similarly $\angle APD = 120°$. Let lines AP, BC meet at G. Then $\angle GPD = 180° - \angle APD = 60° = \angle EPQ$ implies (2).

Comments: If we have (1) and (2), we can write down the two trigonometric forms of Ceva's theorem for points A and E with respect to $\Delta\, QDP$. Cancelling common factors in the two equations leads to

$$\frac{\sin\angle QDA}{\sin\angle ADP} = \frac{\sin\angle PDE}{\sin\angle EDQ}.$$

Then $\angle QDA = \angle PDE$ follows from $f(x) = \sin x / \sin(\angle QDP - x)$ is strictly increasing for $0 < x < \angle QDP$.

Other commended solvers: **CHAN Long Tin** (Cambridge University, Year 2) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 445. For each positive integer n, prove there exists a polynomial $p(x)$ of degree n with integer coefficients such that $p(0)$, $p(1)$, …, $p(n)$ are distinct and each is of the form $2\times 2014^k+3$ for some positive integer k.

Solution. **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Let $a = 2014$. Write $n! = n_1 n_2$, where n_2 is the greatest divisor of $n!$ that is relatively prime to a. Then n_1 and a have the same prime divisors. By Euler's theorem, for $t = \varphi(n_2)$, we have $a^t \equiv 1 \pmod{n_2}$. For the polynomial

$$f(x) = \sum_{i=0}^{n} \frac{x(x-1)\cdots(x-i+1)}{i!}(a^t-1)^i$$

and $j = 0,1,\ldots,n$, we have

$$f(j) = \sum_{i=0}^{j} \binom{j}{i}(a^t-1)^i = a^{tj}.$$

Let s be the maximum of the exponents appeared in the prime factorization of n_1. Then $a^s(a^t-1)/n!$ is a positive integer and $p(x) = 2a^s f(x)+3$ is a polynomial of degree n with integer coefficients such that $p(j) = 2a^{s+tj}+3$ for $j = 0,1,\ldots,n$.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $1/n$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99+½$, prove that it is possible to split the collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite areas; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Notes: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

IMO2014 and Beyond

(*Continued from page 2*)

So far, about 10 Fields' medalists participated in the IMOs, but not everyone was a gold medalist (about half of them were). Even Terry Tao got bronze in his first year, then silver, then gold. Yes, of course I realize some administrators may think otherwise and have different ideas of what it means by sending a team to an IMO.

I heard many theories why we cannot produce even stronger team. Our students have to devote too much time on DSE, in particular SBA. We have no specialized schools, unlike Vietnam and Singapore. Our pool is too small, trainers are no good, training time are not enough, etc. All these are hard to rebuke (no counter-examples?), and not sure how to verify. They may well be so and so what can we do? Indeed in these few years we have strengthened our training process, more tests, asking our members to present and substantiating their views, etc. Indeed we received many suggestions from our former trainees.

We observed a few things by simply looking at the overall results. For instance, despite political trouble in the east, the Ukrainian team still did very good. They ranked 6 out of 101. The Israelites did as well as us (ranked 18). The Koreans, as usual, did very good, but not as formidable as last year. Indeed, Republic of Korea was ranked 7 and the Democratic Republic of Korea was ranked 14. During these 20 or so years, the North Koreans missed the contest altogether for 10 years, but during the times they were around, they did reasonably well. Although we were not as good as the populous countries like China (ranked 1), and USA (ranked 2). We did better than India (ranked 40) and Indonesia (ranked 30). This year we did slightly better than Thailand (ranked 22), the country to host IMO 2015. They have been good, and I was told they put a lot of money into the event and in training their team. We also did better than several traditionally strong countries such as Poland (ranked 28), Iran (ranked 21) and Bulgaria (ranked 37). Indeed Bulgaria has a long tradition of mathematical competitions, and their competition materials are often very well sought. As in the last few years, we still did not do as well as Singapore (ranked 8). However, when I looked closely at their results, I found their gold medalists were not really much better than our silver medalists and I think we can do as well? In short, it is very interesting by simply looking at the results of countries during the years, we may gather some ideas on how we should train our members in the future.

(to be continued)

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2014 Bulgarian National Math Olympiad on May 17-18, 2014.

Problem 1. *(Teodosi Vitanov, Emil Kolev)* Given is a circle k and a point A outside it. The segment BC is a diameter of k. Find the locus of the orthocenter of $\Delta\, ABC$, when BC is changing.

Problem 2. *(Nikolay Beluhov)* Consider a rectangular $n\times$m table where $n \geq 2$ and $m \geq 2$ are positive integers. Each cell is colored in one of the four colors: white, green, red or blue. Call such a coloring interesting if any 2×2 square contains every color exactly once. Find the number of interesting colorings.

Problem 3. *(Alexander Ivanov)* A real nonzero number is assigned to every point in space. It is known that for any tetrahedron τ the number written in the incenter equals the product of the four numbers written in the vertices of τ. Prove that all numbers equal 1.

Problem 4. *(Peter Boyvalenkov)* Find all prime numbers p and q such that

$$p^2 \mid q^3+1 \quad\text{and}\quad q^2 \mid p^6-1.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 20, 2014***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO2014 and Beyond (II)

Leung Tat-Wing

To discuss the IMO2014 problems, let's proceed from the easier problems to the harder problems.

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

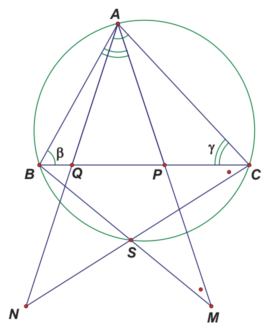
$$a_n < \frac{a_0+a_1+\ldots+a_n}{n} \leq a_{n+1}.$$

This problem is nice and easy. It gave us no problem. All of us got full scores in this problem. Nevertheless the problem is not entirely trivial, and indeed about 100 contestants scored nothing in this problem! First notice the middle term is not an arithmetical mean. Really during the question and answer period, some contestants did ask why the sequence doesn't start at index 1. Moreover the problem is not exactly an algebra problem, as it involves a strictly *increasing* sequence of *integers*. Try small cases, say $n = 1$. Then we need $a_1 < a_0+a_1$ sure, but not necessarily $a_0+a_1 \leq a_2$, why is that so? For $n = 2$, then we need $a_2 < (a_0+a_1+a_2)/2$, or $a_2 < a_0+a_1$, not necessarily true, but say when compared with the case of $n = 1$, if it is false, then $a_0+a_1 \leq a_2$ is true and we have an n satisfying the inequality! And the other side $a_0+a_1+a_2 \leq 2a_3$, why true again? If it is false, look at the left hand side for the case of $n = 3$. After several attempts, we really see what is going on. Indeed the inequality is equivalent to $na_n < a_0+a_1+\cdots+a_n \leq na_{n+1}$. The left hand inequality corresponds to $(a_0+a_1+\cdots+a_n) - na_n > 0$, while the right hand inequality corresponds to $(a_0+a_1+\cdots+a_n)-na_{n+1} \leq 0$, same as $(a_0+a_1+\cdots+a_{n+1})-(n+1)a_{n+1} \leq 0$. Alas, if we define $d_n=(a_0+a_1+\cdots+a_n) - na_n$, then we just have to show there exists a *unique* n such that $d_n > 0 \geq d_{n+1}$! The proof is then complete if we can see (prove) d_n is a strictly *decreasing* sequence of *integers*. Not too bad.

Using induction, or other measures on the expression $(a_0+a_1+\cdots+a_n)/n$, our team members managed to solve the problem.

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

This is the easiest problem in the competition, yet about 30 contestants did not get anything from it. Altogether more than 10 solutions were received, using synthetic geometry, coordinate geometry, complex numbers and the like. Some of us did it by coordinate geometry, setting the foot of A be (0,0), and coordinates $A(0,a)$, $B(b,0)$ and $C(c,0)$. Then get everything out of it via complicated calculations. But indeed if we can draw the picture properly, and do the angle tracings correctly, the problem is really not hard at all.



Indeed suppose BM and NC meet at S. Let $\angle ABC=\angle CAQ=\beta$ and $\angle ACB=\angle BAP=\gamma$, then $\Delta ABP\sim\Delta CAQ$. Hence

$$\frac{BP}{PM}=\frac{BP}{PA}=\frac{AQ}{QC}=\frac{QN}{QC}.$$

Also, $\angle NQC=\angle BQA=\angle APC=\angle BPM$. The last two statements imply $\Delta BPM\sim\Delta NQC$, hence $\angle BMP=\angle NCQ$. Then we also have $\Delta BPM\sim\Delta BSC$!

Finally, we have $\angle CSB=\angle MPB=\beta+\gamma=180°-\angle ABC$. So $\angle CSB+\angle BAC=180°$ and we are done.

Problem 2. Let $n \geq 2$ be an integer. Consider a $n\times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k\times k$ square which does not contain a rook on any of its k^2 unit squares.

All of us managed to give (basically) the correct answer ($\lceil \sqrt{n-1} \rceil$) and knew essentially how to tackle the question. There were gaps here and there and few points eventually deducted, but in my opinion, not really serious mistakes. Here n rooks are placed in a $n\times n$ board so that they are not attacking each other, and this time we ask for the largest possible gap (square with no rook). Of course the k^2 squares should be congruent to others and the "gap" square should be in one piece. Indeed several candidates had the same concern. This is really a classical chess board problem and I am not at all sure if the question was asked before somewhere.

First, given a $n\times n$ board with n rooks non-attacking (peaceful configuration). Suppose l is such that $l^2 < n$, then we can find a $l\times l$ square with no rook in it. Indeed there is a rook in the first *column*, consider the l consecutive *rows* starting with the row where the particular rook is placed. Now remove the first $n - l^2$ columns of this piece (hence at least one rook is removed). The remaining $l\times l^2$ piece can be decomposed into l $l\times l$ pieces of squares, but contain at most $l-1$ rooks, hence we have an empty $l\times l$ square.

Now we want to construct a peaceful configuration with largest possible square of size $\lceil \sqrt{n-1} \rceil \times \lceil \sqrt{n-1} \rceil$. Most of us see what the configuration should look like. We first let n be of the form l^2. Label the square with row i and column j as (i,j), with $0 \leq i \leq l-1$ and $0 \leq j \leq l-1$. The rooks are then placed on the positions $(il+j, jl+i)$, $0 \leq i,j \leq l-1$. One can easily check that any $l\times l$ square contains a rook.

Now comes where the most common gap lies. If $n < l^2$, we need to produce a peaceful configuration with no rook in any $l\times l$ square. The idea is of course to remove columns and rows from the previous construction. Only when (say) the top row and the leftmost column removed, *two* rooks may be removed, we have to put a rook back to an appropriate position (naturally where it should be) to return to a peaceful configuration!

(A 9×9 peaceful configuration with 2×2 squares as largest possible empty squares.)

								X
					X			
		X						
							X	
				X				
	X							
						X		
			X					
X								

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $1/n$.Given a finite collection of such coins (of not necessarily different denominations) with total value at most 99+½, prove that it is possible to split the collection into 100 or fewer groups, such that each group has total value at most 1.

I am happy to see how our students handled this problem. In short, they used various grouping and induction techniques and tricks, and changed the problem to a format they can handle, thus solved the problems. Even though our arguments were sometimes rather unclear and convoluted, thus some points deducted because of gaps and other things, four of us essentially solved the problem. Indeed the main idea of solving the problem is by "merging" or "cleaning" the set of coins. Clearly if the process can still be completed after merging the coins, it can be done before merging!

Indeed the problem can be generalized as follows. Given coins of total value at most N−½, they can be split into N groups each of value at most 1. The problem then can be completed by the following steps.

(i) Two coins of values $1/(2k)$ may be merged into a coin of value $2\times 1/(2k)=1/2$, thus for every even number m, we may assume there is at most one coin of value $1/m$.

(ii) For every odd number m, there are at most $m-1$ coins of such value, otherwise they can be merged to form a coin of value 1 first.

(iii) Coins of value 1 must form a group of itself. Thus if there are d coins of value 1 in a group of N coins, we might as well consider a group of $N-d$ coins of values less than 1.

(iv) Now consider coins of values $1/(2k-1)$ and $1/(2k)$, with $k=1,2,\ldots,N$. We first place them into N groups according different values of k. In each group, the total value is at most

$$(2k-2)\times\frac{1}{2k-1}+\frac{1}{2k}=1-\frac{1}{2k-1}+\frac{1}{2k}<1.$$

The total value of all N groups is at most N−½. By taking average, there exists a group of total value at most

$$\frac{1}{N}(N-\frac{1}{2})=1-\frac{1}{2N}.$$

(v) All the remaining coins are of values less than $1/(2N)$. We may put them one by one into each group, as long as the value of each group does not exceed $1-1/(2N)$ and we are done!

The problem is meant to be a number theory problem, but is really more like a combinatorial problem. Our members managed to give different proofs to this problem and it is very nice. But indeed it is natural to consider coins of larger values (greedy method) first then consider coins of small values (a lot of them).

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90^\circ$, $\angle THC - \angle DTC = 90^\circ$. Prove that line BD is tangent to the circumcircle of triangle TSH.

In these few years, problems of this kind appear rather frequently. Proving a certain line is tangent to a certain (hidden) circle, or two (hidden) circles will touch each other, or the like, are generally not too easy. Still one should be able to handle them by first finding out some related geometric properties, and then obtain final results still by using only basic geometric properties and techniques.

Let us look at this problem. It is not easy to draw an accurate and nice picture, let alone proving it.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 20, 2014.***

Problem 451. Let P be an n-sided convex polygon on a plane and $n>3$. Prove that there exists a circle passing through three consecutive vertices of P such that every point of P is inside or on the circle.

Problem 452. Find the least positive real number r such that for all triangles with sides a,b,c, if $a \ge (b+c)/3$, then

$$c(a+b-c) \le r\,((a+b+c)^2+2c(a+c-b)).$$

Problem 453. Prove that there exist infinitely many pairs of relatively prime positive integers a,b with $a>b$ such that b^2-5 is divisible by a and a^2-5 is divisible by b.

Problem 454. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 respectively. Let P be a point of intersection of Γ_1 and Γ_2. Let line AB be an external common tangent to Γ_1, Γ_2 with A on Γ_1, B on Γ_2 and A, B, P on the same side of line O_1O_2. There is a point C on segment O_1O_2 such that lines AC and BP are perpendicular. Prove that $\angle APC=90°$.

Problem 455. Let a_1, a_2, a_3, … be a permutation of the positive integers. Prove that there exist infinitely many positive integer n such that the greatest common divisor of a_n and a_{n+1} is at most $3n/4$.

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Solutions

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Problem 446. If real numbers a and b satisfy $3^a+13^b=17^a$ and $5^a+7^b=11^b$, then prove that $a<b$.

Solution. **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Elaine LAM** (Tsuen Wan Secondary School), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **PANG Lok Wing, YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff) and **Simon YAU.**

If $a \ge b$, then $3^a+13^a \ge 3^a+13^b=17^a$. (*) Since $3/17<13/17<1$, the function $f(x)=(3/17)^x+(13/17)^x$ is strictly decreasing. By (*), $f(a) \ge 1> f(1)$. So $a<1$.

Next, $5^b+7^b \le 5^a+7^b = 11^b$. (**) Since $5/11 < 7/11 < 1$, the function $g(x) = (5/11)^x + (7/11)^x$ is strictly decreasing. By (**), $g(b) \le 1 < g(1)$. So $b>1>a$, contradiction.

Other commended solvers: **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 447. For real numbers x, y, z, find all possible values of $\sin(x+y)+\sin(y+z)+\sin(z+x)$ if

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}.$$

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff)**, Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Let $S=x+y+z$. Cross multiply and transfer all terms to one side. We get

$$\begin{aligned}0 &= \sin S\cos x-\cos S\sin x+\sin S\cos y\\ &\quad -\cos S\sin y+\sin S\cos z-\cos S\sin z\\ &= \sin(S-x)+\sin(S-y)+\sin(S-z)\\ &= \sin(y+z)+\sin(z+x)+\sin(x+y).\end{aligned}$$

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 448. Prove that if s,t,u,v are integers such that $s^2-2t^2+5u^2-3v^2=2tv$, then $s=t=u=v=0$.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Assume s,t,u,v are not all zeros. By cancelling all common factors of s,t,u,v, we may assume they are relatively prime. We can rewrite the equation as

$$2(s^2+5u^2) = (2t+v)^2+5v^2. \qquad (\dagger)$$

For $0 \le x, y \le 4$, we have $2x^2 \equiv y^2 \pmod 5$ if and only if $x \equiv y \equiv 0 \pmod 5$. (‡) So $s^2+5u^2 \equiv 2t+v \equiv 0 \pmod 5$, which implies $s=5m$ and $2t+v=5n$ for some integers m,n. Substituting these into (†), we get $2(5m^2+u^2)=5n^2+v^2$. By (‡), u, v are divisible by 5. Then s,t,u,v are divisible by 5, contradicting they are relatively prime. So s,t,u,v are all zeros.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India),

Problem 449. Determine the smallest positive integer k such that no matter how $\{1,2,3,\ldots,k\}$ are partitioned into two sets, one of the two sets must contain two distinct elements m, n such that mn is divisible by $m+n$.

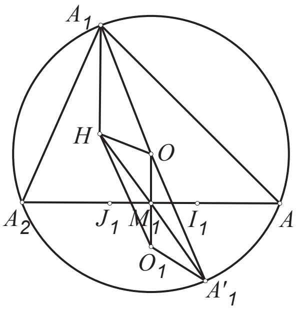
Solution. **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Call distinct positive integers m,n a <u>good</u> pair if mn is divisible by $m+n$. Collect all good pairs with $m,n\le 40$. We will try to separate m,n first. Let $A=\{1,2,3,5,8, 10, 12, 13, 14, 18, 19, 21, 22, 23, 30, 31,32,33,34\}$ and $B=\{4, 6, 7, 9, 11, 15, 16, 17, 20, 24, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39\}$. Each of A and B do not contain any good pair. For $1 \le k \le 39$, we can remove integers greater than k from A and B to get 2 disjoint subsets of $\{1, 2, \ldots, k\}$ with no good pair in each subset.

For k=40, put 6, 12, 24, 40, 10, 15 and 30 around a circle. Notice any two consecutive terms in this circle is a good pair. No matter how we divide $\{1,2,\ldots,40\}$ into 2 disjoint subsets, one of the subsets will contain at least 4 of 7 numbers in the circle. So there will be a good pair in that subset. Therefore, 40 is the desired least integer.

Other commended solvers: **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam).

Problem 450. (*Proposed by Michel BATAILLE*) Let $A_1A_2A_3$ be a triangle with no right angle and O be its circumcenter. For i = 1,2,3, let the reflection of A_i with respect to O be A_i' and the reflection of O with respect to line $A_{i+1}A_{i+2}$ be O_i (subscripts are to be taken modulo 3). Prove that the circumcenters of the triangles OO_iA_i' (i = 1,2,3) are collinear.



Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4).

Notice that O_1 is the reflection of O with respect to the midpoint M_1 of A_2A_3. By the nine point circle theorem (see <u>Math Excalibur</u>, *vol*.3, *no* 1, *p*,1), AH, OM_1 are parallel and their lengths are 2:1. Now A_1O=OA_1'. So, in $\triangle A_1A_1'H$, M_1 is the midpoint of $A_1'H$, i.e. H is the reflection of A_1' with respect to M_1.

Let I_1 be the circumcenter of $\triangle OO_1A_1'$. Then I_1 lies on the perpendicular bisector A_2A_3 of OO_1. Reflect I_1 with respect to M_1 to J_1. Then J_1 also lies on A_2A_3. With respect to M_1, J_1 is the circumcenter of the reflection of $\triangle OO_1A_1'$, i.e. $\triangle OO_1H$. So, J_1 also lies on the perpendicular bisector of OH.

Define I_2, I_3, J_2, J_3 similarly. As J_2, J_3 also lie on the perpendicular bisector of OH by a similar proof, J_1, J_2, J_3 are collinear. Then by Menelaus' theorem,

$$\frac{A_2J_1}{J_1A_3}\cdot\frac{A_3J_2}{J_2A_1}\cdot\frac{A_1J_3}{J_3A_2}=-1.$$

As A_3I_1/I_1A_2=A_2J_1/J_1A_3 (due to I_1, J_1 are reflection of the midpoint of A_2A_3) and similarly for I_2, J_2, I_3, J_3, we have

$$\frac{A_3I_1}{I_1A_2}\cdot\frac{A_1I_2}{I_2A_3}\cdot\frac{A_2I_3}{I_3A_1}=-1.$$

By the converse of Menelaus' Theorem, I_1, I_2, I_3 are collinear as desired.

Other commended solvers: **Andrea FANCHINI** (Cantú, Italy), **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **Samiron SADHUKHAN** (Kendriya Vidyalaya, Barrackpore, Kolkata, India), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Olympiad Corner

(***Continued from page 1***)

Problem 5. *(Nikolay Nikolov)* Find all functions $f:\mathbb{Q}^+\to\mathbb{R}^+$ such that

$f(xy) = f(x+y)(f(x)+f(y))$ for any $x,y\in\mathbb{Q}^+$.

Problem 6. *(Nikolay Beluhov)* The quadrilateral $ABCD$ is inscribed in the circle k. The lines AC and BD meet in E and the lines AD and BC meet in F. Show that the line through the incenters of ΔABE and ΔABF and the line through the incenters of ΔCDE and ΔCDF meet on k.

IMO2014 and Beyond (II)

(***Continued from page 2***)

First, let the line passing through C and is perpendicular to SC meets AB at Q. Then $\angle SQC$=90°−$\angle BSC$=180°−$\angle SHC$. So C, H, S, Q are concyclic. Moreover SQ is a diameter of this circle, thus the circumcenter K of SHC lies on AB. Likewise, circumcenter L of the circle CHT lies on AD. To show the circumcircle of the triangle SHT is tangent to BD, it suffices to show the perpendicular bisectors of HS and HT meet at AH. But the two perpendicular bisectors coincide with the angle bisectors of AKH and ALH, thus by the bisector theorem, it suffices to show AK/KH=AL/LH. Let M be the midpoint of CH, then B,C,M,K are concyclic, L,C,M,D are concyclic. By the sine law, AK/AL= $\sin\angle ALK$ / $\sin\angle AKL$ = $(DM/CL)/(BM/CK)$ = CK/CL = KH/LH.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite areas; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

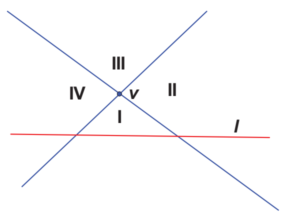
Notes: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

I have to admit that I don't like this problem at all. Indeed it was meant to be an "open end" problem, that students may produce different results with different degrees of difficulty. But when I first saw the problem, I thought we should give an algorithm, say a greedy algorithm, or other heuristic that gives good pattern (with as many blue colored lines as possible), and then analyze the pattern and give an estimate. Not so. (I guess I have become kind of intuitionist.) I doubt if there was any algorithmic solution anyway. Indeed in the official solution, a best possible solution is assumed, surely it exists, but we were not told how to get there.

Let me reproduce a part of the proof as follows. Given a set of n lines colored blue and red, and the lines colored blue is as large as possible (maximality argument), so that every finite region still has at least one boundary line colored red. Assume k lines are colored blue. Call a vertex which is the intersection of two blue lines *blue* as well, so there are ${}_kC_2$ blue vertices.

Now take any red line l, using the maximality argument, there exists at least one region with this red line l as the *only* red side, (for if all regions have two or more red lines, surely we can change one more red line to blue). In this region there is at least one blue vertex v since any finite region has at least three lines. We then associate the blue vertex with the red line. Now finally every blue vertex v belongs to four regions, (some may be unbounded), hence it may be associated with at most four red lines. Therefore the total number of red lines is at most $4{}_kC_2$=$2k(k-1)$.

On the other hand, there are $n-k$ red lines, thus, $n-k\le 2k(k-1)$. Solving for n, we get $n\le 2k^2-k\le 2k^2$. Hence, $k\ge\lceil\sqrt{n/2}\,\rceil$ and we get an estimate on the number of blue lines!



By putting some weights on the blue vertices, or by refining local analysis, one may get the stronger result $k\ge\sqrt{n}$.

Mathematical Excalibur

Volume 19, Number 3 *November 2014 – January 2015*

Olympiad Corner

Below are the problems of the IMO2015 Hong Kong Team Selection Test 2 held on 25th October, 2014.

Problem 1. Assume the dimensions of an answer sheet to be 297 mm by 210 mm. Suppose that your pen leaks and makes some non-intersecting ink stains on the answer sheet. It turns out that the area of each ink stain does not exceed 1 mm^2. Moreover, any line parallel to an edge of the answer sheet intersects at most one ink stain. Prove that the total area of the ink stains is at most 253.5 mm^2. (You may assume a stain is a connected piece.)

Problem 2. Let $\{a_n\}$ be a sequence of positive integers. It is given that $a_1=1$, and for every $n\geq 1$, a_{n+1} is the smallest positive integer greater than a_n which satisfies the following condition: for any integers i, j, k, with $1 \leq i, j, k \leq n+1$, $a_i+a_j \neq 3a_k$. Find a_{2015}.

Problem 3. Let ABC be an equilateral triangle, and let D be a point on AB between A and B. Next, let E be a point on AC with DE parallel to BC. Further, let F be the midpoint of CD and G the circumcentre of $\triangle ADE$. Determine the interior angles of $\triangle BFG$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 31, 2015***.

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Variations and Generalisations to the Rearrangement Inequality

Law Ka Ho

A. The rearrangement inequality

In *Math Excalibur*, vol. 4, no. 3, we can find the following

Theorem 1 (Rearrangement inequality) Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two increasing sequences of real numbers. Then amongst all **random sums** of the form

$$a_1b_{\sigma_1} + a_2b_{\sigma_2} + \cdots + a_nb_{\sigma_n},$$

where $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a permutation of $(1,2,\ldots,n)$,

- the greatest is the **direct sum** $a_1b_1+a_2b_2+\cdots+a_nb_n$;
- the smallest is the **reverse sum** $a_1b_n+a_2b_{n-1}+\cdots+a_nb_1$.

A well-known corollary of the rearrangement inequality is the following

Theorem 2 (Chebyshev's inequality) With the same setting in Theorem 1, the quantity

$$\frac{(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)}{n}$$

lies between the direct sum and the reverse sum, again with equality if and only if at least one of the two sequences is constant.

B. A variation --- from 'sum' to 'product'

The different 'sums' in the rearrangement inequality are in fact 'sums of products'. For this reason we shall from now on call them **P-sums**, to remind ourselves that we take products and then sum them up. Naturally, we ask what happens if we look at 'product of sums' (**S-products**) instead.

A little trial *suggests* that, opposite to the case of P-sums, the direct S-product is minimum while the reverse S-product is maximum. For example we may take the sequences $1\leq2\leq3\leq4$ and $5\leq6\leq7\leq8$. The direct S-product of these sequences is $(1+5)(2+6)(3+7)(4+8) = 5760$ and the reverse S-product of the sequences is $(1+8)(2+7)(3+6)(4+5) = 6561$. We can also check some random S-products, e.g we have $(1+6)(2+5)(3+8)(3+7) = 5929$ and $(1+6)(2+7)(3+8)(4+5) = 6237$.

But then a little further thought shows that this is not quite right. For instance we may take $1\leq2\leq3\leq4$ and $-5\leq-2\leq1\leq2$ and end up with a reverse S-product $(1+2)(2+1)[3+(-2)][4+(-5)]$, which is negative. Yet, some random S-products, such as $[1+(-2)](2+2)(3+1)[4+(-5)]$, can be positive.

It turns out that we have to require the variables to be non-negative for the result to hold.

Theorem 3 (Rearrangement inequality for S-products) Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two increasing sequences of non-negative real numbers. Then amongst all random S-products of the form

$$(a_1+b_{\sigma_1})(a_2+b_{\sigma_2})\cdots(a_n+b_{\sigma_n})$$

where $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a permutation of $(1,2,\ldots,n)$,

- the smallest is the direct S-product $(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)$;
- the greatest is the reverse S-product $(a_1+b_n)(a_2+b_{n-1})\cdots(a_n+b_1)$.

Proof Take any random S-product

$$(a_1+b_{\sigma_1})(a_2+b_{\sigma_2})\cdots(a_n+b_{\sigma_n})$$

which is not the direct S-product. Then there exists $i<j$ such that $b_{\sigma_i} > b_{\sigma_j}$.

Let's see what happens if we swap σ_i and σ_j. In that case only two terms are changed. Consider the two products

$$P_1 = (a_i+b_{\sigma_i})(a_j+b_{\sigma_j}) \text{ and}$$

$$P_2 = (a_i+b_{\sigma_j})(a_j+b_{\sigma_i}).$$

(continued on page 2)

After expanding, cancelling and factoring, we have

$$P_2 - P_1 = (a_i - a_j)(b_{\sigma_j} - b_{\sigma_i}),$$

which is non-positive since $a_i - a_j \le 0$ and $b_{\sigma_i} > b_{\sigma_j}$. So $P_2 \ge P_1$. This means swapping σ_i and σ_j leads to a larger (or equal) S-product. It follows that the direct S-product is the minimum amongst all random S-products. In a similar manner we can prove that the reverse S-product is the maximum.

Example 4 (IMO 1966) In the interior of sides BC, CA, AB of ΔABC, points K, L, M respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of ΔABC.

Solution Let a, b, c denote the lengths of the sides opposite A, B, C respectively. Let also a_1 and a_2 denote the lengths of the two segments after the side with length a is cut into two parts by the point K (i.e. $BK = a_1$ and $KC = a_2$), and similarly for b_1, b_2, c_1, c_2. The six variables a_1, a_2, b_1, b_2, c_1, c_2 can be ordered to form an increasing sequence. By the rearrangement inequality for S-products, the direct S-product

$$(a_1+a_1)(a_2+a_2)(b_1+b_1)(b_2+b_2)(c_1+c_1)(c_2+c_2)$$
$$= 64a_1a_2b_1b_2c_1c_2$$

is less than or equal to the random S-product

$$(a_1+a_2)(a_2+a_1)(b_1+b_2)(b_2+b_1)(c_1+c_2)(c_2+c_1)$$
$$= a^2b^2c^2.$$

Let S denote the area of ΔABC. If triangles AML, BKM, CLK all have areas greater than $S/4$, then using the above result we have

$$\begin{aligned}\left(\frac{S}{4}\right)^3 &< \left(\frac{1}{2}c_1b_2\sin A\right)\left(\frac{1}{2}c_2a_1\sin B\right)\left(\frac{1}{2}a_2b_1\sin C\right)\\ &\le \frac{a^2b^2c^2}{8\cdot 64}\cdot\sin A\sin B\sin C\\ &= \frac{1}{64}\left(\frac{1}{2}ab\sin C\right)\left(\frac{1}{2}bc\sin A\right)\left(\frac{1}{2}ca\sin B\right)\\ &= \left(\frac{S}{4}\right)^3\end{aligned}$$

which is a contradiction.

Example 5 (IMO 1984) Prove that

$$0 \le xy + yz + zx - 2xyz \le 7/27,$$

where x, y and z are non-negative real numbers for which $x+y+z=1$.

Solution The left-hand inequality is pretty easy. We have

$$\begin{aligned}&xy+yz+zx-2xyz\\ &=(xy-xyz)+(yz-xyz)+(zx-xyz)+xyz\\ &=xy(1-z)+yz(1-x)+zx(1-y)+xyz\\ &=xy(x+y)+yz(y+z)+zx(z+x)+xyz \ge 0.\end{aligned}$$

For the right-hand inequality, it is well-known that

$$\begin{aligned}1 &= (x+y+z)^2\\ &= x^2+y^2+z^2+xy+yz+zx\\ &\ge 3(xy+yz+zx)\end{aligned}$$

and so $xy + yz + zx \le 1/3$. By the rearrangement inequality for S-products, we have

$$\begin{aligned}&(1-2x)(1-2y)(1-2z)\\ &\le\left(\frac{1-2x}{2}+\frac{1-2y}{2}\right)\left(\frac{1-2y}{2}+\frac{1-2z}{2}\right)\left(\frac{1-2z}{2}+\frac{1-2x}{2}\right)\\ &=zxy\ .\end{aligned}$$

(The rearrangement inequality for S-products applies only if the three terms on the left hand side are non-negative. However, if this is not true then exactly one of them is negative and the result therefore still holds.) Expanding gives

$$1-2(x+y+z)+4(xy+yz+zx)-8xyz \le xyz$$

or $9xyz \ge 4(xy+yz+zx)-1$. From this, we have

$$\begin{aligned}&xy+yz+zx-2xyz\\ &\le xy+yz+zx-2\left(\frac{4(xy+yz+zx)-1}{9}\right)\\ &=\frac{xy+yz+zx+2}{9}\le\frac{2\tfrac{1}{3}}{9}=\frac{7}{27}.\end{aligned}$$

C. A generalisation — from two sequences to more

Another natural direction of generalising the rearrangement inequality (for P-sums) is to consider the case in which there are more than two sequences. This time we need two subscripts to index the terms, one for the index of the sequence and one for the index of a particular term of a sequence. Again, we need to restrict ourselves to sequences of non-negative numbers (for both P-sums and S-products), otherwise one can easily construct counter- examples. Also, note that there is no such thing as 'reverse P-sum/S-product' when there are more than two sequences.

Theorem 6 (Rearrangement inequality for multiple sequences) Suppose there are m increasing sequences (each with n terms) of non-negative numbers, say, $a_{i1} \le a_{i2} \le \cdots \le a_{in}$, where $i = 1, 2, \ldots, m$. Then

- the direct P-sum $\sum_{j=1}^{n} a_{1j}a_{2j}\cdots a_{mj}$ is greater than or equal to any other random P-sum of the form $\sum_{j=1}^{n} a_{1\sigma_{1j}}a_{2\sigma_{2j}}\cdots a_{m\sigma_{mj}}$;
- the direct S-product $\prod_{j=1}^{n}(a_{1j}+a_{2j}+\cdots+a_{mj})$ is smaller than or equal to any other random S-product of the form $\prod_{j=1}^{n}(a_{1\sigma_{1j}}+a_{2\sigma_{2j}}+\cdots+a_{m\sigma_{mj}})$.

Here $(\sigma_{i1},\sigma_{i2},\ldots,\sigma_{in})$ is a permutation of $(1,2,\ldots,n)$ for $i = 1,2, \ldots, m$.

Remarks.

(1) Theorem 6 is sometimes known as '微微對偶不等式' in Chinese.

(2) A less clumsy way to express Theorem 6 is to use matrices. With the above m sequences we may form the matrices

$$A=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix} \text{ and}$$

$$B=\begin{pmatrix}a_{1\sigma_{11}} & a_{1\sigma_{12}} & \cdots & a_{1\sigma_{1n}}\\ a_{2\sigma_{21}} & a_{2\sigma_{22}} & \cdots & a_{2\sigma_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m\sigma_{m1}} & a_{m\sigma_{m2}} & \cdots & a_{m\sigma_{mn}}\end{pmatrix}.$$

Here each row of A is in ascending order (corresponding to one of the m increasing sequences) while each row of B is a permutation of the terms in the corresponding row of A (corresponding to a permutation of the corresponding sequence). Then Theorem 6 says

- the sum of column products (P-sum) in A is greater than or equal to that in B;
- the product of column sums (S-product) in A is less than or equal to that in B.

(3) The proof of Theorem 6 is essentially the same as that of Theorem 3, and is therefore omitted.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 31, 2015.***

Problem 456. Suppose $x_1, x_2, \ldots, x_n$ are non-negative and their sum is 1. Prove that there exists a permutation σ of $\{1,2,\cdots,n\}$ such that

$$x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)} \le 1/n.$$

Problem 457. Prove that for each $n = 1,2,3,\ldots$, there exist integers a, b such that if integers x, y are relatively prime, then $\sqrt{(a-x)^2+(b-y)^2} > n.$

Problem 458. Nonempty sets A_1, A_2, A_3 form a partition of $\{1,2,\ldots,n\}$. If $x+y=z$ have no solution with x in A_i, y in A_j, z in A_k and $\{i,j,k\}=\{1,2,3\}$, then prove that A_1, A_2, A_3 cannot have the same number of elements.

Problem 459. H is the orthocenter of acute $\triangle ABC$. D,E,F are midpoints of sides BC, CA, AB respectively. Inside $\triangle ABC$, a circle with center H meets DE at P,Q, EF at R,S, FD at T,U. Prove that $CP=CQ=AR=AS=BT=BU$.

Problem 460. If $x,y,z > 0$ and $x+y+z+2 = xyz$, then prove that

$$x+y+z+6 \ge 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right).$$

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Solutions

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Problem 451. Let P be an n-sided convex polygon on a plane and $n>3$. Prove that there exists a circle passing through three consecutive vertices of P such that every point of P is inside or on the circle.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India) and **T.W. LEE** (Alumni of New Method College).

Let R_{XYZ} denote the radius of the circle through vertices X,Y,Z of P. Let circle Γ through vertices A,B,C of P be one with maximal radius. Without loss of generality, we may assume $\angle ABC$ and $\angle ACB < 90°$. If there is a vertex D of P outside Γ, let AD meet Γ at E. Then $\angle ADC < \angle AEC = \angle ABC$. By the extended sine law

$$R_{ADC} = \frac{AC}{2\sin\angle ADC} > \frac{AC}{2\sin\angle ABC} = R_{ABC},$$

contradicting maximality of Γ. So all vertices of P is on or inside Γ.

Let F be the vertex of P next to A (toward C). If F is inside Γ, then $AFCB$ is convex and $\angle AFC+\angle ABC > 180°$. Hence $0° < 180°-\angle AFC < \angle ABC < 90°$. Then

$$R_{AFC} = \frac{AC}{2\sin\angle AFC} > \frac{AC}{2\sin\angle ABC} = R_{ABC},$$

contradiction. So F is on Γ. Similarly, the vertex of P next to A (toward B) is on Γ.

Problem 452. Find the least positive real number r such that for all triangles with sides a,b,c, if $a \ge (b+c)/3$, then

$$c(a+b-c) \le r\,((a+b+c)^2+2c(a+c-b)).$$

Solution. **Jon GLIMMS** and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

Let $I = a+b-c$. Then $a \ge (b+c)/3$ implies $a-b \ge -(a+b-c)/2 = -I/2$ (*)

Using $a+b+c=I+2c$, (*) and the *AM-GM* inequality, we have

$$\begin{aligned} J &= \frac{(a+b+c)^2+2c(a+c-b)}{2c(a+b-c)} \\ &= \frac{I^2+4cI+4c^2}{2cI}+\frac{a+c-b}{I} \\ &= \frac{I}{2c}+2+\frac{3c}{I}+\frac{a-b}{I} \\ &\ge \frac{3}{2}+\frac{I}{2c}+\frac{3c}{I} \ge \frac{3}{2}+2\sqrt{\frac{3}{2}}. \end{aligned}$$

Equality hold if $a = (b+c)/3$ and $I^2=6c^2$, i.e. $a:b:c = 2+\sqrt{6} : 2+3\sqrt{6} : 4$. The least r such that $1/(2J) \le r$ is $(\sqrt{24}-3)/15$.

Problem 453. Prove that there exist infinitely many pairs of relatively prime positive integers a,b with $a>b$ such that b^2-5 is divisible by a and a^2-5 is divisible by b.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

Note $(a,b) = (11,4)$ is a solution. From any solution (a,b) with $a>b\ge 4$, we get $a^2-5=bc$ and $b^2-5=ad$ for some positive integers c and d. Now we show (c,a) is another such solution. First $bc = a^2-5 > a^2-a = a(a-1) \ge ab$ implies $c>a$. If a prime p divides $\gcd(a,c)$, then $a^2-5=bc$ and $b^2-5=ad$ imply $b^2=ad+5=ad+a^2-bc$ is divisible by p. Since $\gcd(a,b)=1$, we get $\gcd(c,a)=1$. Using $\gcd(a,b)=1$ and $a(a+d)=a^2+b^2-5 = b(b+c)$, we see a divides $b+c$. Then a divides $(b+c)(c-b) + (b^2-5) = c^2-5$. So there are infinitely many solutions.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **O Kin Chit** (G. T. (Ellen Yeung College), **WONG Yat** (G. T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 454. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 respectively. Let P be a point of intersection of Γ_1 and Γ_2. Let line AB be an external common tangent to Γ_1, Γ_2 with A on Γ_1, B on Γ_2 and A, B, P on the same side of line O_1O_2. There is a point C on segment O_1O_2 such that lines AC and BP are perpendicular. Prove that $\angle APC=90°$.

Solution. **Serik JUMAGULOV** (Karaganda State University, Qaragandy City, Kazakhstan).

Other than P, let the circles also meet at Q. If $PQ \cap AB = M$, then M is the midpoint of AB as $MA^2 = MP\times MQ = MB^2$. Let $PQ \cap O_1O_2 = K$, $BP\cap AC=N$ and AL be a diameter of the circle with center O_1. Since $PQ \perp O_1O_2$ and $BN\perp AC$, $PNCK$ is cyclic. Now $\angle PBM = 90°-\angle NAB = \angle CAO_1$ and $\angle BPM = \angle KPN = \angle ACO_1$. So $\Delta ACO_1 \sim \Delta BPM$. Then $AC/BP = AO_1/BM = AL/BA$. So $\Delta ACL \sim \Delta BPA$. Then $\angle ALP = \angle BAP = \angle ALC$. So L,C,P are collinear. As AL is a diameter, $\angle APC = 90°$.

Other commended solvers: **Andrea FANCHINI** (Cantú, Italy), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 455. Let $a_1, a_2, a_3, \ldots$ be a permutation of the positive integers. Prove that there exist infinitely many positive integer n such that the greatest common divisor of a_n and a_{n+1} is at most $3n/4$.

Solution. **Jon GLIMMS** and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

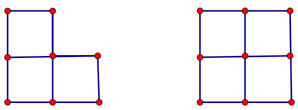
Assume that there exists N such that for all $n\ge N$, $\gcd(a_n,a_{n+1})>3n/4$. Then for all $n \ge 4N$, $a_n \ge \gcd(a_n,a_{n+1}) > 3n/4 \ge 3N$. Since $a_1, a_2, a_3, \ldots$ is a permutation of the positive integers, we see $\{1,2,\cdots,3N\}$ is a subset of $\{a_1, a_2,\cdots, a_{4N-1}\}$. Now the intersection of $\{1,2,\cdots,3N\}$ and $\{a_{2N}, a_{2N+1},\cdots,a_{4N-1}\}$ has at least $3N-(2N-1)$

$= N+1$ elements. By the pigeonhole principle, there exists k such that $2N \le k < 4N-1$ and $a_k, a_{k+1} \le 3N$. Then $\gcd(a_k,a_{k+1}) \le \frac{1}{2}\max\{a_k, a_{k+1}\} \le 3N/2 \le 3k/4$, contradiction.

Olympiad Corner

(*Continued from page 1*)

Problem 4. A 11×11 grid is to be covered completely without overlapping by some 2×2 squares and L-shapes each composed of three unit cells. Determine the smallest number of L-shapes used. (Each shape must cover some grids entirely and cannot be placed outside the 11×11 grid. The L-shapes may be reflected or rotated when placed on the grid.)



Variations and Generalisations

(*Continued from page 2*)

Example 7 Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers whose sum is at most 1/2. Show that $(1-x_1)(1-x_2)\cdots(1-x_n) \ge 1/2$.

Solution Form the $n\times n$ matrix

$$A=\begin{pmatrix} 1-x_1 & 1 & \cdots & 1\\ 1-x_2 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1-x_n & 1 & \cdots & 1\end{pmatrix}$$

whose rows are in ascending order. Consider the matrix

$$B=\begin{pmatrix} 1-x_1 & 1 & \cdots & 1\\ 1 & 1-x_2 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1-x_n\end{pmatrix}$$

in which each row is a permutation of the terms in the corresponding row of A. By the rearrangement inequality for multiple sequences, the P-sum in A is greater than the P-sum in B, i.e.

$$(1-x_1)(1-x_2)\cdots(1-x_n)+n-1 \ge (1-x_1)+(1-x_2)+\cdots+(1-x_n).$$

It follows that

$$(1-x_1)(1-x_2)\cdots(1-x_n) \ge 1-(x_1+x_2+\cdots+x_n) \ge 1-1/2 = 1/2.$$

Example 8 Let $x_1, x_2, \ldots, x_n$ be positive real numbers with sum 1. Show that

$$\frac{x_1x_2\cdots x_n}{(1-x_1)(1-x_2)\cdots(1-x_n)} \le \frac{1}{(n-1)^n}.$$

Solution Without loss of generality assume $x_1 \le x_2 \le \cdots \le x_n$. Form the $(n-1)\times n$ matrix

$$A=\begin{pmatrix} x_1 & x_2 & \cdots & x_n\\ x_1 & x_2 & \cdots & x_n\\ \vdots & \vdots & \ddots & \vdots\\ x_1 & x_2 & \cdots & x_n\end{pmatrix}$$

whose rows are in ascending order. The S-product of A is thus $(n-1)^n x_1x_2\cdots x_n$. Now the matrix B given by

$$B=\begin{pmatrix} x_1 & x_2 & \cdots & x_n\\ x_2 & x_3 & \cdots & x_1\\ \vdots & \vdots & \ddots & \vdots\\ x_{n-1} & x_n & \cdots & x_{n-2}\end{pmatrix}$$

has the property that each of its rows is a permutation of the terms in the corresponding row of A. Furthermore, since $x_1, x_2, \ldots, x_n$ have sum 1, the S-product of B is equal to $(1-x_1)(1-x_2)\cdots(1-x_n)$. By the rearrangement inequality for multiple sequences, we have $(n-1)^n x_1x_2\cdots x_n \le (1-x_1)(1-x_2)\cdots(1-x_n)$.

D. Proofs of some classic inequalities

The rearrangement inequality for multiple sequences can be used to prove a number of classic inequalities. We look at some such examples in this final section.

Theorem 9 (Bernoulli inequality) For real numbers $x_1, x_2, \ldots, x_n$, where either all are non-negative or all are negative but not less than -1, we have

$$\prod_{i=1}^{n}(1+x_i) \ge 1+\sum_{i=1}^{n}x_i.$$

Proof Without loss of generality assume $x_1 \le x_2 \le \cdots \le x_n$. Suppose $x_1, x_2, \ldots, x_n$ are all non-negative. Consider the $n\times n$ matrices

$$A=\begin{pmatrix} 1 & 1 & \cdots & 1+x_1\\ 1 & 1 & \cdots & 1+x_2\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1+x_n\end{pmatrix} \quad\text{and}$$

$$B=\begin{pmatrix} 1+x_1 & 1 & \cdots & 1\\ 1 & 1+x_2 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1+x_n\end{pmatrix}.$$

Then A and B satisfy the properties stated in Theorem 6. Thus the P-sum in A is greater than or equal to that in B,

i.e. $n-1+\prod_{i=1}^{n}(1+x_i) \ge \sum_{i=1}^{n}(1+x_i)$.

This gives $\prod_{i=1}^{n}(1+x_i) \ge 1+\sum_{i=1}^{n}x_i$.

The proof in the latter case (in which $x_1, x_2, \ldots, x_n$ are negative but not less than -1) is essentially the same; just move the rightmost column of A to the leftmost.

Theorem 10 (Generalised Chebyshev's inequality) For m increasing sequences (each with n terms) of non-negative real numbers, say, $a_{i1} \le a_{i2} \le \cdots \le a_{in}$, where $i=1,2,\ldots, m$,

the direct P-sum $\sum_{j=1}^{n} a_{1j}a_{2j}\cdots a_{mj}$ is greater than or equal to

$$\frac{1}{n^{m-1}}\prod_{i=1}^{m}\left(a_{i1}+a_{i2}+\cdots+a_{in}\right).$$

Proof Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}.$$

Now we can randomly form a matrix B as follows. The first row of B is the same as that of A. Each other row of B is obtained by shifting the corresponding row of A to the right by k places, where k is randomly chosen from $0, 1, 2, \ldots, n-1$. (For instance, if $k=1$, then the second row of B will be $(a_{2\,n}, a_{2\,1}, \cdots, a_{2\,n-1}$.) Thus a total of n^{m-1} different possible B's can be formed. Each of them has a P-sum less than or equal to that of A, according to Theorem 6. The sum of all the P-sums for these n^{m-1} is precisely

$$\prod_{i=1}^{m}\left(a_{i1}+a_{i2}+\cdots+a_{in}\right),$$

which should therefore be less than or equal to n^{m-1} times the P-sum of A, i.e. n^{m-1} times the direct P-sum. This gives us the desired result.

Mathematical Excalibur

Volume 19, Number 4 — *February 2015 – March 2015*

Olympiad Corner

Below are the problems of the Team Selection Test 1 for the Dutch IMO team held in June, 2014.

Problem 1. Determine all pairs (a,b) of positive integers satisfying

$$a^2+b \mid a^2b+a \quad\text{and}\quad b^2-a \mid ab^2+b.$$

Problem 2. Let $\triangle ABC$ be a triangle. Let M be the midpoint of BC and let D be a point on the interior of side AB. The intersection of AM and CD is called E. Suppose that $|AD|=|DE|$. Prove that $|AB|=|CE|$.

Problem 3. Let a, b and c be rational numbers for which $a+bc$, $b+ac$ and $a+b$ are all non-zero and for which we have

$$\frac{1}{a+bc}+\frac{1}{b+ac}=\frac{1}{a+b}.$$

Prove that $\sqrt{(c-3)(c+1)}$ is rational.

Problem 4. Let $\triangle ABC$ be a triangle with $|AC|=2|AB|$ and let O be its circumcenter. Let D be the intersection of the angle bisector of $\angle A$ and BC. Let E be the orthogonal projection of O on AD and let $F\neq D$ be a point on AD satisfying $|CD|=|CF|$. Prove that $\angle EBF=\angle ECF$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 10, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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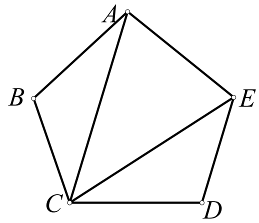
Polygonal Problems

Kin Yin Li

In geometry textbooks, we often come across problems about triangles and quadrilaterals. In this article we will present some problems about n-sided polygons with $n > 4$. This type of problem appears every few years in math olympiads of many countries.

<u>*Example 1.*</u> Prove that if $ABCDE$ is a convex pentagon with all sides equal and $\angle A\geq\angle B\geq\angle C\geq\angle D\geq\angle E$, then it is a regular pentagon.

<u>*Solution.*</u>



Since

$$AC=2AB\sin\frac{\angle B}{2}\geq 2CD\sin\frac{\angle D}{2}=CE,$$

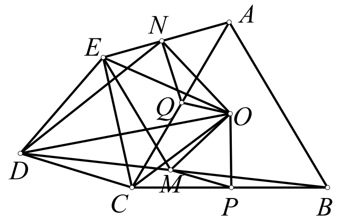
we get $\angle AEC\geq\angle EAC$. Next,

$$\angle EAC=\angle A-\frac{180^\circ-\angle B}{2}=\angle A+\frac{\angle B}{2}-90^\circ$$
$$\geq\angle E+\frac{\angle D}{2}-90^\circ=\angle E-\frac{180^\circ-\angle D}{2}$$
$$=\angle AEC$$

Hence, $\angle EAC=\angle AEC$. Then equality holds everywhere above so that $\angle A=\angle E$ and we are done.

<u>*Example 2.*</u> (*Bulgaria*, 1979) In convex pentagon $ABCDE$, $\triangle ABC$ and $\triangle CDE$ are equilateral. Prove that if O is the center of $\triangle ABC$ and M, N are midpoints of BD, AE respectively, then $\triangle OME\sim\triangle OND$.

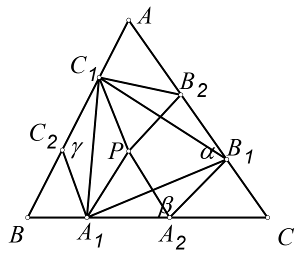
<u>*Solution.*</u>



Let P, Q be the midpoints of BC, AC respectively. Observe that $\angle COP=60^\circ$, $OC=2OP$, $PM\|CD$, $\angle DCE=60^\circ$ and $EC = DC = 2MP$. Then rotating about O by 60° clockwise and follow by doubling distance from O, we see $\triangle OPM$ goes to $\triangle OCE$. Hence $\angle EOM=\angle COP=60^\circ$ and $OE=2OM$. Similarly we can rotate about O by 60° counterclockwise and double distance from O to bring $\triangle OQN$ to $\triangle OCD$. Then $\angle DON = 60^\circ$, $OD = 2ON$ and so $\triangle OME\sim\triangle OND$.

<u>*Example 3.*</u> (*IMO* 2005) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, so that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that A_1B_2, B_1C_2 and C_1A_2 are concurrent.

<u>*Solution.*</u>



Let P be the point inside $\triangle ABC$ such that $\triangle A_1A_2P$ is equilateral. Observe that $A_1P\|C_1C_2$ and $A_1P=C_1C_2$. So $A_1PC_1C_2$ is a rhombus. Similarly, $B_1PB_2B_1$ is a rhombus. So $\triangle C_1B_2P$ is equilateral. Let $\alpha=\angle B_2B_1A_2$, $\beta=\angle B_1A_2A_1$ and $\gamma=\angle C_1C_2A_1$. Then α and β are external angles of $\triangle CB_1A_2$ with $\angle C=60^\circ$. So $\alpha+\beta=240^\circ$. Now $\angle B_2PA_2=\alpha$ and $\angle C_1PA_1=\gamma$. So $\alpha+\gamma=360^\circ-(\angle C_1PB_2+\angle A_1PA_2)=240^\circ$. So $\beta=\gamma$. Similarly, $\angle C_1B_2B_1=\beta$. Hence, $\triangle A_1A_2B_1$, $\triangle B_1B_2C_1$ and $\triangle C_1C_2A_1$ are congruent, which implies $\triangle A_1B_1C_1$ is equilateral. Since sides of $A_1A_2B_1B_2C_1C_2$ have equal lengths, lines A_1B_2, B_1C_2 and C_1A_2 are the perpendicular bisectors of the sides of $\triangle A_1B_1C_1$ and the result follows.

(continued on page 2)

Example 4. (*Czechoslovakia*, 1974) Prove that if a circumscribed hexagon *ABCDEF* satisfies

$$AB=BC,\ CD=DE \text{ and } EF=FA,$$

then the area of $\triangle ACE$ is less than or equal to the area of $\triangle BDF$.

Solution. Let O be the circumcenter of hexagon *ABCDEF* and R be the radius of the circumcircle. Let

$$\alpha=\angle CAE, \beta=\angle AEC, \gamma=\angle ACE.$$

From the given conditions on the sides, we get

$$\angle AOB=\angle BOC=\beta,$$
$$\angle COD=\angle DOE=\alpha,$$
$$\angle EOF=\angle FOA=\gamma.$$

Let $[XYZ]$ denote the area of $\triangle XYZ$. We have

$$[ACE]=\frac{EC\cdot CA\cdot AE}{4R} = \frac{2R\sin\alpha\cdot 2R\sin\beta\cdot 2R\sin\gamma}{4R} = 2R^2\sin\alpha\sin\beta\sin\gamma.$$

Similarly,

$$[BDF]=2R^2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

Now for positive α, β, γ satisfying $\alpha+\beta+\gamma=180^\circ$, we have

$$\begin{aligned}
&\sin^2\alpha\sin^2\beta\sin^2\gamma\\
&=(\sin\alpha\sin\beta)(\sin\gamma\sin\alpha)(\sin\beta\sin\gamma)\\
&=\prod_{cyc}\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}\\
&\le\prod_{cyc}\frac{1-\cos(\alpha+\beta)}{2}\\
&=\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\beta+\gamma}{2}\sin^2\frac{\gamma+\alpha}{2}.
\end{aligned}$$

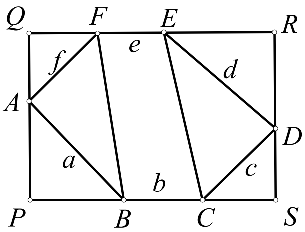
Therefore, $[ACE]\le[BDF]$.

Example 5. (*IMO* 1996) Let *ABCDEF* be a convex hexagon such that *AB* is parallel to *DE*, *BC* is parallel to *EF* and *CD* is parallel to *FA*. Let R_A, R_C, R_E be the circumradii of triangles *FAB, BCD, DEF* respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A+R_C+R_E\ge\frac{P}{2}.$$

Solution. Let a, b, c, d, e, f denote the lengths of the sides *AB, BC, CD, DE, EF, FA* respectively. By the parallel conditions, we have $\angle A=\angle D$, $\angle B=\angle E$, $\angle C=\angle F$.

Consider rectangle *PQRS* such that A is on *PQ; F,E* are on *QR; D* is on *RS* and *B,C* are on *SP*.



We have $BF\ge PQ=SR$. So $2BF\ge PQ+SR$, which is the same as

$$2BF\ge(a\sin B+f\sin C)+(c\sin C+d\sin B).$$

Similarly,

$$2BD\ge(c\sin A+b\sin B)+(c\sin B+f\sin A),$$
$$2DF\ge(c\sin C+d\sin A)+(a\sin A+b\sin C).$$

Next, by the extended sine law,

$$R_A=\frac{BF}{2\sin A}, R_C=\frac{BD}{2\sin C}, R_E=\frac{DE}{2\sin E}.$$

Then using the inequalities and equations above, we have

$$\begin{aligned}
&R_A+R_C+R_E\\
&\ge\frac{a}{4}\left(\frac{\sin B}{\sin A}+\frac{\sin A}{\sin B}\right)+\cdots+\frac{f}{4}\left(\frac{\sin A}{\sin F}+\frac{\sin F}{\sin A}\right)\\
&\ge\frac{a+b+c+d+e+f}{2}=\frac{P}{2}.
\end{aligned}$$

Example 6. (*Great Britain*, 1988) Let four consecutive vertices *A, B, C, D* of a regular polygon satisfy

$$\frac{1}{AB}=\frac{1}{AC}+\frac{1}{AD}.$$

Determine the number of sides of the polygon.

Solution. Let the circumcircle of the polygon have center O and radius R. Let $\alpha=\angle AOB$, then $0<3\alpha=\angle AOD<360^\circ$. So $0<\alpha<120^\circ$. Also, from

$$AB=2R\sin\frac{\alpha}{2},\qquad AC=2R\sin\alpha,$$
$$AD=2R\sin\frac{3\alpha}{2},$$

we get

$$\frac{1}{\sin\frac{\alpha}{2}}=\frac{1}{\sin\alpha}+\frac{1}{\sin\frac{3\alpha}{2}}.$$

Clearing denominators, we have

$$\begin{aligned}
0&=\sin\alpha\sin\frac{3\alpha}{2}-\left(\sin\alpha+\sin\frac{3\alpha}{2}\right)\sin\frac{\alpha}{2}\\
&=\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\frac{5\alpha}{2}\right)-\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\frac{3\alpha}{2}\right)\\
&\qquad-\frac{1}{2}(\cos\alpha-\cos2\alpha)\\
&=\frac{1}{2}\left(\left(\cos\frac{3\alpha}{2}+\cos2\alpha\right)-\left(\cos\alpha+\cos\frac{5\alpha}{2}\right)\right)\\
&=\cos\frac{7\alpha}{4}\left(\cos\frac{\alpha}{4}-\cos\frac{3\alpha}{4}\right)\\
&=2\cos\frac{7\alpha}{4}\sin\frac{\alpha}{4}\sin\frac{\alpha}{2}.
\end{aligned}$$

Then $7\alpha/4=90^\circ$, that is $\alpha=360^\circ/7$. So the polygon has 7 sides.

Example 7. (*Austria*, 1973) Prove that if the angles of a convex octagon are all equal and the ratio of all pairs of adjacent sides is rational, then each pair of opposite sides has equal length.

Solution. Without loss of generality, we may assume the sides of such a polygon $A_1A_2\ldots A_8$ are rational (since the conclusion is the same for octagons similar to such an octagon). Now the sum of all angles of the octagon is $6\times180^\circ$. Hence each angle is 45°.

Let v_n be the vector from A_n to A_{n+1} for $n=1,2,\ldots,8$ (with $A_9=A_1$). Then the angle between v_n and v_{n+1} at the origin is 45°. Observe that the sum of these vectors is zero since we start at A_1 and traverse the octagon once to return to A_1.

Let i and j be a pair of unit vectors perpendicular to each other at the origin. By rotation, we may assume v_1 is a vector in the i direction and v_3 is in the j direction. Then $v_1+v_5=xi$ and $v_3+v_7=yj$ for some rational x and y. Also,

$$v_2+v_4+v_6+v_8=r\sqrt{2}i\pm r\sqrt{2}j$$

for some rational r. Then

$$(x+r\sqrt{2})i+(y\pm r\sqrt{2})j=\sum_{n=1}^{8}v_n=0.$$

Since, x and r are rational, we must have $x=r=0$. That is, $v_5=-v_1$. By rotating the i, j vectors by 45°, similarly we get $v_6=-v_2$. Then also $v_7=-v_3$ and $v_8=-v_4$. The result follows.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 10, 2015.***

Problem 461. Inside rectangle $ABCD$, there is a circle. Points W, X, Y, Z are on the circle such that lines AW, BX, CY, DZ are tangent to the circle. If AW=3, BX=4, CY=5, then find DZ with proof.

Problem 462. For all $x_1, x_2, \ldots, x_n \geq 0$, let $x_{n+1} = x_1$, then prove that

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \geq \frac{n}{\sqrt{2}}.$$

Problem 463. Let S be a set with 20 elements. N 2-element subsets of S are chosen with no two of these subsets equal. Find the least number N such that among any 3 elements in S, there exist 2 of them belong to one of the N chosen subsets.

Problem 464. Determine all positive integers n such that for n, there exists an integer m with 2^n-1 divides m^2+289.

Problem 465. Points A, E, D, C, F, B lie on a circle Γ in clockwise order. Rays AD, BC, the tangents to Γ at E and at F pass through P. Chord EF meets chords AD and BC at M and N respectively. Prove that lines AB, CD, EF are concurrent.

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Solutions

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Problem 456. Suppose $x_1, x_2, \ldots, x_n$ are non-negative and their sum is 1. Prove that there exists a permutation σ of $\{1,2,\cdots,n\}$ such that

$$x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)} \leq 1/n.$$

Solution. **CHAN Long Tin** (Cambridge University, Year 3), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Samiron SADHUKHAN** (Kendriya Vidyalaya, India) and **WONG Yat** (G. T. (Ellen Yeung) College).

Assume the contrary is true. Let $\sigma(n+1) = \sigma(1)$ for all permutations σ. For $1 \leq i < j \leq n$, the terms x_ix_j and x_jx_i appear a total of $2n(n-2)!$ times in

$$\sum_{n\in S_n}\sum_{k=1}^{n} x_{\sigma(k)}x_{\sigma(k+1)}.$$

So, we have

$$\begin{aligned}\frac{n!}{n} &< \sum_{\sigma\in S_n}\sum_{k=1}^{n} x_{\sigma(k)}x_{\sigma(k+1)}\\ &= 2n(n-2)!\sum_{1\leq i<j\leq n} x_ix_j\\ &= n(n-2)!\left[\left(\sum_{i=1}^{n} x_i\right)^2-\sum_{i=1}^{n} x_i^2\right]\\ &= n(n-2)!\left(1-\sum_{i=1}^{n} x_i^2\right).\end{aligned}$$

This simplifies to (*) $\sum_{i=1}^{n} x_i^2 < \frac{1}{n}$. However, by the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2 = 1,$$

which contradicts (*).

Problem 457. Prove that for each n = 1,2,3,…, there exist integers a, b such that if integers x, y are relatively prime, then $\sqrt{(a-x)^2+(b-y)^2} > n.$

Solution. **Samiron SADHUKHAN** (Kendriya Vidyalaya, India) and **WONG Yat** (G. T. (Ellen Yeung) College).

There are $(2n+1)^2$ ordered pairs (r,s) of integers satisfying $|r|, |s| \leq n$. Assign a <u>distinct</u> prime number $p_{r,s}$ to each such (r,s). By the Chinese remainder theorem, there exist integers a,b such that for all integers r; s satisfying $|r|, |s| \leq n$, we have $a \equiv r \pmod{p_{r,s}}$ and $b \equiv s \pmod{p_{r,s}}$.

Let integers x, y be relatively prime. Assume (x,y) has distance at most n from (a,b). Then $|a-x| \leq n$ and $|b-y| \leq n$. Let $a-x=r$ and $b-y=s$. Then $x=a-r$ and $y=b-s$ are multiples of $p_{r,s}$, contradicting $\gcd(x,y) = 1$. Therefore,

$$\sqrt{(a-x)^2+(b-y)^2} > n.$$

Problem 458. Nonempty sets A_1, A_2, A_3 form a partition of $\{1,2,\ldots,n\}$. If $x+y=z$ have no solution with x in A_i, y in A_j, z in A_k and $\{i,j,k\}=\{1,2,3\}$, then prove that A_1, A_2, A_3 cannot have the same number of elements.

Solution. **Oliver GEUPEL** (Brühl, NRW, Germany) and **John GLIMMS**.

Without loss of generality, say $1 \in A_1$ and the smallest element in $A_2 \cup A_3$ is $b \in A_2$. Let the elements in A_3 be $c_1, c_2, \ldots, c_k$ in increasing order.

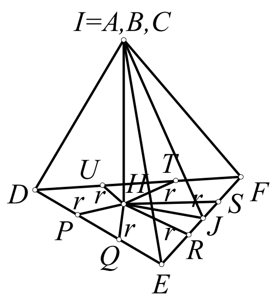
Assume $c_{i+1}-c_i=1$ for some i. Then take i to be the smallest possible. Since $b \in A_2$, the equations $(c_i-b)+b=c_i$ and $(c_i-b+1)+b=c_{i+1}$ imply c_i-b and c_i-b+1 are both not in A_1.

Since $1 \in A_1$ and $(c_i-b)+1= c_i-b+1$, so either c_i-b+1 and c_i-b both are in A_2 or both are in A_3. Since i is smallest such that $c_{i+1}-c_i=1$, so c_i-b+1 and c_i-b cannot be in A_3. However, c_i-b+1 and c_i-b in A_2, $b-1$ in A_1 (by property of b) and $(b-1)+(c_i-b+1)=c_i$ lead to contradiction. So $c_{i+1}-c_i \geq 2$ for all i.

Finally, since $1+(c_i-1)=c_i$, we get $c_i-1 \notin B$. Hence $c_i-1 \in A$. Then A_1 contains 1, c_1-1, c_2-1, ... , c_k-1. Therefore, A_1 has more elements than A_3.

Problem 459. H is the orthocenter of acute $\triangle ABC$. D,E,F are midpoints of sides BC, CA, AB respectively. Inside $\triangle ABC$, a circle with center H meets DE at P,Q, EF at R,S, FD at T,U. Prove that $CP=CQ=AR=AS=BT=BU$.

Solution. **John GLIMMS.**



Let lines AH and FE meet at J. From $AH \perp BC$ and $BC \| FE$, we get FE is perpendicular to AJ and HJ. By folding along DE, EF and FD, we can make a tetrahedron having $\triangle DEF$ as the base and points A, B, C meet at a point I. Then FE is perpendicular to IJ and HJ. So FE is perpendicular to the plane through I,J,H. Then $FE \perp IH$. Similarly, $DE \perp IH$. Then the plane through D,E,F is perpendicular to IH. By Pythagoras' theorem, $IH^2+r^2 = CP^2 = CQ^2 = AR^2 = AS^2 = BT^2 = BV^2$, where r is the radius of the circle.

Other commended solvers: **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Andrea FANCHINI**

(Cantú, Italy), **William FUNG**, **Oliver GEUPEL** (Brühl, NRW, Germany), **MANOLOUDIS Apostolis** (4 High School of Korydallos, Piraeus, Greece), **Samiron SADHUKHAN** (Kendriya Vidyalaya, India), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 460. If $x,y,z > 0$ and $x+y+z+2 = xyz$, then prove that

$$x+y+z+6 \geq 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right).$$

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **CHAN Long Tin** (Cambridge University, Year 3), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Oliver GEUPEL** (Brühl, NRW, Germany), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Vijaya Prasad NULLARI** (Retired Principal, AP Educational Service, India), **Nicuşor ZLOTA** ("Traian Vuia" Technical College,Focşani, Romania) and **Titu ZVONARU** (Comăneşti, Romania).

Let

$$a=\frac{1}{1+x}, b=\frac{1}{1+y}, c=\frac{1}{1+z}.$$

Using $x+y+z+2 = xyz$, we get $a+b+c = 1$. Then $x = (1-a)/a = (b+c)/a$ and similarly $y=(c+a)/b$ and $z=(a+b)/c$. By the AM-GM inequality, we have

$$\begin{aligned} & x+y+z+6 \\ &= \sum_{cyc}\frac{b+c}{a}+6 \\ &= \sum_{cyc}\left(\frac{c+a}{c}+\frac{a+b}{b}\right) \\ &\geq 2\sum_{cyc}\sqrt{\frac{(c+a)(a+b)}{bc}} \\ &= 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right). \end{aligned}$$

Other commended solvers: **Paolo PERFETTI** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **WONG Yat** (G. T. (Ellen Yeung) College).

Olympiad Corner

(*Continued from page 1*)

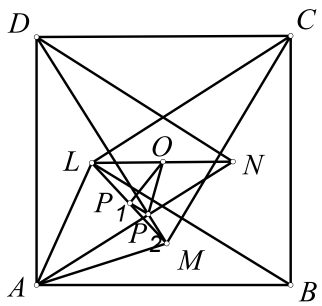
Problem 5. On each of the 2014^2 squares of a 2014×2014-board a light bulb is put. Light bulbs can be either on or off. In the starting situation a number of light bulbs are on. A move consists of choosing a row or column in which at least 1007 light bulbs are on and changing the state of all 2014 light bulbs in this row or column (from on to off or from off to on). Find the smallest non-negative integer k such that from each starting situation there is a finite sequence of moves to a situation in which at most k light bulbs are on.

Polygonal Problems

(*Continued from page 2*)

<u>*Example 8.*</u> (*IMO* 1997) Equilateral triangles *ABK, BCL, CDM, DAN* are constructed inside the square *ABCD*. Prove that the midpoints of the four segments *KL, LM, MN, NK* and the midpints of the eight segments *AK, BK, BL, CL, CM, DM, DN, AN* are the twelve vertices of a regular dodecagon.

<u>*Solution.*</u>



Let us denote the midpoints of segments *LM, AN, BL, MN, BK, CM, NK, CL, DN, KL, DM, AK* by P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, respectively. To prove the dodecagon

$$P_1P_2P_3P_4P_5P_6P_7P_8P_9P_{10}P_{11}P_{12}$$

is regular, we observe that $BL=BA$ and $\angle ABL=30°$. Then $\angle BAL=75°$. Similarly $\angle DAM=75°$. So

$$\angle LAM=\angle BAL+\angle DAM-\angle BAD=60°.$$

Along with $AL=AM$, we see triangle ALM is equilateral.

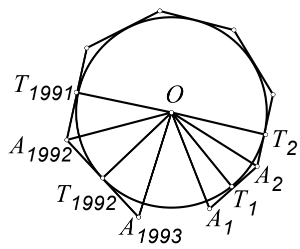
Looking at triangles OLM and ALN, we get $OP_1=½LM$, $OP_2=½AL$ and $OP_2\parallel AL$. Hence, $OP_1=OP_2$, $\angle P_1OP_2=\angle P_1AL=30°$, $\angle P_2OM=\angle DAL=15°$ and $\angle P_2OP_3 = 2\angle P_2OM = 30°$. By symmetry, we can conclude that the dodecagon is regular.

<u>*Example 9.*</u> (*IMO* 1992, *Shortlisted Problem from India*) Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its sides lengths are 1,2,3,…,1992 in some order;

(ii) the polygon is circumscribable about a circle.

<u>*Solution.*</u> For a positive number r, let us draw a circle of radius r and let us draw a polygonal path $A_1A_2\ldots A_{1993}$ such that for i=1 to 1992, side A_iA_{i+1} is tangent to the circle at a point T_i and $T_{1992}A_{1993} = A_1T_1$, $T_1A_2 = A_2T_2$, … , $T_{1991}A_{1992} = A_{1992}T_{1992}$.



To achieve condition (i), we need A_1A_2, A_2A_3, …, $A_{1992}A_{1993}$ to be a permutation of 1, 2, …, 1992. This can be done as follow:

If $i\equiv 1 \pmod 4$, then let $A_iT_i=1/2$.
If $i\equiv 3 \pmod 4$, then let $A_iT_i=3/2$.
If $i\equiv 0,2 \pmod 4$, then let $A_iT_i=i-3/2$.

We can check that the lengths of A_iA_{i+1} for i=1 to 1992 are 1, 2, 4, 3, 5, 6, 8, 7,…, 1989,1990,1992,1991.

To achieve condition (ii), we define a function

$$\begin{aligned} f(r) &= \sum_{i=1}^{1992}\angle A_iOA_{i+1} \\ &= 2\sum_{i=1}^{1992}\arctan\frac{A_iT_i}{r}. \end{aligned}$$

Observe that $f(r)$ is a continuous function on $(0,\infty)$. As r tends to 0, $f(r)$ tends to infinity and as r tends to infinity, $f(r)$ tends to 0. By the intermediate value theorem, there exists r such that $f(r) = 2\pi$. Then $A_{1993}=A_1$ and $A_1A_2\ldots A_{1992}$ is a desired polygon.

We remark that if 1992 is replaced by other positive integers of the form $4k$, then there are such $4k$-sided polygon.

Mathematical Excalibur

Volume 19, Number 5 — *April 2015 – June 2015*

Olympiad Corner

Below are the problems of the 2015 Canadian Mathematical Olympiad held in January 28, 2015.

Notation: If V and W are two points, then VW denotes the line segment with endpoints V and W as well as the length of this segment.

Problem 1. Let $\mathbb{N} = \{1,2,3,\ldots\}$ denote the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2+n$ for every positive integer n.

Problem 2. Let ABC be an acute-angled triangle with altitudes AD, BE and CF. Let H be the orthocenter, that is, the point where the altitudes meet. Prove that

$$\frac{AB\cdot AC+BC\cdot BA+CA\cdot CB}{AH\cdot AD+BH\cdot BE+CH\cdot CF}\le 2.$$

Problem 3. On a $(4n+2)\times(4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 27, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

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Tournament of the Towns

Kin Yin Li

In 1980, Kiev, Moscow and Riga participated in a mathematical problem solving contest for high school students, later called the Tournament of the Towns. At present thousands of high school students from dozens of cities all over the world participate in this contest. In this article, we present some very interesting math problems from this contest. At the end of the article, there are some information on where interested readers can find past problems and solutions of this contest.

Here are some examples we enjoy.

Example 1. (*Junior Questions, Spring 1981, proposed by A. Andjans*) Each of 64 friends simultaneously learns one different item of news. They begin to phone one another to tell them their news. Each conversation last exactly one hour, during which time it is possible for two friends to tell each other all of their news. What is the minimum number of hours needed in order for all of the friends to know all the news?

Solution. More generally, suppose there are 2^n friends. After n rounds, the most anyone can learn are 2^n pieces of gossip. Hence n rounds are necessary. We now prove by induction on n that n rounds are also sufficient. For n=1, the result is trivial. Suppose the result holds up to $n-1$ for some $n\ge 2$. Consider the next case with 2^n friends. Have them call each other impairs in the first round. After this, divide them into two groups, each containing one member from each pair who had exchanged gossip. Each group has 2^{n-1} friends who know all the gossip among them. By the induction hypothesis, $n-1$ rounds are sufficient for everyone within each group to learn everything. This completes the induction argument. In particular, with 64 friends, 6 rounds are both necessary and sufficient.

Example 2. (*Senior Questions, Spring 1983, proposed by A. Andjans*) There are K boys placed around a circle. Each of them has an even number of sweets. At a command each boy gives half of his sweets to the boy on his right. If, after that, any boy has an odd number of sweets, someone outside the circle gives him one more sweet to make the number even. This procedure can be repeated indefinitely. Prove that there will be a time at which all boys have the same number of sweets.

Solution. Suppose initially the maximum number of sweets a boy has is $2m$, and the minimum is $2n$. We may as well assume that $m>n$. After a round of exchange and possible augmentation, we claim that the most any boy can have is $2m$ sweets. This is because he could have kept at most m sweets, and received m more in the exchange, but will not be augmented if he already has $2m$ sweets.

On the other hand, at least one boy who had $2n$ sweets will have more than that, because as long as $m>n$, one of these boys will get more than he gives away. It follows that while the maximum cannot increase, the minimum must increase until all have the same number of sweets.

Example 3. (*Junior Questions, Autumn 1984*) Six musicians gathered at a chamber music festival. At each scheduled concert some of these musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled in order to enable each musician to listen, as a member of the audience, to all the other musicians?

(continued on page 2)

<u>Solution</u>. Let the musicians be A, B, C, D, E and F. Suppose there are only three concerts. Since each of the six must perform at least once, at least one concert must feature two or more musicians. Say both A and B perform in the first concert. They must still perform for each other. Say A performs in the second concert for B and B in the third for A. Now C, D, E and F must all perform in the second concert, since it is the only time B is in the audience. Similarly, they must all perform in the third. The first concert alone is not enough to allow C, D, E and F to perform for one another. Hence we need at least four concerts. This is sufficient, as we may have A, B and C in the first, A, D and E in the second, B, D and F in the third and C, E and F in the fourth.

<u>Example 4.</u> (*Junior Questions, Autumn 1984, proposed by V. G. Ilichev*) On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two different chameleons of different colours meet, they both simultaneously change colour to the third colour (eg. If a grey and a brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

<u>Solution</u>. In this case the numbers of chameleons of each colour at the start have remainders of 0, 1 and 2 when divided by three. Each "meeting" maintains such a situation (not necessarily in any order) as two of these remainders must either be reduced by 1 (or increased by 2) while the other must be increased by 2 (or reduced by 1). Thus at least two colours are present at any stage, guaranteeing the possibility of obtaining all of the three colours in fact by future meetings.

<u>Note.</u> The only way of getting chameleons to be of the same colour would be getting an equal number of two colours first. This would mean getting two with the same remainder on division by three. This would have been possible if we had started with, say 15 of each colour. From this position we can obtain sets with remainders equal to $\{0,0,0\}$, $\{1,1,1\}$ and $\{2,2,2\}$.

<u>Example 5.</u> (*Junior Questions, Spring 1985, proposed by S. Fomin*) There are 68 coins, each coin having a different weight that that of each other. Show how to find the heaviest and lightest coin in 100 weighings on a balance beam.

<u>Solution 1</u>. First divide into 34 pairs and perform 34 weighings, each time identifying the heavier and lighter coins. Put all the heavier coins into one group and the lighter coins into another. Divide the group with heavier coins into 17 pairs, and perform 17 weighings on these to identify the 17 heavier coins. Continue this process with the group of heavier coins each time. If there is an odd number of coins at any stage, the odd coin out must be carried over to the following stage. There will be a total of 17+8+4+2+1+1=3 such weighings required for identifying the heaviest coin.

A similar 33 weighings of the lighter group will identify the lightest coin. The total number of weighing is thus 34+33+33=100, as required.

<u>Solution 2</u>. More generally, we show that $3n-2$ weighings are sufficient for $2n$ coins. We first divide the coins into n pairs, and use n weighings to sort them out into a "heavy" pile and a "light" pile. The heaviest coin is among the n coins in the "heavy" pile. Each weighing eliminates 1 coin. Since there are n coins, $n-1$ weighings are necessary and sufficient. Similarly, $n-1$ weighings will locate the lightest coin in the "light" pile. Thus the task can be accomplished in $3n-2$ weighings.

<u>Example 6.</u> (*Junior Questions, Spring 1987, proposed by D. Fomin*) A certain number of cubes are painted in six colours, each cube having six faces of different colours (the colours in different cubes may be arranged differently). The cubes are placed on a table so as to form a rectangle. We are allowed to take out any column of cubes, rotate it (as a whole) along its long axis and place it in a rectangle. A similar operation with rows is also allowed. Can we always make the rectangle monochromatic (i.e. such that the top faces of all the cubes are the same colour) by means of such operations?

<u>Solution</u>. The task can always be accomplished, and we can select the top colour in advance, say red. By fixing a cube, we mean bringing its red face to the top. Given a rectangular block, we fix one cube at a time, from left to right, and from front to back.

Suppose that the cube in the i-th row and the j-th column is the next to be fixed. Suppose that we need to rotate the i-th row. In order not to unfix the first $j-1$ cubes of this row, we rotate each of the first $j-1$ columns so that all red faces are to the left. They remain to the left when the i-th row is rotated. We can now refix the first $j-1$ columns.

Similarly, if we need to rotate the j-th column, we can go through an analogous three-step process.

<u>Example 7.</u> (*Senior Questions, Autumn 1987, proposed by A. Andjans*) A certain town is represented as an infinite plane, which is divided by straight lines into squares. The lines are streets, while the squares are blocks. Along a certain street there stands a policeman on each 100^{th} intersection. Somewhere in the town there is a bandit, whose position and speed are unknown, but he can move only along the streets. The aim of the police is to see the bandit. Does there exist an algorithm available to the police to enable them to achieve their aim?

<u>Solution.</u> We assume that (a) there is no limit to how far a policeman can see along the street he is on; (b) there is no overall time limit, and (c) if the bandit is ever on the same street as a policeman he will be seen.

Let i, j and k denote integers, let the North-South streets be $x=i$ for all i, the East-West streets $y=j$ for all j and suppose the k-th policeman is at $(100k,0)$.

For all even k the k-th policeman remains stationary throughout. This traps the bandit in the infinite strip between $x=200k$ and $x=200(k+1)$ for some k, say k^*.

All other policemen first travel along $y=0$ towards $(0,0)$ until they reach the first cross street $x=s$ for which there is a policeman on every street $x=i$ for i between 0 and s. Police are to travel at regulation speed, say one block per minute, but nevertheless there will come a time, dependent only on k^*, when every street $x=i$ on the k^* strip will be policed.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 27, 2015.***

Problem 466. Let k be an integer greater than 1. If k+2 integers are chosen among 1,2,3,…,3k, then there exist two of these integers m,n such that $k<|m-n|<2k$.

Problem 467. Let p be a prime number and q be a positive integer. Take any pq consecutive integers. Among these integers, remove all multiples of p. Let M be the product of the remaining integers. Determine the remainder when M is divided by p in terms of q.

Problem 468. Let $ABCD$ be a cyclic quadrilateral satisfying $BC>AD$ and $CD>AB$. E, F are points on chords BC, CD respectively and M is the midpoint of EF. If $BE=AD$ and $DF=AB$, then prove that $BM\perp DM$.

Problem 469. Let m be an integer greater than 4. On the plane, if m points satisfy no three of them are collinear and every four of them are the vertices of a convex quadrilateral, then prove that all m of the points are the vertices of a m-sided convex polygon.

Problem 470. If a, b, c>0, then prove that

$$\frac{a}{b(a^2+2b^2)}+\frac{b}{c(b^2+2c^2)}+\frac{c}{a(c^2+2a^2)} \ge \frac{3}{ab+bc+ca}.$$

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Solutions

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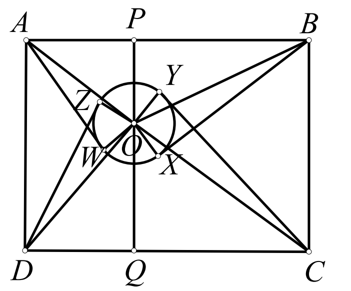
Problem 461. Inside rectangle $ABCD$, there is a circle. Points W, X, Y, Z are on the circle such that lines AW, BX, CY, DZ are tangent to the circle. If AW=3, BX=4, CY=5, then find DZ with proof.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Andrea FANCHINI** (Cantú, Italy), **William FUNG**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Jon GLIMMS**, **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



Let r be the radius of the circle. By Pythagoras' theorem, we have

$$r^2 = AW^2-AO^2 = BX^2-BO^2 = CY^2-CO^2 = DZ^2-DO^2. \quad (*)$$

Let P,Q be the feet of perpendiculars from O to AB, CD respectively. Then

$$AO^2-BO^2 = (AP^2+PO^2)-(BP^2+PO^2) = (DQ^2+QO^2)-(CQ^2+QO^2) = DO^2-CO^2.$$

Using (*), we get $AW^2-BX^2 = AO^2-BO^2 = DO^2-CO^2 = DZ^2-CY^2$. Then

$$DZ=\sqrt{AW^2-BX^2+CY^2}=3\sqrt{2}.$$

Problem 462. For all $x_1, x_2, \ldots, x_n \ge 0$, let $x_{n+1}=x_1$, then prove that

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \ge \frac{n}{\sqrt{2}}.$$

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **DHRUV Nevatia** (10th Standard, Ramanujan Academy, Nashik, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **MAMEDOV Shatlyk** (School of Young Physics and Maths N 21, Dashogus, Turkmenistan), **Corneliu MĂNESCU- AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan), **WADAH Ali** (Ben Badis College, Algeria), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

By squaring both sides or RMS-AM inequality, we have for all $a,b\ge 0$,

$$\sqrt{a^2+b^2}\ge\frac{a+b}{\sqrt{2}}.$$

Applying this, we get

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \ge \sum_{k=1}^{n}\frac{1}{\sqrt{2}}\left(\frac{1}{x_k+1}+(1-\frac{1}{x_{k+1}+1})\right)=\frac{n}{\sqrt{2}}.$$

Problem 463. Let S be a set with 20 elements. N 2-element subsets of S are chosen with no two of these subsets equal. Find the least number N such that among any 3 elements in S, there exist 2 of them belong to one of the N chosen subsets.

Solution. **Jon GLIMMS**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let $x\in S$ be contained in k of the N 2-elements subsets of S, where k is least among the elements of S.

Let $x_1, x_2, \ldots, x_k$ be the other elements in k of the N 2-element subsets with x. As k is least, so each of the x_i's is also contained in at least k of the N 2-element subsets of S.

Also, there are m=19−k elements $w_1, w_2, \ldots, w_m\in S$ not in any of the N 2-element subsets of S with x. For every pair w_r, w_s of these, $\{w_r, w_s\}$ is one of these N 2-element subsets of S (otherwise, no two of x, w_r, w_s form one of the N 2-element subsets). Then

$$N \ge \binom{k+1}{2} + \binom{19-k}{2} = (k-9)^2 + 90 \ge 90.$$

To get the least case of N=90, we divide the 20 elements into two groups of 10 elements. Then take all 2-element subsets of each of the two groups to get 45+45=90 2-element subsets of S.

Problem 464. Determine all positive integers n such that for n, there exists an integer m with 2^n-1 divides m^2+289.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **PANG Lok Wing** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

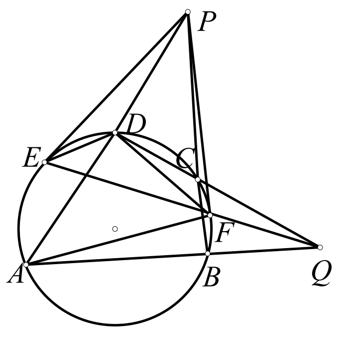
The case n=1 is a solution. For n>1, we first show if a prime q of the form $4k$+3 divides a^2+b^2, then q divides a and b. Assume gcd(q,a)=1. Let $c=a^{q-2}$. Then by Fermat's little theorem, $ac=a^{q-1}\equiv 1$ (mod q). As $q|a^2+b^2$, so $b^2\equiv -a^2$ (mod q). Then $(bc)^2\equiv -(ac)^2\equiv\ -1$ (mod q) and $(bc)^{q-1} = (bc)^{2(2k+1)} \equiv = -1$ (mod q), contradicting Fermat's little theorem. So q divides a (and b similarly).

If n>1, then $2^n-1\equiv 3$ (mod 4). Hence 2^n-1 has a prime divisor $q\equiv 3$ (mod 4). By the fact above, q divides m^2+289 implies q divides m and 17. Then q=17 $\not\equiv 3$ (mod 4), contradiction.

Problem 465. Points A, E, D, C, F, B lie on a circle Γ in clockwise order. Rays AD, BC, the tangents to Γ at E and at F pass through P. Chord EF meets chords AD and BC at M and N respectively. Prove that lines AB, CD, EF are concurrent.

Comments. A number of solvers pointed out if lines AB, CD are parallel, then by symmetry lines AB, CD, EF are all parallel. So below, we present solutions for the case when lines AB and CD intersect at a point.

Solution **1. Jon GLIMMS.**



Let lines AB, CD meet at Q. We have

(1) $\angle AFE = \angle ADE = 180° - \angle PDE$,
(2) $\angle EFD = \angle PED$,
(3) $\angle FDQ = \angle PFC$,
(4) $\angle QAF = \angle FCB = 180° - \angle PCF$.
(5) $\angle DAQ = \angle DCP$,
(6) $\angle QDA = 180° - \angle PDC$.

Then

$$\frac{\sin\angle AFE}{\sin\angle EFD} = \frac{\sin\angle PDE}{\sin\angle PED} = \frac{PE}{PD},$$
$$\frac{\sin\angle FDQ}{\sin\angle QAF} = \frac{\sin\angle PFC}{\sin\angle PCF} = \frac{PC}{PF},$$
$$\frac{\sin\angle DAQ}{\sin\angle QDA} = \frac{\sin\angle DCP}{\sin\angle PDC} = \frac{PD}{PC}.$$

Multiplying these and using PE=PF, we have

$$\frac{\sin\angle AFE}{\sin\angle EFD}\cdot\frac{\sin\angle FDQ}{\sin\angle QAF}\cdot\frac{\sin\angle DAQ}{\sin\angle QDA}$$
$$= \frac{PE}{PD}\cdot\frac{PC}{PF}\cdot\frac{PD}{PC} = 1.$$

Applying the converse of the trigonometric form of Ceva's theorem to $\triangle ADF$ and point Q, we get lines AB, CD, EF are concurrent at Q.

Solution **2. Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India) and **William FUNG**.

Since the tangents to Γ at E and at F intersect at P, line EF is the polar of P. Since lines AD, BC intersect at P, the polar of P (that is, line EF) passes through the intersection of lines AB and CD.

Other commended solvers: **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Olympiad Corner

(*Continued from page 1*)

Problem 3 (Cont'd). In terms of n, what is the largest positive integer k such that there must be a row or a column that the turtle has entered at least k distinct times?

Problem 4. Let ABC be an acute-angled triangle with circum-center O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP\cdot CQ$=$AP\cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.

Problem 5. Let p be a prime number for which $(p-1)/2$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1+\sqrt{2p}$ positive integers n such that n<p and p divides $a^n+b^n+c^n$.

Tournament of the Towns

(*Continued from page 2*)

When this happens the bandit will be trapped on some street $y=j^*$, on a single block between $x=i^*$ and $x=i^*+1$ for some i^*.

For each k, as soon as all streets on the k-th strip are policed, one of the policemen travels north and another travels south. For $k=k^*$ this will inevitably reveal the bandit.

After reading these examples, should anyone want to read more, below are websites, which books on this contest can be ordered or problems and solutions of the recent Tournament of the Towns can be found.

www.amtt.com.au/ProductList.php?pager=1&startpage=1

www.artofproblemsolving.com/community/c3239_tournament_of_towns

www.math.toronto.edu/oz/turgor/archives.php

Mathematical Excalibur

Volume 20, Number 1 *July 2015 – October 2015*

Olympiad Corner

Below are the problems of the *2015 International Mathematical Olympiad* held in July 10-11, 2015.

Problem 1. We say that a finite set $\boldsymbol{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\boldsymbol{S}$, there is a point C in $\boldsymbol{S}$ such that $AC=BC$. We say that $\boldsymbol{S}$ is *center-free* if for any three different points A, B and C in $\boldsymbol{S}$, there is no point P in $\boldsymbol{S}$ such that $PA=PB=PC$.

(a) Show that for all integers $n\geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n\geq 3$ for which there exists a balanced center-free set consisting of n points.

Problem 2. Determine all triples (a,b,c) of positive integers such that each of the numbers

$$ab-c,\ \ bc-a,\ \ ca-b$$

is a power of 2.

(A power of 2 is an integer of the form 2^n, where n is a non-negative integer.)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 27, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2015 – Problem Report

Law Ka Ho

IMO 2015 was held in Chiang Mai, Thailand from July 4 to 16. The examinations were held in the mornings of July 10 and 11 (contestants unable to adhere to this schedule with religious reasons were allowed to be quarantined in the day and sit the Day 2 paper after sunset). The Hong Kong team was consisted of the following students:

CHEUNG Wai Lam (Queen Elizabeth School, Form 5)
KWOK Man Yi (Baptist Lui Ming Choi College, Form 4)
LEE Shun Ming Samuel (CNEC Christian College, Form 4)
TUNG Kam Chuen (La Salle College, Form 6)
WU John Michael (Hong Kong International School, Form 4)
YU Hoi Wai (La Salle College, Form 4)

Cheung and Yu were in the IMO team last year, while the rest are first-timers.

Since Hong Kong will host IMO 2016, we sent a total of 14 observers in addition to the contestants, the leader and the deputy leader.

The following consists mainly of the discussions of the problems, marking schemes, performance etc., rather than of the solutions. The problems can be found from the Olympiad Corner in this issue. (Some readers may want to try the problems before reading this section.)

Problem 1. This is quite a standard question in combinatorial geometry. Clearly odd polygons would work for both (a) and (b). The construction for even n in (a) would take some effort, although there were a number of ways to get it done. In (b), the proof that even n does not work involves a standard double counting technique. The Hong Kong team did very well in this question, with five perfect scores plus a 6 out of 7. This question allows partial progress to various degrees. One may complete the whole question. Those who didn't may just figure out the odd polygons, or in addition they could complete the rest of either part (a) or (b). This is better than an all-or-nothing problem. (The marking scheme does not require students to give any proof that their constructions are balanced and/or centre-free.)

Students raised quite a lot of queries on this question during the contest. The most popular question was whether the point C has to be unique. There were also questions like whether the points must be lattice points, and whether the points A, B, C could be collinear.

Problem 2. This looks like a typical number theory problem. The problem is easy to understand. However, all known solutions involve a heavy amount of considerations of different cases, and very limited number theory techniques were involved. It ended up more like an algebra problem, where one deals with the different algebraic expressions by inequality bounds and so on.

Although the known solutions were not particularly elegant, the answers turned out to be surprisingly nice. While most contestants would get (2,2,2) and (2,2,3) (and its permutations) by trial-and-error or whatever methods, there are two other sets of solutions (3,5,7) and (2,6,11) (and their permutations).

The problem was much more difficult than imagined. Very few students managed to get a complete solution, even among the strongest teams. Most of our team members obtained partial results on this one. The question also killed a lot of the contestants' time, leaving them with little time for the last problem of Day 1.

(continued on page 2)

During the problem selection, there were discussions of whether the note defining what a power of 2 is should be included. Some leaders felt that this destroyed the beauty and elegance of the paper. Some others insisted that it should be there because it would otherwise lead to heaps of questions as for whether 1 is a power of 2. Some even said that in their countries a power of 2 would mean 2 to the power 2 or above!

While discussing the marking scheme, it was decided that no penalty would be levied on students who forgot to list the permutations. In other words, one would not be penalized for saying that there are in total four solutions, namely, (2,2,2), (2,2,3), (3,5,7) and (2,6,11). I also asked for clarification whether points would be deducted for not checking the solutions satisfied the conditions of the problem. The answer was negative.

Problem 3. This is again a difficult question, even for members of some strong teams. Of course, as previously mentioned, most students spent a lot of time dealing with the different cases in Problem 2. So they simply did not have much time left for this one. This could be one of the important reasons for the general poor performance. However, our deputy leader pointed out there is a very simple solution using inversion. Interested readers may wish to try it out.

During problem selection, there had been discussions of whether there should be a note (as in Problem 2) explaining what orthocenter means. It was eventually decided that such note should not appear in the question paper. During the contest, when a question on the meaning of the orthocenter arrived, the leader of UK shouted *"Finally!"*.

Another issue is the possibility of having two different configurations. To avoid making students spend extra time working on the two cases, it was decided to fix one configuration, and so the phrase '*A, B, C, K and Q are all different, and lie on Γ in this order*' was added.

Our team obtained little in this question. Only two students managed to show that Q, M, H are collinear. According to the marking scheme, it is worth 1 point. One of the students, however, did not include much detail of the proof (after all, the question was not to prove that Q, M, H are collinear!), and the coordinator refused to award the point. This went into a long fight. The coordinators referred the case to the problem captain, then the chief coordinator. It turned out that there were many similar cases in which students mentioned the collinearity of the three points but were not accepted by the coordinators as a *proof*.

To prove that Q, M, H are collinear, one simple way is to show that Q, H, A' are collinear (where A' is the point on Γ that is diametrically opposite A), and that H, M, A' are collinear. The coordinators decided that the latter is well-known, but the former requires an explicit mention that $\angle AQH = \angle AQA' = 90^\circ$. To me, it is clear that proving the former is more trivial than the latter. If a student mentioned that A' is the antipodal point of A, then clearly (s)he knew that $\angle AQA'=90^\circ$ (it's the IMO!). Furthermore, $\angle AQH=90^\circ$ is given in the problem. What is the point of penalizing students who failed to copy this again? I didn't really see the consistency in accepting the latter as well-known but requiring such a detailed proof for the former. An urgent Jury Meeting was called to discuss this issue. The motion of sticking to the original marking scheme (i.e. to accept H, M, A' being collinear as well-known but to award 1 point only if $\angle AQH = \angle AQA' = 90^\circ$ is explicitly mentioned) was passed by a narrow margin.

The next day when we went on excursion, the Deputy Leader of Paraguay talked to me saying that many people thought that my speech was really to-the-point (by that time the deputy leaders had moved to the leaders' site and were allowed to sit in the Jury Meetings). But obviously more thought the opposite, as shown by the result of the vote!

Problem 4. This is the first problem of Day 2. It is a geometry problem, phrased carefully to make it as easy as possible. The order of the points was clearly given to ensure that only one configuration is possible. The statement to be proved was also rephrased from the original version so that the word *collinear* could be avoided.

Our team did not do well in this question. Only three students solved it. Another student showed that it suffices to prove $\angle AFK=\angle AGL$, which according to the marking scheme is worth 2 points. This sounds pretty much trivial, and the other two students would probably know it as well (only that they did not write it down because they did not find that useful).

In fact, there had been quite a lot of discussions on this point. Suppose a student showed $\angle AFK = \angle AGL$. How many points should that be worth? According to the original marking scheme, this would be worth 4 points; if a student added that *hence we are done*, that would make it 5; by writing *by symmetry we are done*, that would make it 6. (A perfect score would require some explanation on how symmetry leads to the result.) This led to strong opinion from the leaders. Eventually the (4,5,6) above was revised to (5,6,6).

Problem 5. This is the only question for which no student asked questions. This is interesting because in Problem 1 set notations were deliberately avoided, but in this question notation like $f\colon \mathbb{R} \to \mathbb{R}$ did not lead to any question, which to me is a bit of surprise.

By nature this problem is quite similar to Problem 2. Most students managed to make some partial progress, as one naturally starts by plugging in certain values of x and y into the functional equation, leading to some preliminary discoveries. However not many students obtained full solutions. We are glad everyone in our team got partial marks.

The solution to this problem depends heavily on fixed points, which in hindsight is reasonable considering that the expression $x+f(x+y)$ occurs on both sides. This also justifies starting the problem with setting $y=1$ as it would equate the terms $f(xy)$ and $yf(x)$ on the two sides of the equation. Completing the solution, on the other hand, is much more difficult, as there are too many equations and sometimes it is not clear what to put into which equation.

There were heated debates when discussing the marking scheme to this problem. As there were two functions satisfying the equation, most solutions could be divided into two parts (e.g. according to whether $f(0)=0$ or not). Each part would lead to one solution, and then one needs to check that the two solutions obtained, namely, $f(x)=x$ and $f(x)=2-x$, indeed satisfy the equation in the question. In the original proposal of the marking scheme, the coordinators said that they would accept students directly claiming that the former is a solution, while for the latter, it must be explicitly checked (expanding brackets and showing that the two sides are equal).

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 27, 2015.***

Problem 471. For $n \geq 2$, let $A_1, A_2, \ldots, A_n$ be positive integers such that $A_k \leq k$ for $1 \leq k \leq n$. Prove that $A_1+A_2+\cdots+A_n$ is even if and only if there exists a way of selecting + or – signs such that

$$A_1 \pm A_2 \pm \cdots \pm A_n = 0.$$

Problem 472. There are $2n$ distinct points marked on a line, n of them are colored red and the other n points are colored blue. Prove that the sum of the distances of all pairs of points with same color is less than or equal to the sum of the distances of all pairs of points with different color.

Problem 473. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x,y \in \mathbb{R}$,

$$f(x)\,f(y\,f(x)-1) = x^2 f(y) - f(x).$$

Problem 474. Quadrilateral $ABCD$ is convex and lines AB, CD are not parallel. Circle Γ passes through A, B and side CD is tangent to Γ at P. Circle L passes through C, D and side AB is tangent to L at Q. Circles Γ and L intersect at E and F. Prove that line EF bisects line segment PQ if and only if lines AD, BC are parallel.

Problem 475. Let a, b, n be integers greater than 1. If b^n-1 is a divisor of a, then prove that in base b, a has at least n digits not equal to zero.

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Solutions

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Problem 466. Let k be an integer greater than 1. If $k+2$ integers are chosen among $1,2,3,\ldots,3k$, then there exist two of these integers m,n such that $k<|m-n|<2k$.

Solution. **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania).

Let S be the set of the $k+2$ chosen integers and a be the smallest number in S. Subtracting $a-1$ from each element in S do not change the differences between the elements of S. So, without loss of generality, we can suppose $1 \in S$.

If S contains an element b such that $k+2 \leq b \leq 2k$, then take $m=b$ and $n=1$ to get $k<|m-n|=b-1<2k$. Otherwise, none of the numbers $k+2, k+3, \ldots, 2k$ belong to S. The $k+1$ numbers from $S\setminus\{1\}$ are then among the components of the k pairs $(2,2k+1)$, $(3,2k+2)$, … , $(k+1,3k)$. By the pigeonhole principle, there is a pair containing two numbers m, n from $S\setminus\{1\}$. Then we have $k<|m-n|=2k-1<2k$.

Other commended solvers: **Prithwijit DE** (HBCSE, Mumbai, India), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Simon YAU**.

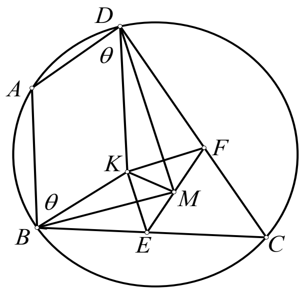
Problem 467. Let p be a prime number and q be a positive integer. Take any pq consecutive integers. Among these integers, remove all multiples of p. Let M be the product of the remaining integers. Determine the remainder when M is divided by p in terms of q.

Solution. **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Mark LAU Tin Wai**, **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

For $r = 0, 1, 2, \ldots, p-1$, among the pq consecutive integers, there are q integers having remainders r when divided by p. Then $M \equiv 1^q 2^q \cdots (p-1)^q = (p-1)!^q \pmod p$. By Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. So $M \equiv (-1)^q \pmod p$. Then the remainder when M is divided by p is 1 if q is even and is $p-1$ if q is odd.

Problem 468. Let $ABCD$ be a cyclic quadrilateral satisfying $BC>AD$ and $CD>AB$. E, F are points on chords BC, CD respectively and M is the midpoint of EF. If $BE=AD$ and $DF=AB$, then prove that $BM \perp DM$.

Solution. **George APOSTOLOPOULOS** (2 High School, Messolonghi, Greece), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India) and **MANOLOUDIS Apostolis** (4 High School of Korydallos, Piraeus, Greece).



Let K be the point such that $ABKD$ is a parallelogram. Let $\theta = \angle ABK = \angle ADK$. Now $BE=AD=BK$, $DF=AB=DK$ and

$$\angle BKE = 90^\circ - \tfrac{1}{2}\angle KBE = 90^\circ - \tfrac{1}{2}(\angle ABC - \theta),$$

$$\angle DKF = 90^\circ - \tfrac{1}{2}\angle KDF = 90^\circ - \tfrac{1}{2}(\angle ADC - \theta),$$

$$\angle BKD = 180^\circ - \theta.$$

Adding these and using $\angle ABC + \angle ADC = 180^\circ$, we get $\angle BKE + \angle BKD + \angle DKF = 270^\circ$. Then $\angle EKF = 90^\circ$, i.e. $KF \perp KE$. So $ME = MK = MF$. Also $BE = BK$ and $DF = DK$. Then $BM \perp KE$ and $DM \perp KF$. So $BM \| KF$ and $DM \| KE$. So $BM \perp DM$.

Other commended solvers: **Prithwijit DE** (HBCSE, Mumbai, India), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 469. Let m be an integer greater than 4. On the plane, if m points satisfy no three of them are collinear and every four of them are the vertices of a convex quadrilateral, then prove that all m of the points are the vertices of a m-sided convex polygon.

***Solution*.** **William FUNG**, **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let S be the set of the m points and C be the set of the vertices of the convex hull H of S. Then S contains C and C has at least 3 elements. Assume there is a point P in S and not in C. Let n be the number of elements in C. Since H is a convex polygon, H can be decomposed into $n-2$ triangles by selecting a vertex and connecting all other vertices to this vertex. Since no three points of S are collinear, P is in the interior of one of these triangles. This contradicts every four of them are the vertices of a convex quadrilateral. So $S=C$, $m=n$ and S is the set of the vertices of a m-sided convex polygon.

Problem 470. If a, b, $c > 0$, then prove that

$$\frac{a}{b(a^2+2b^2)}+\frac{b}{c(b^2+2c^2)}+\frac{c}{a(c^2+2a^2)} \geq \frac{3}{ab+bc+ca}.$$

Solution. **Jon GLIMMS** and **Henry RICARDO** (New York Math Circle, New York, USA).

Let x=1/a, y=1/b and z=1/c. Below all sums are cyclic in the order x,y,z. The desired inequality is the same as

$$\sum \frac{y^2}{z(2x^2+y^2)} \geq \frac{3}{x+y+z}.$$

By Cauchy's inequality, we have

$$\sum \frac{y^2}{z(2x^2+y^2)} \geq \frac{(x^2+y^2+z^2)^2}{\sum y^2 z(2x^2+y^2)}.$$

It suffices to show

$$\frac{(x^2+y^2+z^2)^2}{\sum y^2 z(2x^2+y^2)} \geq \frac{3}{x+y+z}.$$

Cross-multiplying and expanding, this is the same as

$$\sum (x^5+2x^3y^2+x^2y^3+xy^4) \geq \sum (2x^4y+4x^2y^2z). \qquad (*)$$

By the AM-GM inequality, we have

(1) $\sum (x^5+x^3y^2) \geq \sum 2x^4 y,$

(2) $\sum (x^2y^3+xy^4) = \sum (x^2y^3+yz^4) \geq \sum 2y^2z^2x = \sum 2x^2y^2z.$

Next, (3) $\sum (x^3y^2+x^2y^3) \geq \sum 2x^2y^2z$ is the same as $\sum x \sum x^2y^2 \geq 3xyz \sum xy$ after expansion. To get it, we have

$$\sum x \sum xy \geq \sum x \frac{\left(\sum xy\right)^2}{3} \geq 3xyz \sum xy$$

by Cauchy's inequality and the AM-GM inequality. Finally adding up (1), (2), (3), we get (*).

Other commended solvers: **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Olympiad Corner

(Continued from page 1)

Problem 3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$, and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Problem 4. Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Determine all functions f: $\mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+f(x+y))+f(xy) = x+f(x+y)+y\,f(x)$$

for all real numbers x and y.

Problem 6. The sequence a_1, a_2, … of integers satisfies the following conditions:

(i) $1 \leq a_j \leq 2015$ for all $j \geq 1$;

(ii) $k+a_k \neq l+a_l$ for all $1 \leq k \leq l$.

Prove that there exist two positive integers b and N such that

$$\left|\sum_{j=m+1}^{n}(a_j-b)\right| \leq 1007^2$$

for all integers m and n satisfying $n > m \geq N$.

IMO2015–Problem Report

(Continued from page 2)

This led to strong reactions from almost all the leaders, as the process of checking is indeed trivial, so an indication that the student is aware of the need of checking should be sufficient. This was eventually accepted by the coordinators.

Then the Canadian leader suggested that no mark should be deducted at all for omitting the checking. The UK leader said that he was surprised to hear such a suggestion as omitting the checking constitutes a logical error, but he would be happy to let this suggestion go to a vote. The Jury eventually voted against the suggestion. So in the end a student must somehow mention the checking (but need not actually show it) to get full mark for this question.

Interestingly, not checking that the solutions work would also constitute a logical error in Problem 2, but nobody made a suggestion to deduct points in that case. Also, while the coordinators first expected the checking to be explicitly carried out, in Problem 1 the coordinators did not even expect students to do anything to show that their constructed sets are balanced and center-free. It seems that such inconsistency between different problems is a common phenomenon.

Problem 6. Traditionally, Problem 6 is the most difficult problem of the IMO. This year's Problem 6 turned out to be not as difficult. Although only 11 out of the 577 contestants obtained perfect scores, the mean 0.355 for this question was one of the highest in recent years.

One of our team members solved this question. He mentioned that he got the idea by working on small cases first. So after all, this simple rule sometimes helps us solve not-so-simple problems!

At first sight the problem looks like one in mathematical analysis concerning the convergence of a sequence. One may even be tempted to try to prove that the sequence eventually becomes constant, which is not true.

There is an interesting interpretation of this problem (which is probably how this problem came up in the first place). At each second a ball is thrown upward, and the ball thrown at the i-th second will return to the ground after a_i seconds. So the condition $k+a_k \neq l+a_l$ for all $1 \leq k \leq l$ means that no two balls shall return to the ground at the same time. The interested reader may follow this line to see whether a solution could be obtained more easily.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *32nd Balkan Mathematical Olympiad* held in May 5, 2015.

Problem 1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6+b^3c^6+c^3a^6+3a^3b^3c^3 \ge abc(a^3b^3+b^3c^3+c^3a^3) +a^2b^2c^2(a^3+b^3+c^3).$$

Problem 2. Let ABC be a scalene triangle with incenter I and circumcircle (ω). The lines AI, BI, CI intersect (ω) for the second time at the point D, E, F, respectively. The line through I parallel to the sides BC, AC, AB intersect the lines EF, DF, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear.

Problem 3. A jury of 3366 film critics is judging the Oscars. Each critic makes a single vote for his favorite actor, and a single vote for his favorite actress. It turns out that for every integer $n\in\{1,2,\ldots,100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor and the same actress.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 7, 2016***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Divisibility Problems

Kin Y. Li

Divisibility problems are common in many math competitions. Below we will look at some of these interesting problems. As usual, for integers a and b with $a\neq 0$, we will write $a \mid b$ to denote <u>b is divisible by a</u> (or in short <u>a divides b</u>).

In dividing b by a, we get a quotient q and a remainder r, we get $b/a=q+r/a$. Notice that b/a is an integer if and only if r/a is an integer. The following examples exploit this observation.

<u>*Example 1.*</u> (1999 *AIME*) Find the greatest positive integer n such that $(n-2)^2(n+1)/(2n-1)$ is an integer.

<u>*Solution.*</u> The numerator is n^3-3n^2+4. So

$$\frac{n^3-3n^2+4}{2n-1}=\frac{1}{2}n^2-\frac{5}{4}n-\frac{5}{8}+\frac{27/8}{2n-1}.$$

Multiplying by 8, we get

$$\frac{8(n^3-3n^2+4)}{2n-1}=4n^2-10n-5+\frac{27}{2n-1}.$$

Then $2n-1|27$. The greatest such n is 14.

<u>*Example 2.*</u> (*1998 IMO*) Determine all pairs (a,b) of positive integers such that ab^2+b+7 divides a^2b+a+b.

<u>*Solution.*</u> We can think of a as a variable and b as a constant, then do division of polynomials to get

$$\frac{a^2b+a+b}{ab^2+b+7}=\frac{1}{b}a-\frac{7a/b-b}{ab^2+b+7}.$$

Multiplying by $b(ab^2+b+7)$, we get

$$b(a^2b+a+b)= (ab^2+b+7)a-(7a-b^2).$$

If $ab^2+b+7 \mid a^2b+a+b$, then

$$ab^2+b+7 \mid (ab^2+b+7)a-b(a^2b+a+b) =7a-b^2. \quad (*)$$

<u>*Case 1* $(7a-b^2=0)$.</u> Then $7a=b^2$. So $7| b$. Then for some positive integer k, $b=7k$ and $a=7k^2$. We can check $(a,b)=(7k^2,7k)$ are indeed solutions.

<u>*Case 2* $(7a-b^2<0)$.</u> Then $7a < b^2$ and

$$ab^2+b+7 \le |7a-b^2| = b^2-7a.$$

However, $b^2-7a < b^2 < ab^2+b+7$, which leads to a contradiction.

<u>*Case 3* $(7a-b^2>0)$.</u> Then $ab^2+b+7 \le 7a-b^2$. If $b \ge 3$, then $ab^2+b+7 \ge 9a > 7a > 7a-b^2$, contradicting (*).

So $b = 1$ or 2. If $b = 1$, then (*) yields $a+8 \mid 7a-1 = 7(a+8)-57$. Hence, $a+8 \mid 57$, which leads to $a = 11$ or 49. Then we can check $(a,b) = (11,1)$ and $(49,1)$ are solutions. If $b=2$, then (*) yields $4a+9 \mid 7a-4$. Now

$$4a+9 \le 7a-4 < 8a+18 = 2(4a+9).$$

So $4a+9 = 7a-4$, contradicting a is an integer.

<u>*Example 3.*</u> (*2003* IMO) Determine all pairs of positive integers (a,b) such that $a^2/(2ab^2-b^3+1)$ is a positive integer.

<u>*Solution.*</u> Suppose $a^2/(2ab^2-b^3+1)=k$ is a positive integer. Then $a^2-2kb^2a+kb^3-k = 0$. Multiplying by 4 and completing squares, we get

$$(2a-2kb^2)^2 = (2kb^2-b)^2 + (4k-b^2). \quad (**)$$

Let $M = 2a-2kb^2$ and $N = 2kb^2-b$.

<u>*Case 1* $(4k-b^2 = 0)$.</u> Then b is even and $M = \pm N$. If $M =-N$, then $b=2a$. If $M=N$, then $2a = 4kb^2-b = b^4-b$. We get $(a,b) = (b/2,b)$ or $((b^4-b)/2,b)$ with b even. These are easily checked to be solutions.

<u>*Case 2* $(4k-b^2 > 0)$.</u> Then $M^2 > N^2$ and $N = 2kb^2-b = b(2kb-1) \ge 1(2-1) = 1$. So $M^2 \ge (N+1)^2$. Hence, by (**)

$$\begin{aligned}4k-b^2 &= M^2 - N^2\\ &\ge (N+1)^2-N^2=2N+1\\ &= 4kb^2-2b+1,\end{aligned}$$

which implies $4k(b^2-1) + (b-1)^2 \le 0$.

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Then $b = 1$, $k = a/2$ and $(a,b) = (2k,1)$ are easily checked to be solutions for all positive integer k.

Case 3 ($4k-b^2 < 0$). Then $M^2 \le (N-1)^2$. By (**),

$$\begin{aligned}4k-b^2 &= M^2 - N^2\\ &\le (N-1)^2-N^2=-2N+1\\ &= -4kb^2+2b+1.\end{aligned}$$

This implies

$$\begin{aligned}0 &\le (1-4k)b^2+2b+(1-4k)\\ &= (1-4k)\left(b+\frac{1}{1-4k}\right)^2+\frac{8k(2k-1)}{1-4k}<0,\end{aligned}$$

which is a contradiction.

Exercise 1. Find all positive integers n, a, and b such that

$$n^b-1 \mid n^a+1.$$

For divisibility problems involving exponential terms, like 2^n, often we will need to do modulo arithmetic and apply Fermat's little theorem. A useful fact is *if $m>n\ge 0$, then there exist integers s, t such that $\gcd(m,n) = ms+nt$.* (Proof. If $n=0$, then let $s=1$, $t=0$. Suppose it is true for all r with $0 \le r < n$. Then $m=qn+r$, where $q=[m/n]$. We have

$$\begin{aligned}\gcd(m,n) &= \gcd(m,r) = ms+rt\\ &= ms+(m-qn)t = m(s+t)+n(-qt).\ )\end{aligned}$$

So if $d = \gcd(m,n)$ and a^m, $a^n \equiv 1 \pmod k$, then $a^d \equiv 1 \pmod k$ by the fact.

Example 4. (*1972 Putnam Exam*) Show that if n is an integer greater than 1, then 2^n-1 is not divisible by n.

Solution. Assume there exists an integer $n > 1$ such that $n \mid 2^n-1$. Since 2^n-1 is odd, n must be odd. Let p be the least prime divisor of n. Then $p \mid 2^n-1$, which is the same as $2^n\equiv 1 \pmod p$. By Fermat's little theorem, $2^{p-1}\equiv 1 \pmod p$. Let $d = \gcd(n,p-1)$. Then $2^d \equiv 1 \pmod p$. By the definition of p, since $d \mid n$ and $d \le p-1 < p$, we get $d = 1$. Then $2 = 2^d \equiv 1 \pmod p$ lead to a contradiction.

Having seen the last example, here comes a hard problem that one needs to know the last example to get a start.

Example 5. (*1990 IMO*) Determine all integer $n>1$ such that $(2^n+1)/n^2$ is an integer.

Solution. Since 2^n+1 is odd, n must be odd. Let p be the least prime divisor of n. Then $p|2^n+1$, which implies $(2^n)^2\equiv(-1)^2 \equiv 1 \pmod p$. By Fermat's little theorem, $2^{p-1}\equiv 1 \pmod p$. Let $d = \gcd(2n,p-1)\ge 2$. Then $2^d\equiv 1 \pmod p$. By the definition of p, we get $\gcd(n,p-1)=1$. This gives $d=2$ and $4=2^d\equiv 1 \pmod p$ gives $p=3$. Then $n = 3^k m$ for some $k \ge 1$ and m satisfying $\gcd(3,m)=1$.

Using $x^3+1=(x+1)(x^2-x+1)$ for $x=2^m$, 2^{3m}, 2^{9m}, …, we have

$$2^n+1=(2^m+1)\prod_{j=0}^{k-1}(2^{2\cdot 3^j m}-2^{3^j m}+1).\quad (*)$$

For odd c, $2^c\equiv 2,-1,-4 \pmod 9$ implies $2^{2c}-2^c+1\equiv 3 \pmod 9$. From the binomial expansion, we see $2^m+1 = (3-1)^m+1 \equiv 3m \equiv 3$ or $6 \pmod 9$. So each of the factor on the right side of (*) is divisible by 3, but not by 9. So $2^n+1=3^{k+1}s$ for some integer s satisfying $\gcd(3,s)=1$. Now $n^2 = 3^{2k}m^2 \mid 2^n+1 = 3^{k+1}s$, which implies $k=1$ and $n=3m$.

Assume $m>1$. Let q be the least prime divisor of m. Now q is odd and $q>3$. Then $\gcd(m,q-1)=1$. Since $q \mid m \mid n$, we have $q^2 \mid n^2 \mid 2^n+1$. Then 2^{q-1} and $2^{2n}\equiv 1 \pmod q$ lead to $2^w\equiv 1 \pmod q$, where $w = \gcd(2n, q-1)$. Then $w|2n=6m$. Also, from $w|q-1$ and $\gcd(m,q-1) = 1$, we get $w|6$. Now $q>3$, $w=1,2,3,6$ and $2^w\equiv 1 \pmod q$ imply $q=7$. Then $7=q \mid 2^n+1$, but $2^n\equiv 1, 2, 4\not\equiv -1 \pmod 7$, contradiction. Therefore, $m=1$ and $n=3$. Indeed, $3^2=9 \mid 2^3+1$.

Exercise 2. (1999 *IMO*) Find all pairs of positive integers (x,p) such that p is prime, $x\le 2p$, and x^{p-1} divides $(p-1)^x+1$.

In the following examples, we will see there is a very clever trick in solving certain divisibility problems.

Example 6. (*1988 IMO*) Let a and b positive integers such that $ab+1$ divides a^2+b^2. Show that $(a^2+b^2)/(ab+1)$ is square of an integer.

Solution. Let $k = (a^2+b^2)/(ab+1)$. Assume there is a case k is an integer, but not a square. Among all such cases, consider the case when $\max\{a,b\}$ is least possible. Note $a=b$ implies $0 < k = 2a^2/(a^2+1) < 2$, which implies $k=1=1^2$. So in the least case, $a\ne b$, say $a>b$. Now $k = (a^2+b^2)/(ab+1)>0$ and it can be rewritten as $a^2-kba+b^2-k=0$. Note $k\ne b^2$ implies $a\ne 0$.

Other than a, let c be the second root of $x^2-kbx+b^2-k=0$. Then $k = (c^2+b^2)/(cb+1)$, $a+c=kb$ and $ac=b^2-k$. So $c=kb-a=(b^2-k)/a$ is an integer. Now $cb+1=(c^2+b^2)/k > 0$ and $c=(b^2-k)/a\ne 0$ imply c is a positive integer. Finally, $c = (b^2-k)/a < (a^2-k)/a < a$. Now $k = (c^2+b^2)/(cb+1)$ is an integer, not a square and $\max\{b,c\} < a = \max\{a,b\}$. This contradicts $\max\{a,b\}$ is the least.

Example 7. (*2007 IMO*) Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Solution. We can consider a as variable and b as constant to do a division as in example 2, but a nicer way is as follows: from $(4a^2-1)b=a(4ab-1)+(a-b)$, we get

$$(4a^2-1)^2b^2=J(4ab-1)+(a-b)^2,$$

where $J=a^2(4ab-1)+2a(a-b)$. Observe that $\gcd(b^2,4ab-1)=1$ (otherwise prime $p \mid \gcd(b^2,4ab-1)$ would imply $p|b$ and $p|4ab-(4ab-1)=1$). Hence,

$$4ab-1|(4a^2-1)^2 \Leftrightarrow 4ab-1|(a-b)^2.$$

Now $k=(a-b)^2/(4ab-1) > 0$ and it can be rewritten as $a^2-(4k+2)ba+b^2+k=0$.

Assume there exists (a,b) such that k is an integer and $a\ne b$, say $a>b$. Among all such cases, consider the case when $a+b$ is least possible.

Other than a, let c be the second root of $x^2-(4k+2)bx+b^2+k=0$. Then $k = (c-b)^2/(4cb-1)$, $a+c = (4k+2)b$ and $ac = b^2+k$. So $c = (4k+2)b-a = (b^2+k)/a$ is a positive integer. So (c,b) is another case k is an integer. Since $a+b$ is least possible, we would have $c \ge a > b$. Now $c = (b^2+k)/a \ge a$ leads to $k \ge a^2-b^2$. Then

$$(a-b)^2 = k(4ab-1) \ge (a^2-b^2)(4ab-1).$$

Canceling $a-b$ on both sides, we get

$$a-b \ge (a+b)(4ab-1) > a,$$

a contradiction.

The next example is short and cute.

Example 8. (*2005 IMO Shortlisted Problem*) Let a and b be positive integers such that a^n+n divides b^n+n for every positive integer n. Show that $a=b$.

Solution. Assume $a\ne b$. For $n = 1$, we have $a+1|b+1$ and so $a < b$. Let p be a prime greater than b. Then let $n = (a+1)(p-1)+1$. By Fermat's little theorem, $a^n=(a^{p-1})^{a+1}a\equiv a \pmod p$.

So $a^n+n \equiv a+n = (a+1)p \equiv 0 \pmod p$. Then $p \mid a^n+n \mid b^n+n$. By Fermat's little theorem,

$$0\equiv b^n+n=(b^{p-1})^{a+1}b+n\equiv b-a \pmod p,$$

which contradicts $0 < a < b < p$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 7, 2016.***

Problem 476. Let p be a prime number. Define sequence a_n by a_0=0, a_1=1 and a_{k+2}=$2a_{k+1}-pa_k$. If one of the terms of the sequence is -1, then determine all possible value of p.

Problem 477. In $\triangle ABC$, points D, E are on sides AC, AB respectively. Lines BD, CE intersect at a point P on the bisector of $\angle BAC$.

Prove that quadrilateral $ADPE$ has an inscribed circle if and only if AB=AC.

Problem 478. Let a and b be a pair of coprime positive integers of opposite parity. If a set S satisfies the following conditions:

(1) $a, b \in S$;
(2) if $x,y,z\in$S, then $x+y+z\in$S,

then prove that every positive integer greater than $2ab$ belongs to S.

Problem 479. Prove that there exists infinitely many positive integers k such that for every positive integer n, the number $k2^n+1$ is composite.

Problem 480. Let m, n be integers with $n > m > 0$. Prove that if $0 < x < \pi/2$, then

$$2|\sin^n x-\cos^n x| \le 3|\sin^m x-\cos^m x|.$$

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Solutions

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Problem 471. For $n \ge 2$, let $A_1, A_2, \ldots, A_n$ be positive integers such that $A_k \le k$ for $1 \le k \le n$. Prove that $A_1+A_2+\cdots+A_n$ is even if and only if there exists a way of selecting + or – signs such that

$$A_1 \pm A_2 \pm \cdots \pm A_n = 0.$$

Solution. **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If $A_1 \pm A_2 \pm \cdots \pm A_n = 0$, then using $A_i \equiv \pm A_i$ (mod 2), we get $A_1+A_2+\cdots+A_n \equiv 0$ (mod 2). Hence $A_1+A_2+\cdots+A_n$ is even.

Conversely, we will prove by induction that for t from n to 1 that there exists a way of selecting signs so that

$$0 \le S_t = \pm A_t \pm A_{t+1} \pm \cdots \pm A_n \le t.$$

The case t=n is $0 < A_n \le n$. Suppose the case t=k is true, that is

$$0 \le S_k = \pm A_k \pm A_{k+1} \pm \cdots \pm A_n \le k.$$

If $A_{k-1} \le S_k$, then let $S_{k-1} = -A_{k-1}+S_k$ and we have $0 \le S_{k-1} = S_k - A_{k-1} \le k-1$. If $A_{k-1} > S_k$, then let $S_{k-1}=A_{k-1}-S_k$ (here $-S_k$ means reversing all the signs of S_k) and we have $0 \le S_{k-1} \le A_{k-1} \le k-1$. This completes the induction.

The case t=1 gives us $0 \le \pm A_1 \pm A_2 \pm \cdots \pm A_n \le 1$. As $\pm A_1 \pm A_2 \pm \cdots \pm A_n$ is an even integer, $\pm A_1 \pm A_2 \pm \cdots \pm A_n = 0$.

Problem 472. There are $2n$ distinct points marked on a line, n of them are colored red and the other n points are colored blue. Prove that the sum of the distances of all pairs of points with same color is less than or equal to the sum of the distances of all pairs of points with different color.

Solution. **Jon GLIMMS**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Raul A. SIMON** (Chile).

Let the points be on the real axis with red points having coordinates $x_1<x_2<\cdots<x_n$ and the blue points having coordinates $y_1<y_2<\cdots<y_n$. Let S_n denote the sum of distances of all pairs of points with same color and D_n denote the sum of distances of all pairs of points with different color. We will prove $S_i \le D_i$ for all i by induction. Now $S_1=0\le|x_1-y_1|=D_1$. Suppose $S_n \le D_n$. For case n+1,

$$\begin{aligned} S_{n+1}-S_n &= \sum_{i=1}^{n}(x_{n+1}-x_i)+(y_{n+1}-y_i) \\ &\le |x_{n+1}-y_{n+1}|+\sum_{i=1}^{n}|x_{n+1}-y_i|+|y_{n+1}-x_i| \\ &= D_{n+1}-D_n. \end{aligned}$$

Then $S_{n+1}-D_{n+1} \le S_n - D_n \le 0$. So $S_{n+1} \le D_{n+1}$.

Problem 473. Determine all functions f: $\mathbb{R} \to \mathbb{R}$ such that for all $x,y\in\mathbb{R}$,

$$f(x)\,f(y\,f(x)-1) = x^2 f(y) - f(x).$$

Solution. **Coco YAU** (Pui Ching Middle School).

The zero function is a solution. Suppose f is a solution that is not the zero function. Then there exists $a\in\mathbb{R}$ such that $f(a)\neq 0$.

Denote the functional equation by (*). Setting x=0 in (*), we get

$$f(0)(\,f(yf(0)-1)+1\,)=0.$$

If $f(0) \neq 0$, then $f(yf(0)-1) = -1$. Since $\{yf(0)-1: y\in\mathbb{R}\}=\mathbb{R}$, we can see f is the constant function -1. Then (*) with x=1 yields $(-1)^2 = -1^2+1$, which is a contradiction. So $f(0)$=0.

Now setting x=a, y=0 in (*), we can get

$$f(-1) = -1.$$

Also, if $f(b)$=0, then setting x=b and y=a, we get b=0. Hence,

$$f(x) = 0 \Leftrightarrow x = 0.$$

Next by setting x=y=1 in (*), we get $f(1)f(f(1)-1)=0 \Leftrightarrow f(1)-1=0 \Leftrightarrow f(1)=1$.

Setting x=1 in (*), we get

$$f(y-1)=f(y)-1. \tag{1}$$

Applying (1) to $f(yf(x)-1)$ in (*), we can simplify (*) to

$$f(x)\,f(yf(x)) = x^2 f(y). \tag{2}$$

Setting x=-1 in (2), we get $-f(-y)$=$f(y)$. So f is an odd function.

Applying induction to (1), we get for n = 1,2,3,…,

$$f(y-n) = f(y)-n. \tag{3}$$

Setting y=0, this gives $f(-n)= -n$. As f is odd, we get $f(n)$=n for all integers n. Setting x=n in (2), we get

$$f(ny) = nf(y). \tag{4}$$

Setting y=$1/n$ and y=$1/m$ we get 1=$nf(1/n)$ and $f(n/m)$=$nf(1/m)$=n/m. So $f(x) = x$ for all rational x.

Setting y=1 in (2), we get

$$f(x)\,f(\,f(x)) = x^2. \tag{5}$$

Setting x,y to be $f(x)$ in (2), we also get

$$f(f(x))\,f(\,f(x)\,f(f(x)\,) = f(x)^2 f(f(x)).$$

Cancelling $f(f(x))$ on both sides, we get

$$f(x)^2 = f(\,f(x)\,f(f(x))) = f\,(x^2),$$

where the second equality follows from applying f to both sides of (5). Then we see w>0 implies $f(w)$>0.

For irrational $w > 0$, assume $f(w) > w$. Take rational q=n/m such that m>0 and $f(w) > q > w$. We have $m(q-w) > 0$. So $f(n-mw) = f(m(q-w)) > 0$. As f is odd, using (4) and (3), we get

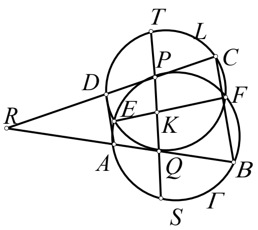
$$mf(w)-n = f(mw)-n = f(mw-n) < 0,$$

which contradicts $f(w) > q$. Similarly, $f(w) < w$ will lead to a contradiction. Therefore, $f(w)=w$ for all w and we can check (*) holds in this case.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 474. Quadrilateral $ABCD$ is convex and lines AB, CD are not parallel. Circle Γ passes through A, B and side CD is tangent to Γ at P. Circle L passes through C, D and side AB is tangent to L at Q. Circles Γ and L intersect at E and F. Prove that line EF bisects line segment PQ if and only if lines AD, BC are parallel.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Let EF meet PQ at K. Extend PQ to meet Γ and L at S and T respectively. Let lines AB, CD meet at R. We have

$$RP^2=RA\cdot RB \text{ and } RQ^2=RC\cdot RD. \quad (*)$$

By the intersecting chord theorem, we have $KP\cdot KS=KE\cdot KF=KQ\cdot KT$. Then $KP(KQ+QS) = KQ(KP+PT)$. Cancel $KP\cdot KQ$. We have

$$KP\cdot QS= KQ\cdot PT.$$

Then

$$\begin{aligned} & KP=KQ \\ \Leftrightarrow\ & QS=PT \\ \Leftrightarrow\ & PQ\cdot QS=QP\cdot PT \\ \Leftrightarrow\ & AQ\cdot QB=DP\cdot PC. \end{aligned}$$

Using $AQ=RQ-RA$, $QB=RB-RQ$, $DP=RP-RD$, $PC=RC-RP$ and (*), we get

$$\begin{aligned} & AQ\cdot QB=DP\cdot PC \\ \Leftrightarrow\ & RQ(RA+RB)=RP(RC+RD) \\ \Leftrightarrow\ & RC\cdot RD(RA+RB)^2=RA\cdot RB(RC+RD)^2 \\ \Leftarrow\ & (RA\cdot RC-RB\cdot RD)^2=0 \\ \Leftrightarrow\ & \frac{RA}{RB}=\frac{RD}{RC} \\ \Leftrightarrow\ & AD \parallel BC. \end{aligned}$$

Problem 475. Let a, b, n be integers greater than 1. If b^n-1 is a divisor of a, then prove that in base b, a has at least n digits not equal to zero.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Among all numbers that are multiples of b^n-1, suppose the least number of nonzero digits in base b of these numbers is s. Let A be one of these numbers with least digit sum, say

$$A=a_1b^{n_1}+a_2b^{n_2}+\cdots+a_sb^{n_s},$$

where $n_1 > n_2 > \cdots > n_s \ge 0$ and $1 \le a_i < b$ for $i=1,2,\cdots,s$.

Assume there are i,j such that $1\le i<j\le s$ and $n_i\equiv n_j\equiv r \pmod n$ with $0\le r\le n-1$. Then consider

$$B=A-a_ib^{n_i}-a_jb^{n_j}+(a_i+a_j)b^{nn_1+r}.$$

From $b^n \equiv 1 \pmod{b^n-1}$, we get $B \equiv 0 \pmod{b^n-1}$. If $a_i+a_j<b$, then the number of nonzero digits of B in base b is $s-1$, contradicting the choice of A. So we must have $b \le a_i+a_j < 2b$. Let $a_i+a_j = b+q$, where $0\le q<b$. Then

$$\begin{aligned} B = {} & b^{nn_1+r+1}+qb^{nn_1+r}+a_1b^{n_1}+\cdots \\ & +a_{i-1}b^{n_i-1}+a_{i+1}b^{n_{i+1}}+\cdots \\ & +a_{j-1}b^{n_{j-1}}+a_{j+1}b^{n_{j+1}}+\cdots+a_sb^{n_s}. \end{aligned}$$

Then the digit sum of B is

$$\begin{aligned} & \sum_{k=1}^{s}a_k-(a_i+a_j)+1+q \\ & =\sum_{k=1}^{s}a_k+1-b \\ & <\sum_{k=1}^{s}a_k, \end{aligned}$$

which is the digit sum of A. This contradicts the choice of A. So n_1, n_2, …, $n_s \pmod n$ are pairwise distinct. Then $s\le n$.

Assume $s<n$. Then let $n_i\equiv r_i \pmod n$ with $0\le r_i<n$ and consider

$$C=a_1b^{r_1}+a_2b^{r_2}+\cdots+a_sb^{r_s}.$$

Since $b^{n_i}\equiv b^{r_i} \pmod{b^n-1}$, so C is a multiple of b^n-1. Now $s<n$ implies

$$\begin{aligned} 0<C & \le (b-1)b+(b-1)b^2 \\ & \quad +\cdots+(b-1)b^{n-1} \\ & <b^n-1, \end{aligned}$$

contradiction. Therefore, $s = n$.

Other commended solvers: **Mark LAU Tin Wai** (Pui Ching Middle School) and **LEUNG Kit Yat** (St. Paul's College, Hong Kong).

Olympiad Corner

(*Continued from page 1*)

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d}\,\{n\sqrt{d}\}>\frac{5}{2}$$

where $\{x\}$ denotes the fractional part of the real number x. The fractional part of a real number x is x minus the greatest integer less than or equal to x.

Divisibility Problems

(*Continued from page 2*)

Solution of Exercise 1. Let $a=qb+r$ with $0 \le r \le b-1$. Then

$$\frac{n^a+1}{n^b-1}=n^r\sum_{j=0}^{q-1}n^{bj}+\frac{n^r+1}{n^b-1}.$$

So we need to find when $n^b-1 \mid n^r+1$. If $b=1$, then $r=0$ and we get $n=2,3$. If $b>1$, then $n>1$ and $n^b\ge 4$. For $n^b>4$, we have $0<n^r+1\le n^{b-1}+1\le n^b/2+1<n^b-1$, hence no solution. For $n^b\le 4$, we have three cases, namely $(n,b,a) = (2,2,2k-1)$, $(3,1,k)$ and $(2,1,k)$, where $k=1,2,3,\ldots$.

Solution of Exercise 2. For $x<3$ or $p<3$, the solutions are $(x,p)=(2,2)$ and $(1,\text{prime})$. For x and $p\ge 3$, since p is odd, $(p-1)^x+1$ is odd, so x is odd. Let q be the least prime divisor of x, which must be odd. We have $q \mid x \mid x^{p-1} \mid (p-1)^x+1$. So $(p-1)^x\equiv -1 \pmod q$. By Fermat's little theorem, $(p-1)^{q-1}\equiv 1 \pmod q$. By the definition of q, we have $\gcd(x,q-1)=1$. Then there are integers a,b such that $ax=b(q-1)+1$ is odd. Then a is odd. Now

$$p-1\equiv(p-1)^{b(q-1)+1}=(p-1)^{ax}\equiv -1 \pmod q$$

implies $q|p$. So $q=p$. Since x is odd, $p = q \mid x$ and the problem require the condition $x\le 2p$, we must have $x=p$ for the cases $x,p \ge 3$. Observe that

$$p^{p-1}\mid(p-1)^p+1=p^2(mp+1)$$

for some m. Then $p-1\le 2$. So $x=p=3$ is the only solution.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Second Round of the *32nd Iranian Math Olympiad*.

Problem 1. A local supermarket is responsible for the distribution of 100 supply boxes. Each box is ought to contain 10 kilograms of rice and 30 eggs. It is known that a total of 1000 kilograms of rice and 3000 eggs are in these boxes, but in some of them the amount of either item is more or less than the amount required. In each step, supermarket workers can choose two arbitrary boxes and transfer any amount of rice or any number of eggs between them. At least how many steps are required so that, starting from any arbitrary initial condition, after these steps the amount of rice and the number of eggs in all these boxes are equal?

Problem 2. Square $ABCD$ is given. Points N and P are selected on sides AB and AD, respectively, such that $PN = NC$, and point Q is selected on segment AN such that $\angle NCB = \angle QPN$. Prove that $\angle BCQ = ½\angle PQA$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 29, 2016***.

For individual subscription for the next five issues for the 15-16 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Coloring Problems

Kin Y. Li

In some math competitions, there are certain combinatorial problems that are about partitioning a board (or a set) into pieces like dominos. We will look at some of these interesting problems. Often clever ways of assigning color patterns to the squares of the board allow simple solutions. Below, a <u>$m\times n$ rectangle</u> will mean a m-by-n or a n-by-m rectangle.

<u>Example 1.</u> A 8×8 chessboard with the the northeast and southwest corner unit squares removed is given. Is it possible to partition such a board into thirty-one dominoes (where a domino is a 1×2 rectangle)?

<u>Solution.</u> For such a board, we can color the unit squares alternatively in black and white, say black is color 1 and white is color 2. Then we have the following pattern.

1	2	1	2	1	2	1	
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
	1	2	1	2	1	2	1

Each domino will cover two adjacent squares, one with color 1 and the other with color 2. If 31 dominoes can cover the board, there should be 31 squares with color 1 and 31 squares with color 2. However, in the board above there are 32 squares of color 1 and 30 squares of color 2. So the task is impossible.

<u>Example 2.</u> Eight 1×3 rectangles and one 1×1 square covered a 5×5 board. Prove that the 1×1 square must be over the center unit square of the board.

<u>Solution.</u> Let us paint the 25 unit squares of the 5×5 board with colors A, B and C as shown on the top of the next column.

A	B	C	A	B
B	C	A	B	C
C	A	B	C	A
A	B	C	A	B
B	C	A	B	C

There are 8 color A squares, 9 color B squares and 8 color C squares. Each 1×3 rectangle covers a color A, a color B and a color C square. So the 1×1 square piece must be over a color B square.

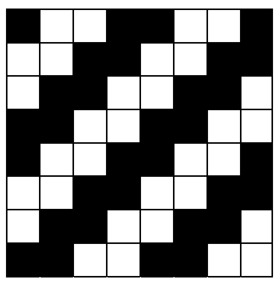
Next, we rotate the <u>*coloring*</u> of the board (not the board itself) clockwise 90˚ around the center unit square.

B	A	C	B	A
C	B	A	C	B
A	C	B	A	C
B	A	C	B	A
C	B	A	C	B

Then observe that the 1×1 square piece must still be over a color B square due to reasoning used in the top paragraph. However, the only color B square that remains color B after the 90˚ rotation is the center unit square. So the 1×1 square piece must be over the center unit square.

<u>Example 3.</u> Can a 8×8 board be covered by fifteen 1×4 rectangles and one 2×2 square without overlapping?

<u>Solution.</u> Consider the following coloring of the 8×8 board.



(continued on page 2)

In the coloring of the board, there are 32 white and 32 black squares respectively. By simple checking, we can see every 1×4 rectangle will cover 2 white and 2 black squares. The 2×2 square will cover *either* 1 black and 3 white squares *or* 3 black and 1 white squares. Assume the task is possible. Then the 16 pieces together should cover *either* 31 black and 33 white squares *or* 33 black and 31 white squares, which is a contradiction to the underlined statement above.

In coloring problems, other than assigning different colors to all the squares, sometimes assigning different numerical values for different types of squares can be useful in solving the problem. Below is one such example.

Example 4. Let m,n be integers greater than 2. Color every 1×1 square of a $m\times n$ board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then call this pair of squares a *distinct pair*. Let S be the number of distinct pairs in the $m\times n$ board. Prove that whether S is odd or even depends only on the 1×1 squares on the boundary of the board excluding the 4 corner 1×1 squares.

Solution. We first divide the 1×1 squares into three types. Type 1 squares are the four 1×1 squares at the corners of the board. Type 2 squares are the 1×1 squares on the boundary of the board, but not the type 1 squares. Type 3 squares are the remaining 1×1 squares.

Assign every white 1×1 square the value 1 and every black 1×1 square the value −1. Let the type 1 squares have values a, b, c, d respectively. Let the type 2 squares have values $x_1, x_2, \ldots, x_{2m+2n-8}$ and the type 3 squares have values $y_1, y_2, \ldots, y_{(m-2)(n-2)}$.

Next for every pair of 1×1 squares sharing a common edge, write the product of the values in the two squares on their common edge. Let H be the product of these values on all the common edges. For every type 1 square, it has two neighbor squares sharing a common edge with it. So the number in a type 1 square appears two times as factors in H. For every type 2 square, it has three neighbor squares sharing a common edge with it. So the number in a type 2 square appears three times as factors in H. For every type 3 square, it has four neighbor squares sharing a common edge with it. So the number in a type 3 square appears four times as factors in H. Hence,

$$\begin{aligned} H &= (abcd)^2(x_1x_2\cdots x_{2m+2n-8})^3(y_1y_2\cdots y_{(m-2)(n-2)})^4 \\ &= (x_1x_2\cdots x_{2m+2n-8})^3. \end{aligned}$$

If $x_1x_2\cdots x_{2m+2n-8} = 1$, then $H = 1$ and there are an even number of distinct pairs in the board. If $x_1x_2\cdots x_{2m+2n-8} = -1$, then $H = -1$ and there are an odd number of distinct pairs in the board. So whether S is even or odd is totally determined by the set of type 2 squares.

Next we will look at problems about coloring elements of some sets.

Example 5. There are 1004 distinct points on a plane. Connect each pair of these points and mark the midpoints of these line segments black. Prove that there are at least 2005 black points and there exists a set of 1004 distinct points generating exactly 2005 black midpoints of the line segments connecting pairs of them.

Solution. From 1004 distinct points, we can draw $k={}_{1004}C_2$ line segments connecting pairs of them. Among these, there exists a *longest* segment AB. Now the midpoints of the line segments joining A to the other 1003 points lie inside or on the circle center at A and radius ½AB. Similarly, the midpoints of the line segments joining B to the other 1003 points lie inside or on another circle center at B and radius ½AB. These two circles intersect only at the midpoint of AB. Then there are at least 2×1003−1=2005 black midpoints generated by the line segments.

To construct an example of a set of 1004 points generating exactly 2005 black midpoints, we can simply take 0, 2, 4, …, 2006 on the x-axis. Then the black midpoints generated are exactly the point at 1, 2, 3, …, 2005 of the x-axis..

Example 6. Find all ways of coloring all positive integers such that

(1) every positive integer is colored either black or white (but not both) and

(2) the sum of two numbers with distinct colors is always colored black and their product is always colored white.

Also, determine the color of the product of two white numbers.

Solution. Other than coloring all positive integers the same color, we have the following coloring satisfying conditions (1) and (2). We claim if m and n are white numbers, then mn is a white number. To see this, assume there are m, n both white, but mn is black. Let k be black. By (1), $m+k$ is black and $(m+k)n = mn+kn$ is white. On the other hand, kn is white and mn is black. So by (2), $mn+kn$ would also be black, which is a contradiction.

Next, let j be the *smallest* white positive integer. From (2) and the last paragraph, we see every sj is white, where s is any positive integer. We will prove every positive integer p that is not a multiple of j is black. Suppose $p=qj+r$, where q is a nonnegative integer and $0 < r < j$. Since j is the smallest white integer, so r is black. When q=0, $p=r$ is black. When $q \ge 1$, qj is white and so by (2), $p=qj+r$ is black.

Example 7. In the coordinate plane, a point (x,y) is called a *lattice point* if and only if x and y are integers. Suppose there is a convex pentagon $ABCDE$ whose vertices are lattice points and the lengths of its five sides are all integers. Prove that the perimeter of the pentagon $ABCDE$ is an even integer.

Solution. Let us color every lattice point of the coordinate plane either black or white. If $x+y$ is even, then color (x,y) white. If $x+y$ is odd, then color (x,y) black. Notice (x,y) is assigned a color different from its four *neighbors* $(x\pm1,y)$ and $(x,y\pm1)$.

Now for each of the five sides, say AB, of the pentagon $ABCDE$, let A be at (x_1, y_1) and B be at (x_2, y_2). Also let T_{AB} to be at (x_1,y_2). Then ΔABT_{AB} is a right triangle with AB as the hypotenuse or it is a line segment (which we can consider as a degenerate right triangle).

Since each lattice point is assigned a color different from any one of its four neighbors, the polygonal path

$$AT_{AB}BT_{BC}CT_{CD}DT_{DE}ET_{EA}A$$

has even length. For positive integers a, b, c satisfying $a^2+b^2=c^2$, since $n^2\equiv n \pmod 2$, we get $a+b\equiv c \pmod 2$. It follows the perimeter of $ABCDE$ and the length of $AT_{AB}BT_{BC}CT_{CD}DT_{DE}ET_{EA}A$ are of the same parity. So the perimeter of $ABCDE$ is even.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 29, 2016.***

Problem 481. Let $S=\{1,2,\ldots,2016\}$. Determine the least positive integer n such that whenever there are n numbers in S satisfying every pair is relatively prime, then at least one of the n numbers is prime.

Problem 482. On $\triangle ABD$, C is a point on side BD with $C\neq B,D$. Let K_1 be the circumcircle of $\triangle ABC$. Line AD is tangent to K_1 at A. A circle K_2 passes through A and D and line BD is tangent to K_2 at D. Suppose K_1 and K_2 intersect at A and E with E inside $\triangle ACD$. Prove that $EB/EC=(AB/AC)^3$.

Problem 483. In the open interval $(0,1)$, n distinct rational numbers a_i/b_i $(i=1,2,\ldots,n)$ are chosen, where $n>1$ and a_i, b_i are positive integers. Prove that the sum of the b_i's is at least $(n/2)^{3/2}$.

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices A, B and C. For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Problem 485. Let m and n be integers such that $m>n>1$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1, a_2, \ldots, a_n\}$ is a subset of S. It is known that every two numbers in T do not both divide any number in S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{n}.$$

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Solutions

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Problem 476. Let p be a prime number. Define sequence a_n by $a_0=0$, $a_1=1$ and $a_{k+2}=2a_{k+1}-pa_k$. If one of the terms of the sequence is -1, then determine all possible value of p.

Solution. **Jon GLIMMS** and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

Observe that $p\neq 2$ (otherwise beginning with a_2, the rest of the terms will be even, then -1 cannot appear). On one hand, using the recurrence relation, we get

$$a_{k+2}\equiv 2a_{k+1}\equiv\cdots\equiv 2^{k+1}a_1=2^{k+1} \pmod p.$$

If $a_m=-1$ for some $m\geq 2$, then letting $k=m-2$, we get

$$-1=a_m\equiv 2^{m-1} \pmod p. \qquad (*)$$

On the other hand, using the recurrence relation again, we also have

$$a_{k+2}\equiv 2a_{k+1}-a_k \pmod{p-1},$$

which implies $a_{k+2}-a_{k+1}\equiv a_{k+1}-a_k\equiv\cdots\equiv a_1-a_0=1 \pmod{p-1}$. Then

$$-1=a_m\equiv m+a_0=m \pmod{p-1},$$

which implies $p-1$ divides $m+1$. By Fermat's little theorem and (*), we get

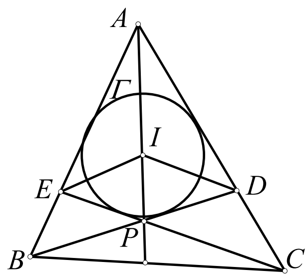
$$1\equiv 2^{m+1}\equiv 4\cdot 2^{m-1}\equiv -4 \pmod p.$$

Then $p=5$. Finally, if $p=5$, then $a_3=-1$.

Problem 477. In $\triangle ABC$, points D, E are on sides AC, AB respectively. Lines BD and CE intersect at a point P on the bisector of $\angle BAC$.

Prove that quadrilateral $ADPE$ has an inscribed circle if and only if $AB=AC$.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Jafet Alejandro Baca OBANDO** (IDEAS High School, Nicaragua) and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Suppose $ADPE$ has an inscribed circle Γ. Since the center of Γ is on the bisector of $\angle BAC$, the center is on line AP. Similarly, AP also bisects $\angle DPE$, so $\angle APE=\angle APD$. It also follows that $\angle APB=\angle APC$, since $\angle EPB=\angle DPC$. By ASA, we get $\triangle APB\cong\triangle APC$ with AP common. Then $AB=AC$.

Conversely, if $AB=AC$, then $\triangle ABC$ is symmetric with respect to AP. Thus, lines BP and CP (hence also D and E) are symmetric with respect to AP. By symmetry, the bisectors of $\angle ADP$ and $\angle AEP$ meet at a point I on AP. Then the distances from I to lines EA, EP, DP, DA are the same. So $ADPE$ has an inscribed circle with center I.

Other commended solvers: **Mark LAU Tin Wai** (Pui Ching Middle School), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 478. Let a and b be a pair of coprime positive integers of opposite parity. If a set S satisfies the following conditions:

(1) $a, b\in S$;
(2) if $x,y,z\in S$, then $x+y+z\in S$,

then prove that every positive integer greater than $2ab$ belongs to S.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

Without loss of generality, we assume that a is odd and b is even. Let $n>2ab$. Since a and b are coprime, the equation $ax\equiv n \pmod b$ has a solution satisfying $0\leq x<b$. Then $y=(n-ax)/b$ is a positive integer. Now

$$a=\frac{2ab-ab}{b}<\frac{n-ax}{b}=y\leq\frac{2ab}{b}=2a.$$

Let $x'=x+b$, $y'=y-a$, then x', y' are positive and $ax'+by'=n$. Observe $x+y$ and $x'+y'=x+y+b-a$ are of opposite parity. So we may assume $x+y$ is odd (otherwise take $x'+y'$). Then $x+y\geq 3$ and by (1) and (2),

$$n=a+\cdots+a+b+\cdots+b\in S,$$

where a appeared x times and b appeared y times.

Other commended solvers: **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5) and **Mark LAU Tin Wai** (Pui Ching Middle School).

Problem 479. Prove that there exists infinitely many positive integers k such that for every positive integer n, the number $k2^n+1$ is composite.

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

By the Chinese remainder theorem, there exist infinitely many positive integers k such that

$$\begin{aligned} k &\equiv 1 \pmod{3},\\ k &\equiv 1 \pmod{5},\\ k &\equiv 3 \pmod{7},\\ k &\equiv 10 \pmod{13},\\ k &\equiv 1 \pmod{17},\\ k &\equiv -1 \pmod{241}. \end{aligned}$$

If $n\equiv 1$ (mod 2), then $k2^n+1\equiv 2+1\equiv 0$ (mod 3). Otherwise $2|n$. If $n\equiv 2$ (mod 4), then $k2^n+1\equiv 2^2+1\equiv 0$ (mod 5). Otherwise $4|n$. If $n\equiv 4$ (mod 8), then $k2^n+1 \equiv 2^4+1\equiv 0$ (mod 17). Otherwise $8|n$. Then we have three cases:

Case 1: $n\equiv 8$ (mod 24). By Fermat's little theorem, $2^{24} = (2^{12})^2 \equiv 1$ (mod 13). So $2^n = 2^{8+24m} \equiv 256 \equiv -4$ (mod 13) and $k2^n+1 \equiv 10(-4)+1 \equiv 0$ (mod 13).

Case 2: $n\equiv 16$ (mod 24). Since $2^{24} = (2^3)^8 \equiv 1$ (mod 7), we have $2^n = 2^{16+24m} \equiv 2^{1+3(5+8m)} \equiv 2$ (mod 7) and $k2^n+1 \equiv 3\cdot 2+1 \equiv 0$ (mod 7).

Case 3: $n\equiv 0$ (mod 24). Since $2^{24} = (2^8)^3 \equiv 15^3 \equiv 225\cdot 15 \equiv -16\cdot 15 \equiv 1$ (mod 241). So $2^n = 2^{24m} \equiv 1$ (mod 241) and Then $k2^n+1 \equiv -1+1 = 0$ (mod 241).

Comment: We may wonder why modulo 3, 5, 7, 13, 17, 241 work. It may be that in dealing with $n\equiv 8, 16, 0$ (mod 24), we want $2^{24}\equiv 1$ (mod p) for some useful primes p. Then we notice

$$\begin{aligned} 2^{24}-1 &= (2^3-1)(2^3+1)(2^6+1)(2^{12}+1)\\ &= 7\cdot 3^2\cdot 5\cdot 13\cdot 17\cdot 241. \end{aligned}$$

Other commended solvers: **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Prishtina Math Gymnasium Problem Solving Group** (Republic of Kosova), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 480. Let m, n be integers with $n > m > 0$. Prove that if $0 < x < \pi/2$, then

$$2|\sin^n x - \cos^n x| \le 3|\sin^m x - \cos^m x|.$$

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

If $x=\pi/4$, both sides are 0. Since the inequality for x and $\pi/2-x$ are the same, we only need to consider $0 < x < \pi/4$. Let $k\ge 0$. Define $a_k=\cos^k x-\sin^k x$. We have $a_k\ge 0$. For $k\ge 2$, we have

$$\begin{aligned} a_k &= (\cos^k x-\sin^k x)(\cos^2 x+\sin^2 x)\\ &= a_{k+2}+\sin^2 x\cos^2 x\, a_{k-2}\\ &\ge a_{k+2}. \end{aligned}$$

Let $m \ge 2$. For the case $n-m = 2,4,6,\ldots$, we have $3a_m \ge 2a_m \ge 2a_n$. Next, for the case $n-m = 1,3,5,\ldots$, observe that

$$(\cos x+\sin x)a_m=a_{m+1}+\sin x\cos x\, a_{m-1}.$$

Using this, we have

$$\begin{aligned} &3a_m \ge 2a_{m+1}\\ \Leftrightarrow\ &3a_m \ge 2[(\cos x+\sin x)a_m - \sin x\cos x\, a_{m-1}]\\ \Leftrightarrow\ &[3-2\sqrt{2}\sin(x+\tfrac{\pi}{4})]a_m \ge -2\sin x\cos x\, a_{m-1}, \end{aligned}$$

which is true as the left side is positive and the right side is negative. Then $3a_m \ge 2a_{m+1} \ge 2a_n$.

Finally, for the case $m=1$, we get $3a_1\ge 2a_2$ from $3 > 2\sqrt{2} \ge 2(\cos x + \sin x) = 2a_2/a_1$. Then $3a_1 \ge 2a_2 \ge 2a_n$ for $n = 2, 4, 6, \ldots$. Also, we get $3a_1\ge 2a_3$ from $3 \ge 2+\sin 2x = 2a_3/a_1$. Then $3a_1 \ge 2a_3 \ge 2a_n$ for $n = 3, 5, 7, \ldots$.

Other commended solvers: **Nicușor ZLOTA** ("Traian Vuia" Technical College, Focșani, Romania).

Olympiad Corner

(Continued from page 1)

Problem 3. Let x, y and z be nonnegative real numbers. Knowing that $2(xy+yz+zx) = x^2+y^2+z^2$, prove

$$\frac{x+y+z}{3} \ge \sqrt[3]{2xyz}.$$

Problem 4. Find all of the solutions of the following equation in natural numbers:

$$n^{n^n} = m^m.$$

Problem 5. A non-empty set S of positive real numbers is called **powerful** if for any two distinct elements of it like a and b, at least one of the numbers a^b or b^a is an element of S.

a) Present an example of a powerful set having four elements.

b) Prove that a finite powerful set cannot have more than four elements.

Problem 6. In the **Majestic Mystery Club (MMC)**, members are divided into several groups, and groupings change by the end of each week in the following manner: in each group, a member is selected as king; all of the kings leave their respective groups and form a new group. If a group has only one member, that member goes to the new group and his former group is deleted. Suppose that MMC has n members and at the beginning all of them form a single group. Prove that there comes a week for which thereafter each group will have at most $1+\sqrt{2n}$ members.

Coloring Problems

(Continued from page 2)

Example 8. Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are divided into two groups, each having at least one number. Prove that there always exists a three term arithmetic progression (AP in short) in one of the two groups.

Solution. Assume no three term AP is in any of the two groups. Color numbers in one group red and the other group blue. Since 5/2>2, among 1, 3, 5, 7, 9, there exist three of them assigned the same color, say they are red. By assumption, they are not the terms of an AP. Below are the possibilities of these red numbers: {1,3,7}, {1,3,9}, {1,5,7}, {1,7,9}, {3,5,9} or {3,7,9}.

If 1,3,7 are red, then as 1,2,3 and 1,4,7 and 3,5,7 are AP, so 2, 4, 5 are blue. As 4,5,6 and 2,5,8 are AP, so 6, 8 are red. So 6,7,8 is a red AP, contradiction.

If 1,3,9 are red, then as 1,2,3 and 1,5,9 and 3,6,9 are AP, so 2, 5, 6 are blue. As 4,5,6 and 5,6,7 an AP, so 4,7 are red. Then 1,4,7 is a red AP, contradiction.

If 1,5,7 are red, then as 1,3,5 and 5,6,7 and 1,5,9 are AP, so 3,6,9 are blue. Then 3, 6, 9 is a blue AP, contradiction.

If 1,7,9 are red, then as 1,4,7 and 1,5,9 and 7,8,9 are AP, so 4, 5, 8 are blue. As 3,4,5 and 4,5,6 an AP, so 3,6 are red. Then 3,6,9 is a red AP, contradiction.

If 3,5,9 are red, then as 1,5,9 and 3,4,5 and 5,7,9 are AP, so 1,4,7 are blue. Then 1,4,7 a blue AP, contradiction.

If 3,7,9 are red, then as 3,5,7 and 3,6,9 and 7,8,9 are AP, so 5, 6, 8 are blue. As 2,5,8 and 4,5,6 are AP, so 2, 4 are red. So 2,3,4 is a red AP, contradiction.

Mathematical Excalibur

Volume 20, Number 4 *March 2016 – July 2016*

Olympiad Corner

Below are the problems of the *28th Asian Pacific Math Olympiad*, which was held in March 2016.

Problem 1. We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A=90^\circ$ and $AB=AC$.

Problem 2. A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1}+2^{a_2}+\cdots+2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Problem 3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 14, 2016***.

For individual subscription for the next five issues for the 15-16 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inequalities of Sequences

Kin Y. Li

There are many math competition problems on inequalities. While most symmetric inequalities can be solved by powerful facts like the Muirhead and Schur inequalities, there are not many tools for general inequalities involving sequences. Below we will first take a look at some relatively easy examples on inequalities of sequences.

<u>*Example 1*</u>. (1997 *Chinese Math Winter Camp*) Let $a_1, a_2, a_3, \ldots$ be a sequence of nonnegative numbers. If for all positive integers m and n, $a_{n+m} \le a_n+a_m$, then prove that

$$a_n \le ma_1+\left(\frac{n}{m}-1\right)a_m.$$

<u>*Solution.*</u> Let $n = mq + r$, where q, r are integers and $0 \le r < m$. We have

$$\begin{aligned} a_n &\le a_{mq}+a_r \le qa_m+a_r \\ &= \frac{n-r}{m}a_m+a_r \\ &= \left(\frac{n}{m}-1\right)a_m+\frac{m-r}{m}a_m+a_r \\ &\le \left(\frac{n}{m}-1\right)a_m+\frac{m-r}{m}ma_1+ra_1 \\ &= \left(\frac{n}{m}-1\right)a_m+ma_1. \end{aligned}$$

<u>*Example 2.*</u> (*IMO* 2014) Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that

$$a_n < \frac{a_0+a_1+\cdots+a_n}{n} \le a_{n+1}.$$

<u>*Solution.*</u> For $n = 1,2,3,\ldots$, define

$$d_n = (a_0+a_1+\cdots+a_n)-na_n.$$

We have

$$\begin{aligned} &na_{n+1}-(a_0+a_1+\cdots+a_n) \\ &=(n+1)a_{n+1}-(a_0+a_1+\cdots+a_{n+1}) \\ &=-d_{n+1}. \end{aligned}$$

In terms of d_i's, the required conclusion is the same as $d_n > 0 \ge d_{n+1}$ for some unique $n \ge 1$.

Now observe that $d_1 = (a_0+a_1) - a_1 > 0$. Also the d_i's are strictly decreasing as

$$\begin{aligned} &d_{n+1}-d_n \\ &=\sum_{i=1}^{n+1}a_i-(n+1)a_{n+1}-\sum_{i=1}^{n}a_i-na_n \\ &=n(a_n-a_{n+1})<0. \end{aligned}$$

Finally, from $d_1 > 0$, the d_i's are integers and strictly decreasing, there must be a first non-positive d_i. So $d_n > 0 \ge d_{n+1}$ for some unique $n \ge 1$.

<u>*Example 3.*</u> (1980 *Austrian-Polish Math Competition*) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers satisfying the inequality

$$|a_{k+m}-a_k-a_m| \le 1 \quad \text{for all } k, m.$$

Show that the following inequality holds for all positive integers k and m,

$$\left|\frac{a_k}{k}-\frac{a_m}{m}\right| < \frac{1}{k}+\frac{1}{m}.$$

<u>*Solution.*</u> Observe that multiplying by km, the desired inequality is the same as $|ma_k-ka_m| < m+k$. To get this, we will prove for a fixed m, $|a_{km}-ka_m| < k$ holds for all positive integer k by induction. The case $k = 1$ is $|a_m-a_m|=0<1$. Suppose the k-th case is true. Then

$$\begin{aligned} &|a_{(k+1)m}-(k+1)a_m| \\ &=|a_{km+m}-a_{km}-a_m+a_{km}-ka_m| \\ &\le |a_{km+m}-a_{km}-a_m|+|a_{km}-ka_m| \\ &\le 1+|a_{km}-ka_m|<1+k. \end{aligned}$$

This completes the inductive step. Now interchanging k and m, similarly we also have $|a_{km}-ma_k| < m$. Then

$$\begin{aligned} |ma_k-ka_m| &\le |a_{km}-ma_k|+|a_{km}-ka_m| \\ &\le m+k \end{aligned}$$

and we are done.

(continued on page 2)

Example 4. (2006 *IMO Shortlisted Problem*) The sequence of real numbers a_0, a_1, a_2, … is defined recursively by

$$a_0=-1,\ \sum_{k=0}^{n}\frac{a_{n-k}}{k+1}=0 \quad for \quad n\ge 1.$$

Show that $a_n>0$ for $n\ge 1$.

Solution. Setting n=1, we find a_1=1/2. For $n\ge 1$, reversing the order of the terms in the given sum, we have

$$\sum_{k=0}^{n}\frac{a_k}{n-k+1}=0 \quad and \quad \sum_{k=0}^{n+1}\frac{a_k}{n-k+2}=0.$$

Suppose a_1 to a_n are positive. Then

$$\begin{aligned}0&=(n+2)\sum_{k=0}^{n+1}\frac{a_k}{n-k+2}-(n+1)\sum_{k=0}^{n}\frac{a_k}{n-k+1}\\&=(n+2)a_{n+1}+\sum_{k=0}^{n}\left(\frac{n+2}{n-k+2}-\frac{n+1}{n-k+1}\right)a_k.\end{aligned}$$

Notice the k=0 term in the last sum is 0. Solving for a_{n+1}, we get

$$\begin{aligned}a_{n+1}&=\frac{1}{n+1}\sum_{k=1}^{n}\left(\frac{n+1}{n-k+1}-\frac{n+2}{n-k+2}\right)a_k\\&=\frac{1}{n+2}\sum_{k=1}^{n}\frac{k}{(n-k+1)(n-k+2)}a_k\end{aligned}$$

is positive as a_1 to a_n are positive.

Next we will study certain examples that require more observation and possibly involve some calculations of limit of sequences.

Example 5. (1988 *IMO Shortlisted Problem*) Let a_1, a_2, a_3, … be a sequence of nonnegative real numbers such that

$$a_k-2a_{k+1}+a_{k+2}\ge 0 \quad \text{and} \quad \sum_{j=1}^{k}a_j\le 1$$

for all k = 1, 2, …. Prove that

$$0\le a_k-a_{k+1}<\frac{2}{k^2}$$

for all k = 1, 2, ….

Solution. We claim $0\le a_k-a_{k+1}$ for all k. (Otherwise assume for some k, we have $a_k-a_{k+1}<0$. From $a_k-2a_{k+1}+a_{k+2}\ge 0$, we get $a_{k+1}-a_{k+2}\le a_k-a_{k+1}<0$. It follows $a_k<a_{k+1}<a_{k+2}<\cdots$. Then

$$a_k+a_{k+1}+a_{k+2}+\cdots$$

diverges to infinity, which leads to a contradiction.)

Let $b_k=a_k-a_{k+1}$. Then for all positive integer k, we have $b_k\ge b_{k+1}\ge 0$. Now we have

$$\begin{aligned}b_k\sum_{i=1}^{k}i\le\sum_{i=1}^{k}ib_i&=\sum_{i=1}^{k}ia_i-\sum_{i=1}^{k}ia_{i+1}\\&=\sum_{i=1}^{k}ia_i-\sum_{i=2}^{k+1}(i-1)a_i\\&=\sum_{i=1}^{k}a_i-ka_{k+1}\le\sum_{i=1}^{k}a_i\le 1.\end{aligned}$$

Therefore,

$$b_k\le 1\Big/\sum_{i=1}^{k}i=\frac{2}{k(k+1)}<\frac{2}{k^2}.$$

Example 6. (1970 *IMO*) Let $1=a_0\le a_1\le a_2\le\cdots\le a_n\le\cdots$ be a sequence of real numbers. Consider the sequence defined by

$$b_n=\sum_{k=1}^{n}\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}.$$

Prove that :

(a) For all positive integers n, $0\le b_n\le 2$.

(b) Given an arbitrary $0\le b<2$, there is a sequence a_0, a_1, …, a_n, … of the above type such that $b_n>b$ is true for infinitely many natural numbers n.

Solution. (a) For all k, we have

$$\begin{aligned}0&\le\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}\\&=\frac{(\sqrt{a_k}+\sqrt{a_{k-1}})(\sqrt{a_k}-\sqrt{a_{k-1}})}{a_k\sqrt{a_k}}\\&\le 2\frac{\sqrt{a_k}-\sqrt{a_{k-1}}}{\sqrt{a_ka_{k-1}}}\\&=2\left(\frac{1}{\sqrt{a_{k-1}}}-\frac{1}{\sqrt{a_k}}\right).\end{aligned}$$

Then

$$\begin{aligned}0\le b_n&\le 2\sum_{k=1}^{n}\left(\frac{1}{\sqrt{a_{k-1}}}-\frac{1}{\sqrt{a_k}}\right)\\&=2\left(1-\frac{1}{\sqrt{a_n}}\right)<2.\end{aligned}$$

(b) Let $0<q<1$. Then $a_n=q^{-2n}$ for n = 0, 1, 2, … satisfy $1=a_0<a_1<a_2<\cdots$ and the sequence $b_n=q(1+q)(1-q^n)$ has

$$\lim_{n\to\infty}b_n=q(1+q).$$

For an arbitrary $0\le b<2$, take q satisfy

$$\frac{\sqrt{1+4b}-1}{2}<q<1.$$

Then $0<q<1$ and $q(1+q)>b$. So eventually the sequence b_n (on its way to $q(1+q)$) will be greater than b.

Example 7. (2006 *IMO Shortlisted Problem*) Prove the inequality

$$\sum_{1\le i<j\le n}\frac{a_ia_j}{a_i+a_j}\le\frac{n}{2(a_1+a_2+\cdots+a_n)}\sum_{1\le i<j\le n}a_ia_j$$

for positive real numbers a_1, a_2, …, a_n.

Solution. Let S be the sum of the n numbers. Let L and R be the left and the right expressions in the inequality. Observe that

$$\sum_{1\le i<j\le n}(a_i+a_j)=(n-1)\sum_{k=1}^{n}a_k=(n-1)S$$

and

$$\begin{aligned}L&=\sum_{1\le i<j\le n}\frac{a_ia_j}{a_i+a_j}\\&=\sum_{1\le i<j\le n}\frac{1}{4}\left(a_i+a_j-\frac{(a_i-a_j)^2}{a_i+a_j}\right)\\&=\frac{n-1}{4}S-\frac{1}{4}\sum_{1\le i<j\le n}\frac{(a_i-a_j)^2}{a_i+a_j}.\end{aligned}$$

Next we will write the expression R in two ways. On one hand, we have

$$R=\frac{n}{2S}\sum_{1\le i<j\le n}a_ia_j=\frac{n}{4S}\left(S^2-\sum_{i=1}^{n}a_i^2\right).$$

On the other hand,

$$\begin{aligned}R&=\frac{n}{4S}\sum_{1\le i<j\le n}\left(a_i^2+a_j^2-(a_i-a_j)^2\right)\\&=\frac{n(n-1)}{4S}\sum_{i=1}^{n}a_i^2-\frac{n}{4S}\sum_{1\le i<j\le n}(a_i-a_j)^2.\end{aligned}$$

Multiplying the first of these equations by $n-1$ and adding it to the second equation, then dividing the sum by n, we get

$$R=\frac{n-1}{4}S-\frac{1}{4}\sum_{1\le i<j\le n}\frac{(a_i-a_j)^2}{S}.$$

Comparing L and R and using $S\ge a_i+a_j$, we get $L\le R$.

Example 8. (1998 *IMO Longlisted Problem*) Let

$$a_n=[\sqrt{(1+n)^2+n^2}],\quad n=1,2,\ldots,$$

where $[x]$ denotes the integer part of x. Prove that

(a) there are infinitely many positive integers m such that $a_{m+1}-a_m>1$;

(b) there are infinitely many positive integers m such that $a_{m+1}-a_{m=1}=1$.

(continued on page 4)

Problem Corner

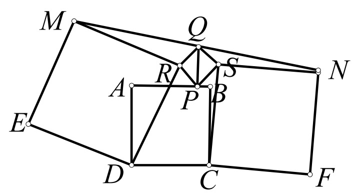
We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 14, 2016.***

Problem 486. Let a_0=1 and

$$a_n=\frac{\sqrt{1+a_{n-1}^2}-1}{a_{n-1}}.$$

for n=1,2,3,…. Prove that $2^{n+2}a_n>\pi$ for all positive integers n.

Problem 487. Let $ABCD$ and $PSQR$ be squares with point P on side AB and $AP>PB$. Let point Q be outside square $ABCD$ such that $AB\perp PQ$ and AB=2PQ. Let $DRME$ and $CSNF$ be squares as shown below. Prove Q is the midpoint of line segment MN.



Problem 488. Let $\mathbb{Q}$ denote the set of all rational numbers. Let f: $\mathbb{Q}\to\{0,1\}$ satisfy f(0)=0, f(1)=1 and the condition $f(x)=f(y)$ implies $f(x)=f((x+y)/2)$. Prove that if $x\geq 1$, then $f(x)=1$.

Problem 489. Determine all prime numbers p such that there exist positive integers m and n satisfying $p=m^2+n^2$ and m^3+n^3-4 is divisible by p.

Problem 490. For a parallelogram $ABCD$, it is known that $\triangle ABD$ is acute and AD=1. Prove that the unit circles with centers A, B, C, D cover $ABCD$ if and only if

$$AB\leq\cos\angle BAD+\sqrt{3}\sin\angle BAD.$$

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Solutions

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Problem 481. Let S={1,2,…,2016}. Determine the least positive integer n such that whenever there are n numbers in S satisfying every pair is relatively prime, then at least one of the n numbers is prime.

Solution. **BOBOJONOVA Latofat** (academic lycuem S.H.Sirojiddinov, Tashkent, Uzbekistan), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **Toshihiro SHIMIZU** (Kawasaki, Japan),**WONG Yat.**

Let k_0=1 and k_i be the square of the i-th prime number. Then $k_{14}=43^2<2016$. Since the numbers $k_0,k_1,\ldots,k_{14}$ are in S and are pairwise coprime, so $n\geq 16$.

Next suppose $A=\{a_1,a_2,\ldots,a_{16}\}\subset S$ with no a_i prime and a_r, a_s are coprime for $r\neq s$.

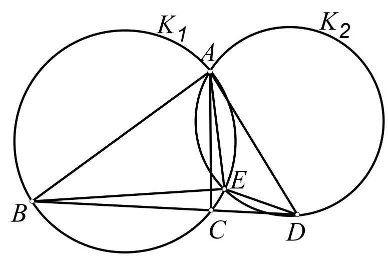
Then in case $1\notin A$, let p_i be the least prime divisor of a_i. We have $a_i\geq p_i^2$. As the a_i's are pairwise coprime, no two p_i's are the same. Now the 15th prime is 47. So the largest p_i is at least 47, which leads to some $a_i\geq p_i^2\geq 47^2>2016$, a contradiction.

Otherwise, $1\in A$. For the 15 numbers in A that is not 1, let a_i be their maximum, then $a_i\geq p_i^2\geq 47^2>2016$, again contradiction. So the least n is 16.

Other commended solvers: **Joe SPENCER.**

Problem 482. On $\triangle ABD$, C is a point on side BD with $C\neq B,D$. Let K_1 be the circumcircle of $\triangle ABC$. Line AD is tangent to K_1 at A. A circle K_2 passes through A and D and line BD is tangent to K_2 at D. Suppose K_1 and K_2 intersect at A and E with E inside $\triangle ACD$. Prove that $EB/EC=(AB/AC)^3$.

Solution. **Jafet Alejandro BACA OBANDO** (IDEAS High School, Nicaragua), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Vijaya Prasad NALLURI** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Line AD tangent to K_1 at A implies $\angle DAC=\angle DBA$. With $\angle ADC=\angle BDA$, we see $\triangle DAC$ is similar to $\triangle DBA$. Now BD/CD = $[DBA]/[DAC]$ = AB^2/CA^2. Then we have

$$\left(\frac{AB}{AC}\right)^3=\frac{BD}{CD}\cdot\frac{AB}{AC}=\frac{BD/AC}{CD/AB}.\quad (*)$$

Next, $\angle DBE=\angle CBE=\angle CAE$ and $\angle BDE=\angle DAE=\angle ACE$ implies $\triangle DBE$ is similar to $\triangle CAE$. Similarly, $\angle ECD=\angle EAB$ and $\angle EDC=\angle EAD=\angle EBA$ implies $\triangle ECD$ is similar to $\triangle EAB$. Then

$$\frac{BD/CA}{CD/AB}=\frac{EB/AE}{EC/EA}=\frac{EB}{EC}.\quad (**)$$

Therefore, combining (*) and (**), we have $EB/EC=(AB/AC)^3$.

Other commended solvers: **BOBOJONOVA Latofat** (academic lycuem S. H. Sirojiddinov, Tashkent, Uzbekistan) and **WONG Yat**.

Problem 483. In the open interval (0,1), n distinct rational numbers a_i/b_i (i=1,2,…,n) are chosen, where n>1 and a_i, b_i are positive integers. Prove that the sum of the b_i's are at least $(n/2)^{3/2}$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

Without loss of generality, we may suppose the numbers a_i/b_i are sorted so that the denominators are in ascending order. We have the following lemma.

<u>*Lemma.*</u> Let k be an integer in [1,n] and b be the denominator of the k-th number. Then we have

$$b\geq\left(\frac{k}{2}\right)^{3/2}-\left(\frac{k-1}{2}\right)^{3/2}.$$

<u>*Proof.*</u> We first consider the number of denominators that are at most b. For every i = 1, 2, …, b, the number of denominators equal to i is at most i−1. Thus,

$$k\leq\sum_{i=1}^{b}(i-1)=\frac{b(b-1)}{2}\leq\frac{b^2}{2}.$$

This implies $b\geq\sqrt{2k}$. We will show

$$\sqrt{2k}\geq\left(\frac{k}{2}\right)^{3/2}-\left(\frac{k-1}{2}\right)^{3/2}.$$

It is equivalent to

$$4\sqrt{k}\geq k\sqrt{k}-(k-1)\sqrt{k-1}$$

or $(k-1)\sqrt{k-1}\geq(k-4)\sqrt{k}$.

For k=1,2,3,4, the left hand side is greater than the right hand side is non-positive. For $k\geq 5$, squaring the inequality, it is equivalent to $(k-1)^3 \geq (k-4)^2 k$ or $5k^2-13k+1\geq 0$. The larger roots of the left hand side is $(13+\sqrt{149})/10$, which is less than 2.6. Then the left hand side is always positive for $k\geq 5$. QED

Using the lemma and summing the cases k=1, 2, …,n, we get the result.

Other commended solvers: **Jim GLIMMS, Joe SPENCER** and **WONG Yat**.

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices *A, B* and *C*. For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

More generally, suppose there are n problems with $n\geq 4$. Let S_n be the maximum number of students who took the test with n problems. If $S_1>3$, then there would exist 2 students with the same choice and 1 problem cannot distinguish these 2 students. Now S_1=3 is certainly possible by given condition. In general if there is a problem which 3 students have different choices, then we say the problem *distinguish* them.

By pigeonhole principle, for problem 1, there is a choice among *A, B, C*, which at most $[S_n/3]$ selected. For the remaining at least $S_n-[S_n/3]$ students, problem 1 does not distinguish any 3 of them. So problem 2 to n will be used to distinguish these remaining students. Then $S_{n-1}\geq S_n-[S_n/3] \geq 2S_n/3$. Hence, $S_n\leq 3S_{n-1}/2$. So $S_2\leq 4$, $S_3\leq 6$ and $S_4\leq 9$.

The following table will show S_4=9 :

Student\problem	I	II	III	IV
1	A	A	A	A
2	A	B	B	B
3	A	C	C	C
4	B	A	C	B
5	B	B	A	C
6	B	C	B	A
7	C	A	B	C
8	C	B	C	A
9	C	C	A	B

Other commended solvers: **Joe SPENCER**.

Problem 485. Let m and n be integers such that $m>n>1$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1, a_2, \ldots, a_n\}$ is a subset of S. It is known that every two numbers in T do not both divide any number in S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{n}.$$

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

For i=1,2,…,n, let

$$T_i=\{\,k\in S : k \text{ is divisible by } a_i\,\}.$$

Then T_i has $[m/a_i]$ elements. Since every pair of numbers in T do not both divide any number in S, so if $i\neq k$, then T_i and T_k are disjoint. Now the number of elements in the union of the sets $T_1, T_2, \ldots, T_n$ is

$$\left[\frac{m}{a_1}\right]+\left[\frac{m}{a_2}\right]+\cdots+\left[\frac{m}{a_n}\right]\leq m.$$

Using $m/a_i<[m/a_i]+1$, we have

$$\sum_{i=1}^{n}\frac{m}{a_i}\leq\sum_{i=1}^{n}\left[\frac{m}{a_i}\right]+\sum_{i=1}^{n}1\leq m+n.$$

Then $m\sum_{i=1}^{n}\frac{1}{a_i}<m+n.$ Therefore,

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{m}<\frac{m+n}{n}.$$

Other commended solvers: **Joe SPENCER**.

Olympiad Corner

(*Continued from page 1*)

Problem 3. (Continued) Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest integer k such that no matter how Starways establishes its flights, the city can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Problem 5. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$(z+1)f(x+y)=f(xf(z)+y)+f(yf(z)+x),$$

for all positive real numbers x, y, z.

Inequalities of Sequences

(*Continued from page 2*)

<u>Solution.</u> For every integer n, we have

$$\begin{aligned}\sqrt{2}n-3&<[\sqrt{2}(n-1)]\\ &<a_n=[\sqrt{2(n^2-n+1/2)}] \quad (*)\\ &\leq[\sqrt{2}n]<\sqrt{2}n.\end{aligned}$$

From this, we get

$$n^2+(n+1)^2-(n-1)^2-n^2=4n>2a_n+1.$$

Hence,

$$\begin{aligned}a_{n+1}&=[\sqrt{n^2+(n+1)^2}]\geq[\sqrt{a_n^2+4n}]\\ &\geq[\sqrt{a_n^2+2a_n+1}]=a_n+1\end{aligned}$$

for n=1,2,3,…. If (a) is false, then there exists N such that

$$a_{k+1}-a_k=1 \quad \text{for all } k\geq N. \qquad (**)$$

So $a_{N+k}=a_N+k$ for k=0,1,2,3,…. By (*), for k=0,1,2,3,…, we have

$$\sqrt{2}(N+k)-3<a_{N+k}=a_N+k,$$

i.e. $(\sqrt{2}-1)k<a_N+3-\sqrt{2}N$. Since N is constant, when k is large, this leads to a contradiction. So (a) must be true.

Next assume (b) is false. By (**), we can see there exists N such that

$$a_{k+1}-a_k\geq 2 \quad \text{for all } k\geq N.$$

Then $a_{N+k}\geq a_N+2k$ for k = 0,1,2,3,…. By (*), we have

$$a_N+2k<\sqrt{2}(N+k),$$

which is the same as

$$(2-\sqrt{2})k<\sqrt{2}N-a_N.$$

This leads to a contradiction when k is large. So (b) must be true.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *2016 IMO Team Selection Contest I* for Estonia.

Problem 1. There are k heaps on the table, each containing a different positive number of stones. Jüri and Mari make moves alternatively; Jüri starts. On each move, the player making the move has to pick a heap and remove one or more stones in it from the table; in addition, the player is allowed to distribute any number of the remaining stones from that heap in any way between other non-empty heaps. The player to remove the last stone from the table wins. For which positive integers k does Jüri have a winning strategy for any initial state that satisfies the conditions?

Problem 2. Let p be a prime number. Find all triples (a,b,c) of integers (not necessarily positive) such that

$$a^b b^c c^a = p.$$

Problem 3. Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ satisfying the equality $f(2^x+2y) = 2^y f(f(x))f(y)$ for every $x,y\in\mathbb{R}$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 21, 2016***.

For individual subscription for the next five issues for the 16-17 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2016

Kin Y. Li

This year Hong Kong served as the host of the International Mathematical Olympiad (IMO), which was held from July 6 to 16. Numerous records were set. Leaders, deputy leaders and contestants from 109 countries or regions participated in this annual event. A total of 602 contestants took part in this world class competition. Among the contestants, 71 were female and 531 were male.

After the two days of competition on July 11 and 12, near 700 contestants and guides from more than 100 countries or regions went to visit Mickey Mouse at the Hong Kong Disneyland for an excursion. That was perhaps the happiest moment in the IMO.

For Hong Kong, due to the hard work of the 6 team members and the strong coaching by Dr. Leung Tat Wing, Dr. Law Ka Ho and our deputy leader Cesar Jose Alaban along with the support of the many trainers and former team members, the team received 3 gold, 2 silver and 1 bronze medals, which was the best performance ever. Also, for the first time since Hong Kong participated in the IMO, we received a top 10 team ranking.

The Hong Kong IMO team members (in alphabetical order) are as follows:

(HKG1) Cheung Wai Lam, Queen Elizabeth School, Silver Medalist,

(HKG2) Kwok Man Yi, Baptist Lui Ming Choi Secondary School, Bronze Medalist,

(HKG3) Lee Shun Ming Samuel, CNEC Christian College, Gold Medalist,

(HKG4) Leung Yui Hin Arvin, Diocesan Boys' School, Silver Medalist,

(HKG5) Wu John Michael, Hong Kong International School, Gold Medalist and

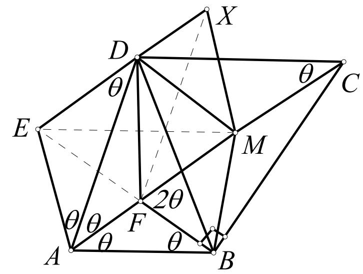
(HKG 6) Yu Hoi Wai, La Salle College, Gold Medalist.

The top 10 teams in IMO 2016 are (1) USA, (2) South Korea, (3) China, (4) Singapore, (5) Taiwan, (6) North Korea, (7) Russia and UK, (9) Hong Kong and (10) Japan.

The cutoffs for gold, silver and bronze medals were 29, 22 and 16 marks respectively. There were 44 gold, 101 silver, 135 bronze and 162 honourable mentions awardees.

Next, we will look at the problems in IMO 2016.

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA=FB$ and F lies between A and C. Point D is chosen such that $DA=DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA=ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM||EX$ and $AE||MX$). Prove that lines BD, FX, and ME are concurrent.



From the statement of the problem, we get a whole bunch of equal angles as labeled in the figure. We have $\triangle ABF\sim\triangle ACD$. Then $AB/AC=AF/AD$. With, $\angle BAC=\theta=\angle FAD$, we get $\triangle ABC\sim\triangle AFD$.

(continued on page 2)

Then $\angle AFD = \angle ABC = 90^\circ + \theta = 180^\circ - \frac{1}{2}\angle AED$. Hence, F is on the circle with center E and radius EA. Then $EF=EA=ED$ and $\angle EFA = \angle EAF = 2\theta = \angle BFC$. So B, F, E are collinear. Also, $\angle EDA = \angle MAD$ implies $ED||AM$. Hence E,D,X are collinear. From M is midpoint of CF and $\angle CBF=90^\circ$, we get $MF=MB$. Next the isosceles triangles EFA and MFB are congruent due to $\angle EFA=\angle MFB$ and $AF=BF$. Then $BM=AE=XM$ and $BE = BF+FE=AF+FM=AM=EX$. So $\triangle EMB \cong \triangle EMX$. As F and D lie on EB and EX respectively and $EF=ED$, we see lines BD and XF are symmetric respect to EM. Therefore, BD, XF, EM are concurrent.

Problem 2. Find all positive integer n for which each cell of an $n\times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note: The rows and columns of an $n\times n$ table are each labeled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1\le i, j\le n$. For $n>1$, the table has $4n-2$ *diagonals* of two types. A diagonal of the first type consists of all cells (i, j) for which $i+j$ is a constant, and a diagonal of the second type consists of all cells (i,j) for which $i-j$ is a constant.

For $n=9$, it is not difficult to get an example such as

$$\begin{pmatrix} I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \end{pmatrix}$$

For $n=9m$, we can divide the $n\times n$ table into $m\times m$ blocks, where in each block we use the 9×9 table above.

Next suppose a $n\times n$ table satisfies the conditions. Then n is a multiple of 3, say $n=3k$. Divide the $n\times n$ into $k\times k$ blocks of 3×3 tables. Call the center entry of the 3×3 tables a *vital entry* and call any row, column or diagonal passing through a vital entry a *vital line*. The trick here is to do double counting on the number N of all ordered pairs (L,c), where L is a vital line and c is an entry on L that contains the letter M. On one hand, there are k occurrences of M in each vital row and each vital column. For vital diagonals, there are

$$1+2+\cdots+(k-1)+k+(k-1)+\cdots+2+1=k^2$$

occurrences of M. So $N=4k^2$. On the other hand, there are $3k^2$ occurrences of M in the whole table. Note each entry belongs to exactly 1 or 4 vital lines. Hence $N\equiv 3k^2 \pmod 3$, making k a multiple of 3 and n a multiple of 9.

Problem 3. Let $P=A_1A_2\ldots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

This is the hardest problem. 548 out of 602 contestants got 0 on this problem.

That $2S$ is an integer follows from the well-known *Pick's formula*, which asserts $S=I+B/2-1$, where I and B are the numbers of interior and boundary points with integral coordinates respectively.

Below we will outline the cleverest solution due to Dan Carmon, the leader of Israel. It suffices to consider the case $n=p^t$ with p prime, $t\ge 1$. By multiplying the denominator and translating, we may assume the center O is a point with integral coordinates, which we can move to the origin. We can further assume the x, y coordinates of the vertices are coprime and there exists i with x_i, y_i not both multiples of p. Then we make two claims:

(1) For $\triangle ABC$ with integral coordinates, suppose $n \mid AB^2, BC^2$ and let S be its area. Then $n \mid 2S$ if and only if $n \mid AC^2$.

(2) For those i such that x_i, y_i not both multiples of p, let Δ be twice the area of triangle $A_{i-1}A_iA_{i+1}$. Then p^t divides Δ.

For (1), note that $2S = \left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$,

$$AC^2 = AB^2+BC^2-2\overrightarrow{BA}\cdot\overrightarrow{BC}$$
$$\equiv -2\overrightarrow{BA}\cdot\overrightarrow{BC} \pmod n$$

and $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|^2+\left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|^2 = AB^2BC^2 \equiv 0 \pmod{n^2}$.

For (2), assume p^t does not divide Δ. Note O is defined by the intersection of the perpendicular bisectors, which can be written as the following system of vectors:

$$\overrightarrow{A_iA_{i+1}}\cdot\overrightarrow{A_iO}=\tfrac{1}{2}A_iA_{i+1}^2,\quad \overrightarrow{A_iA_{i-1}}\cdot\overrightarrow{A_iO}=\tfrac{1}{2}A_iA_{i-1}^2.$$

Say $\overrightarrow{A_iA_{i+1}}=(u_1,v_1)$, $\overrightarrow{A_iA_{i-1}}=(u_2,v_2)$. Using the fact that p^t does not divide $\Delta = |u_1v_2-u_2v_1|$, one can conclude that x_i, y_i are divisible by p by Cramer's rule. The rest of the solution follows by induction on the number of sides of the polygon and the two claims.

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n)=n^2+n+1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

One can begin by looking at facts like

1. $\gcd(P(n),P(n+1))=1$ for all n
2. $\gcd(P(n),P(n+2))=1$ for $n\not\equiv 2 \pmod 7$
3. $\gcd(P(n),P(n+2))=7$ for $n\equiv 2 \pmod 7$
4. $\gcd(P(n),P(n+3))=1$ for $n\not\equiv 1 \pmod 3$
5. $3|\gcd(P(n),P(n+3))$ for $n\equiv 1 \pmod 3$.

Assume $P(a)$, $P(a+1)$, $P(a+2)$, $P(a+3)$, $P(a+4)$ is fragrant. By 1, $P(a+2)$ is coprime to $P(a+1)$ and $P(a+3)$. Next assume $\gcd(P(a),P(a+2))>1$. By 3, $a\equiv 2 \pmod 7$. By 2, $\gcd(P(a+1),P(a+3))=1$. In order for the set to be fragrant, we must have both $\gcd(P(a),P(a+3))$ and $\gcd(P(a+1),P(a+4))$ be greater than 1. By 5, this holds only when a and $a+1\equiv 1 \pmod 3$, which is a contradiction.

For a fragrant set with 6 numbers, we can use the Chinese remainder theorem to solve the system $a\equiv 7 \pmod{19}$, $a+1\equiv 2 \pmod 7$ and $a+2\equiv 1 \pmod 3$. For example, $a=197$. By 3, $P(a+1)$ and $P(a+3)$ are divisible by 7. By 5, $P(a+2)$ and $P(a+5)$ are divisible by 3. Using $19|P(7)=57$ and $19|P(11)=133$, we can check $19|P(a)$ and $19|P(a+4)$. Then $P(a)$, $P(a+1)$, $P(a+2)$, $P(a+3)$, $P(a+4)$, $P(a+5)$ is fragrant.

Problem 5. The equation

$$(x-1)(x-2)\cdots(x-2016)$$
$$=(x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 21, 2016.***

Problem 491. Is there a prime number p such that both p^3+2008 and p^3+2010 are prime numbers? Provide a proof.

Problem 492. In convex quadrilateral $ADBE$, there is a point C within $\triangle ABE$ such that

$\angle EAD+\angle CAB=180°=\angle EBD+\angle CBA$.

Prove that $\angle ADE=\angle BDC$.

Problem 493. For $n \geq 4$, prove that $x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ cannot be factored into a product of two polynomials with rational coefficients, both with degree greater than 1.

Problem 494. In a regular n-sided polygon, either 0 or 1 is written at each vertex. By using non-intersecting diagonals, Bob divides this polygon into triangles. Then he writes the sum of the numbers at the vertices of each of these triangles inside the triangle. Prove that Bob can choose the diagonals in such a way that the maximal and minimal numbers written in the triangles differ by at most 1.

Problem 495. The lengths of each side and diagonal of a convex polygon are rational. After all the diagonals are drawn, the interior of the polygon is partitioned into many smaller convex polygonal regions. Prove that the sides of each of these smaller convex polygons are rational numbers.

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Solutions

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Problem 486. Let $a_0=1$ and

$$a_n=\frac{\sqrt{1+a_{n-1}^2}-1}{a_{n-1}}.$$

for n=1,2,3,…. Prove that $2^{n+2}a_n>\pi$ for all positive integers n.

Solution. **Charles BURNETTE** (Graduate Student, Drexel University, Philadelphia, PA, USA), **Prithwijit DE** (HBCSE, Mumbai, India), **FONG Ho Leung** (Hoi Ping Chamber Secondary School), **Mustafa KHALIL** (Instituto Superior Tecnico, Syria), **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania), **Toshihiro SHIMIZU** (Kawasaki, Japan), **WONG Yat** and **YE Jeff York**, **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

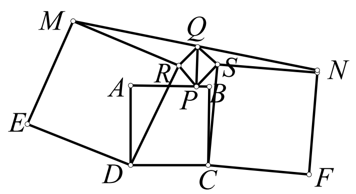
Let $a_n=\tan\theta_n$, where $0 \leq \theta_n<\pi/2$. Then $a_0=1$ implies $\theta_0=\pi/4$. By the recurrence relation of a_n, we get

$$\tan\theta_n=\frac{\sec\theta_{n-1}-1}{\tan\theta_{n-1}}=\frac{1-\cos\theta_{n-1}}{\sin\theta_{n-1}}$$
$$=\frac{2\sin^2(\theta_{n-1}/2)}{2\cos(\theta_{n-1}/2)\sin(\theta_{n-1}/2)}=\tan\frac{\theta_{n-1}}{2}.$$

Then $a_n=\tan\theta_n=\tan\frac{\theta_0}{2^n}=\tan\frac{\pi}{2^{n+2}}>\frac{\pi}{2^{n+2}}$,

which is the desired inequality.

Problem 487. Let $ABCD$ and $PSQR$ be squares with point P on side AB and $AP>PB$. Let point Q be outside square $ABCD$ such that $AB\perp PQ$ and $AB=2PQ$. Let $DRME$ and $CSNF$ be squares as shown below. Prove Q is the midpoint of line segment MN.



Solution. **FONG Ho Leung** (Hoi Ping Chamber Secondary School), **Tran My LE** (Sai Gon University, Ho Chi Minh City, Vietnam) and **Duy Quan TRAN** (University of Medicine and Pharmacy, Ho Chi Minh City, Vietnam), **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Mihai STOENESCU** (Bischwiller, France), **WONG Yat** and **YE Jeff York**.

Let Q be the origin, P be $(0,-2)$ and $B=(x,-2)$. Since $AB\perp PQ$ and $PSQR$ is a square, so $S=(1,-1)$. Using $AB=2PQ=4$, we get $C=(x,-6)$. Since $CS=NS$ and $\angle CSN=90°$, we get $N=(6,2-x)$.

Similarly, $R=(-1,-1)$, $D=(x-4,-6)$ and $\angle DRM=90°$, so $M=(-6, x-2)$. Then the midpoint of MN is $(0,0)=Q$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **Apostolos MANOLOUDIS** (4 High School of Korydallos, Piraeus, Greece) and **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India).

Problem 488. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f\colon \mathbb{Q}\to\{0,1\}$ satisfy $f(0)=0$, $f(1)=1$ and the condition $f(x)=f(y)$ implies $f(x)=f((x+y)/2)$. Prove that if $x\geq 1$, then $f(x)=1$.

Solution. **Jon GLIMMS.**

We first show $f(n)=1$ for n=1,2,3,… by induction. The case $n=1$ is given. For $n>1$, suppose case $n=k-1$ is true. If $f(k)=0=f(0)$, then $f(k)=f((0+k)/2)=f((1+(k-1))/2)=f(k-1)=1$, which is a contradiction.

Assume there exists rational $r>1$ such that $f(r)=0$. Suppose $r=s/t$, where s, t are coprime positive integers. Define $g\colon\mathbb{Q}\to\{0,1\}$ by $g(x)=1-f(w(x))$, where $w(x)=(r-[r])x+[r]$. Observe that the graph of w is a line. So $w((x+y)/2)=(w(x)+w(y))/2$.

If $g(x)=g(y)$, then $f(w(x))=f(w(y))$, which implies

$$f(w(x))=f\left(\frac{w(x)+w(y)}{2}\right)=f\left(w\left(\frac{x+y}{2}\right)\right).$$

So $g(x)=g((x+y)/2)$. Then $g(n)=1$ by induction as f above. Finally, $s>t$ implies $w(t)=(r-[r])t+[r]=s-[r]t+[r]$ is a positive integer. Then $g(t)=1-f(w(t))=0$, contradiction.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan), **WONG Yat** and **YE Jeff York**,

Problem 489. Determine all prime numbers p such that there exist positive integers m and n satisfying $p=m^2+n^2$ and m^3+n^3-4 is divisible by p.

Solution. **Prithwijit DE** (HBCSE, Mumbai, India), **Jon GLIMMS, WONG Yat** and **YE Jeff York**.

Clearly, the case $p=2$ works. For such prime $p>2$, we get $m>1$ or $n>1$. Now we have

$$\begin{aligned}&(3m+3n)p-2(m^3+n^3-4)\\&=(m+n)^3+8\\&=(m+n+2)((m+n)^2-2(m+n)+4)\\&=(m+n+2)(p+2((m-1)(n-1)+1)).\end{aligned}$$

Observe that $p<p+2((m-1)(n-1)+1)<p+2mn\leq p+m^2+n^2=2p$. Then p divides $m+n+2$. So $m^2+n^2\leq m+n+2$, i.e. $(m-1/2)^2+(n-1/2)^2\leq(3/2)^2$. Then

(m,n)=(1,2) or (2,1) and $m^3+n^3-4=5=p$. So $p=2$ and 5 are the solutions.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 490. For a parallelogram *ABCD*, it is known that $\triangle ABD$ is acute and AD=1. Prove that the unit circles with centers *A, B, C, D* cover *ABCD* if and only if

$$AB \le \cos\angle BAD + \sqrt{3}\sin\angle BAD.$$

Solution. **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We first show that the unit circles with centers *A, B, C, D* cover *ABCD* if and only if the circumradius *R* of $\triangle ABD$ is not greater than 1. Since $\triangle ABD$ is acute, its circumcenter *O* is inside the triangle. Then at least one of *B* or *D* is closer than (or equal to) *C* to *O*, since the region in $\triangle CDB$ that is closer to *C* than both *B* and *D* is the quadrilateral *CMO'N*, where *M* is the midpoint of *CD*, *O'* is the circumcenter of $\triangle CDB$ and *N* is the midpoint of *BC*. So for any point *P* in $\triangle ABD$, $\min\{PA,PB,PD\} \le PC$ and the maximal value of $\min\{PA,PB,PD\}$ is attained when $P=O$. So the unit circles with centers *A, B, C, D* cover *ABCD* is equivalent to they cover *O*, which is equivalent to $R\le 1$.

Let $\alpha=\angle BAD$, $\beta=\angle ADB$ and $\gamma=\angle DBA$. By sine law, $AB/\sin\beta=1/\sin\gamma=2R$. Then, we have

$$AB=\frac{\sin\beta}{\sin\gamma}=\frac{\sin(\alpha+\gamma)}{\sin\gamma}$$
$$=\frac{\sin\alpha\cos\gamma+\cos\alpha\sin\gamma}{\sin\gamma}$$
$$=\cos\alpha+\cot\gamma\sin\alpha.$$

Moreover, $R\le 1$ is equivalent to $1\ge 1/(2\sin\gamma)$ or $\sin\gamma\ge 1/2=\sin 30^\circ$ or $\gamma\ge 30^\circ$ or $\cot\gamma\le\sqrt{3}$. Therefore, it is equivalent to $AB\le\cos\alpha+\sqrt{3}\sin\alpha$.

Other commended solvers: **WONG Yat** and **YE Jeff York**.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Prove that for any positive integer n, $2\cdot\sqrt{3}\cdot\sqrt[3]{4}\cdots\sqrt[n-1]{n}>n$.

Problem 5. Let *O* be the circumcenter of the acute triangle *ABC*. Let c_1 and c_2 be the circumcircles of triangles *ABO* and *ACO*. Let *P* and *Q* be points on c_1 and c_2 respectively, such that *OP* is a diameter of c_1 and *OQ* is a diameter of c_2. Let *T* be the intersection of the tangent to c_1 at *P* and the tangent to c_2 at *Q*. Let *D* be the second intersection of the line *AC* and the circle c_1. Prove that points *D, O* and *T* are collinear.

Problem 6. A circle is divided into arcs of equal size by n points ($n\ge 1$). For any positive integer x, let $P_n(x)$ denote the number of possibilities for coloring all those points, using colors from x given colors, so that any rotation of the coloring by $i\cdot 360^\circ/n$, where i is a positive integer less than n, gives a coloring that differs from the original in at least one point. Prove that the function $P_n(x)$ is a polynomial with respect to x.

IMO 2016

(*Continued from page 2*)

For this problem, observe we need to erase at least 2016 factors. Consider erasing all factors $x-k$ with $k\equiv 2,3 \pmod 4$ on the left and $x-k$ with $k\equiv 0,1 \pmod 4$ on the right to get the equation

$$\prod_{j=0}^{503}(x-4j-1)(x-4j-4)=\prod_{j=0}^{503}(x-4j-2)(x-4j-3)$$

There are 4 cases we have to check.

(1) For $x=1,2,\cdots,2016$, one side is 0 and the other nonzero.

(2) For $x\in(4k+1,4k+2)\cup(4k+3,4k+4)$ where $k=0,1,\ldots,503$, if $j=0,1,\ldots,503$ and $j\ne k$, then $(x-4j-1)(x-4j-4)>0$, but if $j=k$, then $(x-4k-1)(x-4k-4)<0$ so that the left side is negative. However, on the right side, each product $(x-4j-2)(x-4j-3)$ is positive, which is a contradiction.

(3) For $x<1$ or $x>2016$ or $x\in(4k,4k+1)$, where $k=0,1,\ldots,503$, dividing the left side by the right, we get

$$1=\prod_{j=0}^{503}\left(1-\frac{2}{(x-4j-2)(x-4j-3)}\right).$$

Note $(x-4j-2)(x-4j-3)>2$ for $j=0,1,\ldots,503$. Then the right side is less than 1, contradiction.

(4) For $x\in(4k+2,4k+3)$, where $k=0,1,\ldots,503$, dividing the left side by the right, we get

$$1=\frac{x-1}{x-2}\frac{x-2016}{x-2015}\prod_{j=1}^{503}\left(1+\frac{2}{(x-4j+1)(x-4j-2)}\right).$$

The first two factors on the right are greater than 1 and the factor in the parenthesis is greater than 1, which is a contradiction.

Problem 6. There are $n>2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hand $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if n is odd.

(b) Prove that Geoff can never fulfill his wish if n is even.

Unlike previous years, this problem 6 was not as hard as problem 3. There were 474 out of 602 contestants, who got 0 on this problem.

Take a disk containing all segments. Extend each segment to cut the boundary of the disk at points A_i, B_i.

(a) For odd n, go along the boundary and mark all these points 'in' and 'out' alternately. For each A_iB_i rename the 'in' point as A_i and 'out' point as B_i. Geoff can put a frog on each of the 'in' points. Let $A_iB_i\cap A_kB_k=P$. There are $n-1$ points on the open segment A_iB_i for every i. On the *open* arc A_iA_k, there is an odd number of points due to the alternate naming of the boundary points. Each of the points on *open* arc A_iA_k is a vertex of some A_xB_x, which intersects a unique point on either open segment A_iP or A_kP. So the number of points on open segments A_iP and A_kP are of opposite parity. Then the frogs started at A_i and A_k cannot meet at *P*.

(b) For even n, let Geoff put a frog on a vertex of a A_iB_i segment, say the frog is at A_i, which is the 'in' point and B_i is the 'out' point. As n is even, there will be two neighboring points labeled A_i and A_k. Let $A_iB_i\cap A_kB_k=P$. Then any other segment A_mB_m intersecting one of the open segments A_iP or A_kP must intersect the other as well. So the number of intersection points by the other segments on open segments A_iP and A_kP are the same. Then the frogs started at A_i and A_k will meet at *P*.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Final Round of the 65th Czech and Slovak Math Olympiad (April 4-5, 2016).

Problem 1. Let $p>3$ be a prime. Find the number of ordered sextuples (a,b,c,d,e,f) of positive integers, whose sum is $3p$, and all the fractions

$$\frac{a+b}{c+d},\frac{b+c}{d+e},\frac{c+d}{e+f},\frac{d+e}{f+a},\frac{e+f}{a+b}$$

are integers.

Problem 2. Let r and r_a be the radii of the inscribed circle and excircle opposite A of the triangle ABC. Show that if $r+r_a=|BC|$, then the triangle is right-angled.

Problem 3. Mathematics clubs are very popular in certain city. Any two of them have at least one common member. Prove that one can distribute rulers and compasses to the citizens in such a way that only one citizen get both (compass and ruler) and any club has to his disposal both, compass and ruler, from its members.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 15, 2017***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Miscellaneous Problems

Kin Y. Li

There are many Math Olympiad problems. Some are standard problems in algebra or in geometry or in number theory or in combinatorics, where there are some techniques for solving them. Then, there are problems that are not so standard, which cross two or more categories. In math problem books, they go under the category of miscellaneous problems. Some of these may arise due to curiosity. Then one may need to combine different facts to explain them. Below are some such problems we hope the readers will enjoy.

Example 1 (1995 USA Math Olympiad). A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational numbers q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculation with infinite precision. All functions are in terms of radians.

Solution. We will show that all numbers of the form $\sqrt{m/n}$, where m, n are positive integers, can be displayed by doing induction on $k=m+n$. (Since $r/s = \sqrt{r^2/s^2}$, these include all positive rational numbers.)

For k=2, pressing cos will display 1. Suppose the statement is true for integer less than k. Observe that if x is displayed, then letting $\theta=\tan^{-1}x$, we see

$$\cos^{-1}(\sin x)=\frac{\pi}{2}-\theta \quad and \quad \tan\left(\frac{\pi}{2}-\theta\right)=\frac{1}{x}.$$

So we can display $1/x=\tan(\cos^{-1}(\sin x))$. Therefore, to display $\sqrt{m/n}$ with $k=m+n$, we may assume $m<n$. By the induction step, since $(n-m)+m = n < k$, $\sqrt{(n-m)/m}$ can be displayed. Then using

$$\phi=\tan^{-1}\sqrt{(n-m)/m} \quad and \quad \cos\phi=\sqrt{m/n},$$

we can display $\sqrt{m/n}$. This completes the induction.

Example 2 (1986 Brazilian Math Olympiad). A ball moves endlessly on a circular billiard table. When it hits the edge it is reflected. Show that if it passes through a point on the table three times, then it passes through it infinitely many times.

Solution. Suppose AB and BC are two successive chords of the ball's path. By the reflection law, $\angle ABO = \angle OBC$. Now $\triangle OAB$ and $\triangle OBC$ are isosceles. So $\angle AOB = \angle BOC$. Hence, $AB = BC$. Then every chord of the path has the same length d.

We now claim that through any given point P inside the circle there are at most two chords with length d. Let AB and CD be a chord containing P, with $AP=a$ and $CP=b$. The power of P with respect to the circle is $PA\cdot PB=PC\cdot PD$, which is $a(d-a)=b(d-b)$. Hence, $a=b$ or $a+b=d$. This means that P always divides the chord containing it in two segments of fixed lengths a and $d-a$. Now if three chords passes through P, the circle with center P and radius a would cut the circle of the billiard table three times, a contradiction.

Thus if the path passes through P more than twice, then on two occasions it must be moving along the same chord AB. That implies $\angle AOB$ is a rational multiple of 2π and hence the path will traverse AB repeatedly.

Example 3. Is there a way to pack 250 $1\times1\times4$ bricks into a $10\times10\times10$ box?

Solution. Assign coordinate (x,y,z) to each of the cells, where $x,y,z= 0,1,\ldots,9$. Let the cell (x,y,z) be given color $x+y+z$ (mod 4). Note each $1\times1\times4$ brick contain all 4 colors exactly once. If the packing is possible, then there are exactly 250 cells of each color. However, a direct counting shows there are 251 cells of color 0, a contradiction. So such packing is impossible.

(continued on page 2)

Example 4 (2013 Singapore Math Olympiad). Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that every two musicians each must play for the other in some concert?

Solution. Let the musicians be A,B,C,D,E,F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A,B,C\}$, $\{A,D,E\}$, $\{B,D,F\}$ and $\{C,E,F\}$ satisfy the requirement. We shall now prove that three concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also played for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C,D,E,F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.

Example 5 (1999 Brazilian Math Olympiad). Prove that there is at least one nonzero digit between the 1,000,000$^{\text{th}}$ and the 3,000,000$^{\text{th}}$ decimal digits of $\sqrt{2}$.

Solution. Let us suppose that all digits between the 1,000,000$^{\text{th}}$ and the 3,000,000$^{\text{th}}$ decimal digits of $\sqrt{2}$ are zeros. Then

$$\sqrt{2}=\frac{n}{10^{1,000,000}}+\varepsilon, \qquad (*)$$

where n is a positive integer and $\varepsilon > 0$ satisfy

$$n<2\cdot 10^{1,000,000}\ and\quad \varepsilon<(10^{-3})^{10^{1,000,000}}.$$

By squaring (*), we can get

$$\begin{aligned}&2\cdot 10^{2,000,000}-n^2\\ &=2n\varepsilon 10^{1,000,000}+\varepsilon^2 10^{2.000.000}.\end{aligned}$$

However, the left side is a positive integer and the right side is less than 1, which is a contradiction.

Example 6 (1995 Russian Math Olympiad). Is it possible to fill in the cells of a 9×9 table with positive integers ranging from 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

Solution. Place 0,1,2,3,4,5,6,7,8 on the first, fourth and seventh rows. Place 3,4,5,6,7,8,0,1,2 on the second, fifth and eigth rows. Place 6,7,8,0,1,2,3,4,5 on the third, sixth and ninth rows. Then every 3×3 square contains 0 to 8. Consider this table and its 90° rotation. For each cell, fill it with the number $9a+b+1$, where a is the number in the cell originally and b is the number in the cell after the table is rotated by 90°. By inspection, 1 to 81 appears exactly once and every 3×3 square has sum 9×36+36+9=369.

Example 7. Can the positive integers be partitioned into infinitely many subsets such that each subset is obtained from any other subset by adding the same integer to each element of the other subset?

Solution. Yes. Let A be the set of all positive integers whose odd digit positions (from the right) are zeros. Let B be the set of all positive integers whose even digit positions (from the right) are zeros. Then A and B are infinite set and the set of all positive integers is the union of $a+B=\{a+b\colon\ b\in B\}$ as a range over the element of A. (For example, 12345 = 2040+10305 ∈ 2040+B.)

Example 8 (2015 IMO Shortlisted Problem proposed by Estonia). In Lineland there are $n\geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so if a bulldozers reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep* town B *away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of the towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

Solution. Let $T_1, T_2,\ldots,T_n$ be the towns enumerated from left to right. Observe first that, if town T_a can sweep away town T_b, then T_a also can sweep away every town located between T_a and T_b.

We prove by induction on n. The case n=1 is trivial. For the induction step, we first observe that the left bulldozer in T_1 and the right bulldozer in T_n are completely useless, so we may forget them forever. Among the other $2n-2$ bulldozers, we choose the largest one. Without loss of generality, it is the right bulldozer of some town T_k with $k<n$.

Surely, with this right bulldozer T_k can sweep away all towns to the right of it. Moreover, none of these towns can sweep T_k away; so they also cannot sweep away any town to the left of T_k. Thus, if we remove the towns T_{k+1}, $T_{k+2},\ldots,T_n$, none of the remaining towns would change its status of being (un)sweepable away by the others.

Applying the induction hypothesis to the remaining towns, we find a unique town among $T_1,T_2,\ldots,T_k$ which cannot be swept away. By the above reasons, it is also the unique such town in the initial situation. Thus the inductive step is established.

Example 9 (1991 Brazilian Math Olympiad). At a party every woman dances with at least one man, and no man dances with every woman. Show that there are men M and M' and women W and W' such that M dances with W, M' dances with W', but M does not dance with W', and M' does not dance with W.

Solution. Let M be one of the men who dance with the maximal number of women, W' one of the women he doesn't dance with, and M' one of the men W' dances with. If M' were to dance with every woman that M dances with, then the maximality of the number of women that M dances with would be contradicted, so there is a woman W that dances with M but not with M'.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 15, 2017.***

Problem 496. Let a,b,c,d be real numbers such that a+sin $b > c$+sin d, b+sin $a > d$+sin c. Prove that $a+b>c+d$.

Problem 497. Let there be three line segments with lengths 1, 2, 3. Let the segment of length 3 be cut into $n\geq 2$ line segments. Prove that among these $n+2$ segments, there exist three of them that can be put to form a triangle where each side is one of the three segments.

Problem 498. Determine all integers $n>2$ with the property that there exists one of the numbers $1,2,\ldots,n+1$ such that after its removal, the n numbers left can be arranged as $a_1,a_2,\ldots,a_n$ with no two of $|a_1-a_2|$, $|a_2-a_3|$, …, $|a_{n-1}-a_n|$, $|a_n-a_1|$ being equal.

Problem 499. Let ABC be a triangle with circumcenter O and incenter I. Let Γ be the escribed circle of $\triangle ABC$ meeting side BC at L. Let line AB meet Γ at M and line AC meet Γ at N. If the midpoint of line segment MN lies on the circumcircle of $\triangle ABC$, then prove that points O, I, L are collinear.

Problem 500. Determine all positive integers n such that there exist $k\geq 2$ positive rational numbers such that the sum and the product of these k numbers are both equal to n.

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Solutions

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Problem 491. Is there a prime number p such that both p^3+2008 and p^3+2010 are prime numbers? Provide a proof.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Prithwijit DE** (HBCSE, Mumbai, India), **EVGENIDIS Nikolaos** (M. N. Raptou High School, Palaiokastrou 10, Agia, Greece), **Karaganda** (Nazarbaev iIntellectual School, Nurligenov Temirlan - 9 grade student), **Koopa KOO**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S6), **Mark LAU**, **Toshihiro SHIMIZU** (Kawasaki, Japan), **Anderson TORRES**, **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

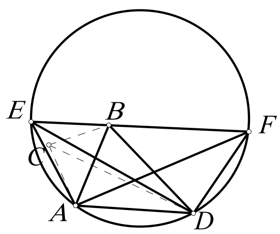
Let p be a prime. If $p\neq 7$, then $p^3\equiv -1$ or 1 (mod 7). Since $2008 \equiv -1$ (mod 7) and $2010\equiv 1$ (mod 7), so either p^3+2008 or p^3+2010 is divisible by 7, hence composite. If $p = 7$, then $p^3+2010 = 2353 = 13\times 181$ is composite. Therefore, there is no such prime.

Problem 492. In convex quadrilateral $ADBE$, there is a point C within $\triangle ABE$ such that

$$\angle EAD+\angle CAB=180^\circ=\angle EBD+\angle CBA.$$

Prove that $\angle ADE=\angle BDC$.

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S6).



Let F be the second intersection of the circumcircle of $\triangle EAD$ and line EB. Then $\angle DBF=180^\circ-\angle EBD=\angle CBA$. Moreover,

$$\begin{aligned}\angle BDF &= 180^\circ-\angle AEB-\angle ADB\\ &= 180^\circ-(360^\circ-\angle EAD-\angle EBD)\\ &= 180^\circ-(\angle CAB+\angle CBA) =\angle BCA.\end{aligned}$$

These two relations give $\triangle BDF\sim\triangle BCA$. So $BD/BF=BC/BA$. Together with $\angle DBF=\angle CBA$, we have $\triangle BDC\sim\triangle BFA$. Then $\angle ADE=\angle AFE=\angle BFA=\angle BDC$.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 493. For $n \geq 4$, prove that $x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ cannot be factored into a product of two polynomials with rational coefficients, both with degree greater than 1.

Solution. **Prithwijit DE** (HBCSE, Mumbai, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let $P_n(x) = x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ and $Q_n(x)= (x-1)P_n(x)= x^{n+1}-2x^n+1$. The cases $n = 2$ or 3 follow directly from the rational root theorem. For $n\geq 4$, the Descartes' rule of signs shows there is a positive root r. It is easy to check $P_n(\sqrt{3}) < 0$. So $r > \sqrt{3}$.

If $P_n(s)=0$ with $|s|>1$, then $Q_n(s)=0$, which implies $|s|^n\,|s-2|=1$. We get $2\leq|s-2|+|s| = |s|^{-n}+|s|$. So $Q_n(|s|) \geq 0$. Since $Q_n(x)<0$ for $1<x<r$, we must have $|s|\geq r$. On the other hand, if $P_n(t)=0$ and $|t|<1$, then $1=|t-2||t|^n \leq 3|t|^n$. It follows that the absolute value of the product of all roots t of $P_n(x)$ with $|t|<1$ is at least 1/3. So r is the only root of $P_n(x)$ with absolute value greater than 1.

Assume $P_n(x)=f(x)g(x)$, where $f(x)$, $g(x)$ are monic polynomials with integer coefficients and $f(r)=0$. Then if $g(x)$ has positive degree, its roots would have absolute value less than 1 and so $|g(0)|<1$. This contradicts the constant term of $g(x)$, being $g(0)$, must be ± 1.

Other commended solvers: **Anderson TORRES**.

Problem 494. In a regular n-sided polygon, either 0 or 1 is written at each vertex. By using non-intersecting diagonals, Bob divides this polygon into triangles. Then he writes the sum of the numbers at the vertices of each of these triangles inside the triangle. Prove that Bob can choose the diagonals in such a way that the maximal and minimal numbers written in the triangles differ by at most 1.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If all numbers written at the vertices of the polygon are equal, then the claim holds trivially. Hence assume that there are both zeros and ones among the numbers at the vertices. We prove by induction that, for every convex polygon, the partition into triangles can be chosen in such a way that Bob writes either 1 or 2 to each triangle.

If n=3, then this claim holds since the sum of the numbers at the vertices of a triangle can be neither 0 nor 3. If n=4, then draw the diagonal that connects the vertices where 0 and 1 are written, respectively, or, if such a diagonal does not exist, then an arbitrary diagonal. In both cases, only sums 1 and 2 can arise. If $n\geq 5$, then choose two consecutive vertices with different labels and a third vertex P that is not neighbor to either of them. Irrespective of whether the label of P is 0 or 1, we can draw the diagonal from it to one of the two consecutive vertices chosen before so that the labels of its endpoints are different. Now the polygon is divided into two convex polygons with smaller number of vertices so that both 0 and 1 occur among their vertex labels. By the induction hypothesis, both polygons can be partitioned into triangles with sum of labels of vertices either 1 or 2.

Other commended solvers: **William FUNG**.

Problem 495. The lengths of each side and diagonal of a convex polygon are rational. After all the diagonals are drawn, the interior of the polygon is partitioned into many smaller convex polygonal regions. Prove that the sides of each of these smaller convex polygons are rational numbers.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Anderson TORRES**.

We only need to show the quadrilateral case, since if this is showed, then the length of any segment of a diagonal connecting a vertex to an intersection point with other diagonal would be rational. Let $ABCD$ be a quadrilateral with all sides and diagonals have rational lengths. Let $\alpha=\angle ABD$ and $\beta=\angle DBC$. Let P be the intersection of AC and BD. Since

$$\cos\alpha=\frac{AB^2+BD^2-AD^2}{2AB\cdot BD},$$

$\cos\alpha$ is rational. Similarly, $\cos\beta$ and $\cos(\alpha+\beta)=\cos\angle ABC$ are rational. Then, since $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, so $\sin\alpha\sin\beta$ is also rational. Also, $\sin^2\beta=1-\cos^2\beta$ is rational. Thus, $\sin\alpha/\sin\beta=\sin\alpha\sin\beta/\sin^2\beta$ is rational. Then, AP/PC = area(ABD)/area(DBC) = $(AB\cdot BD\sin\alpha)/(BD\cdot BC\sin\beta)$ is rational. Therefore, AP and PC are rational. Similarly, PB and PD are rational.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania).

Olympiad Corner

(*Continued from page 1*)

Problem 4. For positive a, b, c, it holds $(a+c)(b^2+ac)=4a$. Find the maximal possible value of $b+c$ and find all triples (a,b,c), for which the value is attained.

Problem 5. There is $|BC|=1$ in a triangle ABC and there is a unique point D on BC such that $|DA|^2=|DB|\cdot|DC|$. Find all possible values of the perimeter of ABC.

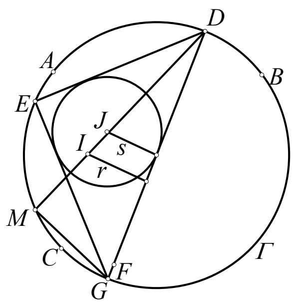
Problem 6. There is a figure of a prince on a field of a 6×6 square chessboard. The prince can in one move jump either horizontally or vertically. The lengths of the jumps are alternately either one or two fields, and the jump on the next field is the first one. Decide whether one can choose the initial field for the prince, so that the prince visits in an appropriate sequence of 35 jumps every field of the chessboard.

Miscellaneous Problems

(*Continued from page 2*)

Example 10. Two triangles have the same incircle. If a circle passes through five of the six vertices of the two triangles, then must it also pass the sixth vertex?

Solution. Let ABC and DEF be the triangles. Let A, B, C, D, E be on the same circle Γ, with radius R and center O. Suppose that F does not belong to Γ. Let $G\neq D$ be the intersection of DF with Γ. Let $\theta=\angle EDF=\angle EDG$. Let I and r be the common incenter and the inradius of $\triangle ABC$ and $\triangle DEF$. Let J and s be the incenter and the inradius of $\triangle DEG$.



We will prove that the incircle of $\triangle ABC$ and $\triangle DEG$ coincide. First, we prove that $I=J$ by showing $IM=JM$. It is well known that $IM=2R\sin(\theta/2)=EM$. From Euler's formula, $OI^2=R^2-2Rr$, which implies that the power of I with respect to Γ is $IM\cdot ID=2Rr$. Since $ID=r/\sin(\theta/2)$, we have $IM=2R\sin(\theta/2)=JM$. So $I=J$. This also proves $r=s$. Hence, the incircle of $\triangle ABC$ and $\triangle DEG$ are the same. Then $F=G$ follows.

Example 11 (1988 Brazilian Math Olympiad). A figure on a computer screen shows n points on a sphere, no four coplanar. Some pairs of points are joined by segments. Each segment is colored red or blue. For each point there is a key that switches the colors of all segments with that point as endpoint. For every three points there is a sequence of key presses that make all three segments between them red. Show that it is possible to make all the segments on the screen red. Find the smallest number of key presses that can turn all the segments red, starting from the worst case.

Solution. Consider three of the points. The parity of the number of blue segments of the triangle with these points as vertices doesn't change while switching the keys. Since it is possible to make all three segments red, the number of blue segments in each triangle is even.

Let P be one of the n points. Let A be the set of points connected to P by red points and B be the set of points connected to P by blue segments. Let $A_1, A_2\in A$. So PA_1 and PA_2 are both red and thus A_1A_2 is red. Now consider $B_1B_2\in B$. Then PB_1 and PB_2 are both blue and B_1B_2 is red. Finally consider $A\in A$ and $B\in B$. PA is red and PB is blue, so AB is blue. Put P in A. All this reasoning shows that segments in the same set are red and segments connecting points in different sets are blue.

Switching all points in set A will make all segments red. Indeed, all segments in A will change twice, one time from each of its edges, all segments connecting points from A and B will change once, turning from blue to red and segments in B won't change. This proves the first part.

For the second part, notice first that one needs to switch each point at most once. Let $|A|=k$ and $|B|=n-k$. If we switch a point from A and b points from B, we change at most $a(n-k)+bk$ blue segments. Suppose without loss of generality that $k\leq n-k$, hence $k\leq[n/2]$. Then $k(n-k)\leq a(n-k)+bk\leq a(n-k)+b(n-k)$, hence $k\leq a+b$. So the number of key presses is at most k and in the worst case, $[n/2]$. This number is needed to make all segments red if $|A|=[n/2]$.

Mathematical Excalibur

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Notes on IMO2017

Tat Wing Leung

International Mathematical Olympiad (IMO) 2017 was held in Rio De Janeiro, Brazil from 12 to 24 July, 2017. Members of Hong Kong Team are as follows.

Tat Wing Leung (Leader)

Tak Wing Ching (Deputy Leader)

Man Yi Mandy Kwok, Shun Ming Samuel Lee, Yui Hin Arvin Leung, Cheuk Hin Alvin Tse, Jeff York Ye, Hoi Wai Yu (Contestants)

All contestants except Alvin Tse are entering universities during the academic year 2017-18. Thus we will have an essentially new team next year.

I went first to Brazil in July 13. Professor Shum Kar Ping, chairman of our Committee also went with me. He was to present the report of IMO2016. It was over quickly. Apparently members of the Advisory Board had nothing more to ask. Luckily it was done.

Upon arrival, I just had to follow the program closely and to attend Jury meetings. As claimed, I did experience *the famous Brazilian Hospitality* (this clause was copied from the program book) and I was quite happy in general.

As in these few years, in choosing the problems, first 4 easy problems, 1 from each of the four categories (Algebra, Combinatorics, Geometry and Number Theory) were selected. Then 4 medium problems, again 1 from each category was selected. Then members of the Jury (leaders) selected two easy problems of two categories, and the 2 medium problems from the two complementary categories were selected. It was claimed this scheme will help to produce a more balanced paper. But after a few years, I do think it is not necessarily true. First almost certain an easy geometry problem will be selected, thus all medium but nice geometry problems will be discarded. It is also almost certain two combinatorics problems will be selected. The papers will then become more predictable. Anyway members still chose this scheme.

Our contestants arrived on July 16. During the opening ceremony, July 17, I had a chance to look at them (from far away). In the opening ceremony, the speech of Marcelo Viana, director of IMPA (Instituto de Mathematica Pura e Applicada) was particularly genuine and moving. He talked about the IMO training and selection in Brazil in these 38 years. (Certainly it was not an easy task to select a team of 6 from 18 million youngsters). Then he also talked about Maryam Mirzakhani, the Iranian Mathematician, who was a 1994 and 1995 IMO gold medalist, 2014 Fields' medalist and passed away prematurely at age 40. Finally, he also talked about the upcoming International Congress of Mathematicians (ICM) 2018, to be held in Brazil.

The next two days (July 18 and 19) are contest days. The contestants had to sit for two 4.5 hours exam during the mornings. In the first half hours of the exams, there were Q&A times. In this year again they adapted a new scheme, namely they had 4 tables, 3 tables for problems 1, 2 and 3 (problems 4, 5 and 6 the next day), and so they were 4 queues. Clearly this is a more efficient scheme than before.

Again the next two days (July 20 and 21) were days of coordination, namely leaders and coordinators would decide the score of a particular problem. We followed the schedule to go to a particular table. We had a very capable deputy leader this year and so he knew well what our team members had done. So the process became relatively easy.

(continued on page 2)

Olympiad Corner

Below are the problems of the 2017 International Mathematical Olympiad (July 18-19, 2017) held in Brazil.

Problem 1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases}$$

for each $n \geq 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y, $f(f(x)f(y)) + f(x+y) = f(xy)$.

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 21, 2017***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Since the problems of IMO2017 is listed in this issue of Excalibur, I shall not reproduce them here nor to copy the proofs. I will only give a few comments of this year's problems. First a few key words came to my mind. My first word is algorithm (or construction). Indeed the proposer has been trying hard to think of a new scenario that when you try to solve the problem, you need to invent a new algorithm to solve the problem. For example, problems 5 and 6 do not need to know a lot of higher math, but you do need to have some sense of ingenuity to think of a new scheme or method to solve a particular problem. In Problem 3, you had to show an algorithm does not exist. The second word is induction, namely in these problems, small cases (cases with smaller parameters) were easy. So one might try to consider if the method of induction does work. It was not obvious. The third word is geometry. In this year, only 3 of us could solve the relatively easy geometry problem. Indeed this year's geometry problem (P4), no new constructions are required, no new transformations (inversion, homothety, etc.) are needed. It is simply correct drawing and angle chasing. So I must admit that we have reverted back to our usual tradition.

Now I will say a few more words on the individual problems and the performance of our team. Problem 1 is a number theory problem. Once a contestant tries a few cases and guess the correct answer ($a_0 \equiv 0 \pmod 3$), then it is not too hard to prove $a_0 \equiv 1,2 \pmod 3$ do not work but $a_0 \equiv 0 \pmod 3$ works. Our team this year is relatively mature and relatively well trained. So all of them solved the problem and we have a perfect score.

Problem 2 is a functional equation, showing $f(f(x)f(y))+f(x+y)=f(xy)$ for all real x and y will imply $f(x)=0$ or $f(x) = \pm(x-1)$. The most troublesome thing is the marking scheme. It is easy to get the first 3 points, but it is real hard to get an extra point, i.e., proving injectivity and onward. A leader secretly showed me the scores of problem 2 of his team, apparently he was dismayed by the performance. I was not sure. At the end I found their team scored 1 more point than us.

For problem 3, I had (and still have) a very serious concern about it. Observe only two contestants scored 7 points (a Russian and an Australian contestant), and also none of the USA team and the Chinese team (plus other teams) together scored any point at all. I suspect many contestants are like me and simply don't know what exactly is going on. Indeed it is not quite sure what it means by "no matter how" and what exactly it means by a tracking device, I was told it is not like the "best strategy". Indeed when you look at the solution, you get the idea such a strategy (or algorithm) does not exist. The solution is roughly as follows. Assume the rabbit moves in a straight line, and with luck (this term appears quite a few times in the solution) the tracking device also moves in a straight line. Because of this happening, the hunter can only move along a straight line (also with no justification but intuition) and follow the rabbit, and after finitely many steps, the distance between the rabbit and the hunter will only increase (easy to show by simple geometry). Thus there is no best strategy. I am still awaiting members to educate me on this problem.

Problem 4 was a relatively easy geometry exercise.

We did best in problem 5 among all teams, (our deputy leader reminded me about this point). Indeed altogether we scored 26 points. So essentially 4 of us solved the problem, while other teams scored at most 23 points. This shows our team does know something about problem solving. Indeed the problem is equivalent to say there are $N(N+1)$ distinct integers randomly placed in a row, say, you can throw away $N(N-1)$ of them, and among the remaining integers, the largest integer and the second largest integer will stick together, so are the third largest and the fourth largest integer will stick together, and so on. Not too hard?

For Problem 6, an ordered pair (x,y) of integers is a primitive point if $\gcd(x,y)=1$. Now given a set of finitely many primitive points (x_i, y_i), $1\le i\le n$, we need to find a homogeneous polynomial $g(x,y)$ such that $g(x_i,y_i)=1$. If there is only one primitive point, then it is trivial, by Euclidean algorithm. The hard part is how to move on by induction. But it is not at all easy.

At the end Shun Ming was awarded a gold medal (25 points), Mandy a silver (23 points), Jeff (18 points), Hoi Wai (17 points) and Cheuk Hin (17 points) all received Bronze medals. Yui Hin (11 points) managed to get a honorable mention. Our rank is 26 among 111 countries/regions. Surely the result was not as good as last year nor as we had hoped for. Nevertheless there were certain things we can say. Indeed this was the 30th consecutive year that we sent teams to IMOs. No matter what, it is not an easy matter and it should be a date to remember. (Better still, we hosted the event in 1994 and 2016). Also Mandy Kwok was the second girls among all girl contestants. IMPA this year gave out 5 prizes to female contestants. Initially I thought Mandy should have a chance to get a prize. Later I found out the prizes were for the top female students who contribute the most to their respective team's score. So I understand why she was not eligible for the prize. Nevertheless I must say we are very glad to see her improving very well in these few years. Finally we managed to get the highest score in Problem 5. I think this is an indication that our team is comparable with any other team. They really don't have much special recipe we don't envisage.

I hasten to say the cut scores of IMO this year cannot be said to be ideal. Indeed the cut scores for gold is 25, for silver 19, and bronze 16. One may say the easy problems (problems 1 and 4) were too easy and the four other problems too hard. The easy problem were too easy. Hence 14 points was not enough for a bronze and the hard problems too hard. Thus 25 points was good enough to get a gold. Really we expect a contestant to solve at least 2 problems ($\ge$14 points) to get a bronze, at least 3 problems ($\ge$21 points) a silver, and at least 4 problems ($\ge$28 points) to get a gold. Some people expect a contestant should solve nearly at least 5 problems to get a gold. Really what is the point to set a problem so that only 2 out of 615 contestants can solve it?

Since we are trailing behind some other Asian countries this year, it was suggested that more money should be put into this activity. In my opinion the stakeholders (members of the Committee, the Academy and the Gifted Section of Education, but most important of all, past and present trainees) should sit together and sort out what exactly do we want, how much money/resource should be put into it and who will contribute what, etc. I suppose it is time to start thinking.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to Dr. *Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 21, 2017.***

Problem 501. Let x, y, s, m, n be positive integers such that $x+y=s^m$ and $x^2+y^2=s^n$. Determine the number of digits s^{300} has in base 10.

Problem 502. Let O be the center of the circumcircle of acute $\triangle ABC$. Let P be a point on arc BC so that A, P are on opposite sides of side BC. Point K is on chord AP such that BK bisects $\angle ABC$ and $\angle AKB > 90°$. The circle Ω passing through C, K, P intersect side AC at D. Line BD meets Ω at E and line PE meets side AB at F. Prove that $\angle ABC = 2\angle FCB$.

Problem 503. Let S be a subset of $\{1,2,\ldots,2015\}$ with 68 elements. Prove that S has three pairwise disjoint subsets A, B, C such that they have the same number of elements and the sums of the elements in A, B, C are the same.

Problem 504. Let $p>3$ be a prime number. Prove that there are infinitely many positive integers n such that the sum of k^n for $k=1,2,\ldots,p-1$ is divisible by p^3.

Problem 505. Determine (with proof) the least positive real number r such that if z_1, z_2, z_3 are complex numbers having absolute values less than 1 and sum 0, then

$$|z_1z_2+z_2z_3+z_3z_1|^2+|z_1z_2z_3|^2<r.$$

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Solutions

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Problem 496. Let a,b,c,d be real numbers such that $a+\sin b > c+\sin d$, $b+\sin a > d+\sin c$. Prove that $a+b>c+d$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

For $x\ge 0$, $|\sin x|\le x$. Let $s = a-c$ and $t = d-b$. We have

$$\begin{aligned} s = a-c &> \sin d-\sin b \\ &= 2\cos[(d+b)/2]\sin[(d-b)/2] \\ &\ge -2|\sin(t/2)| \end{aligned}$$

and

$$\begin{aligned} t = d-b &< \sin a-\sin c \\ &= 2\cos[(a+c)/2]\sin[(a-c)/2] \\ &\le 2\,|\sin(s/2)|. \end{aligned}$$

If $s\ge 0$, then $t< 2|\sin(s/2)|\le s$. Similarly, if $t\le 0$, then $s > -2|\sin(-t/2)| \ge -2(-t/2) = t$.

Finally, if $s < 0 < t$, then $-s < 2|\sin(t/2)| \le t$ and $t < 2|\sin(s/2)| = |\sin(-s/2)| \le -s$, which leads to a contradiction.

Comment: The above solution avoided calculus as it used $\sin x \le x$ for $0\le x\le 1$, which followed by taking points A, B on a unit circle with center O such that $\angle AOB = 2x$, then the length $2x$ of arc AB is greater than the length $2\sin x$ of chord AB.

Other commended solvers: **Jason FONG** and **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School).

Problem 497. Let there be three line segments with lengths 1, 2, 3. Let the segment of length 3 be cut into $n\ge 2$ line segments. Prove that among these $n+2$ segments, there exist three of them that can be put to form a triangle where each side is one of the three segments.

Solution. **William FUNG**, **Mark LAU** (Pui Ching Middle School), **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Note line segments with lengths $a\le b\le c$ form a triangle if and only if $a+b>c$. Let $a_1\le a_2\le\cdots\le a_n$ be the lengths of such n segments with sum equals to 3. Assume there exists i such that $a_i>1$. If $1<a_i<2$, then the segments with length $1,a_i,2$ forms a triangle since $1+a_i>2$. If $2\le a_i$, then the segments with length $1,2,a_i$ forms a triangle since $1+2>a_i$. It remains to consider the case all $a_i\le 1$. Then $i\ge 3$.

Assume no 3 of these segments form a triangle. Then $a_1+a_2\le a_3$, $a_2+a_3\le a_4$, …, $a_{n-2}+a_{n-1}\le a_n$, $a_n+1\le 2$. Adding these and cancelling $a_3,\ldots,a_n,1$ on both sides, we have

$$3+a_2=(a_1+a_2+\cdots+a_n)+a_2\le 2,$$

which yields $a_2\le -1$, a contradiction.

Problem 498. Determine all integers $n>2$ with the property that there exists one of the numbers $1,2,\ldots,n+1$ such that after its removal, the n numbers left can be arranged as $a_1,a_2,\ldots,a_n$ with no two of $|a_1-a_2|$, $|a_2-a_3|$, …, $|a_{n-1}-a_n|$, $|a_n-a_1|$ being equal.

Solution. **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School), **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

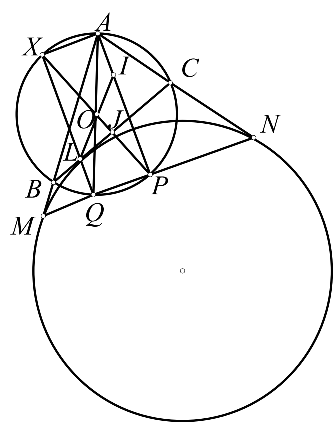
Since no two of $|a_1-a_2|$, $|a_2-a_3|$, …, $|a_{n-1}-a_n|$, $|a_n-a_1|$ being equal and each is at most n, they must be $1,2,\ldots,n$ in some order. So $|a_1-a_2|+|a_2-a_3|+\cdots+|a_{n-1}-a_n|+|a_n-a_1|=1+2+\cdots+n=n(n+1)/2$. From $a\equiv |a| \pmod 2$ and $(a_1-a_2)+(a_2-a_3)+\cdots+(a_{n-1}-a_n)+(a_n-a_1)=0$, we see $|a_1-a_2|+|a_2-a_3|+\cdots+|a_{n-1}-a_n|+|a_n-a_1|$ is even. For $n(n+1)/2$ to be even, this implies $n\equiv 0$ or $-1 \pmod 4$.

In the case $n=4k$, remove $k+1$ and let $a_1=4k+1$, $a_2=1$, $a_3=4k$, $a_4=2$, …, $a_{2k-1}=3k+2$, $a_{2k}=k$, $a_{2k+1}=3k+1$, $a_{2k+2}=k+2$, $a_{2k+3}=3k$, $a_{2k+4}=k+3$, …, $a_{4k-1}=2k+2$ and $a_{4k}=2k+1$.

In the case $n=4k-1$, remove $3k$ and let $a_1=4k$, $a_2=1$, $a_3=4k-1$, $a_4=2$, …, $a_{2k-1}=3k+1$, $a_{2k}=k$, $a_{2k+1}=3k-1$, $a_{2k+2}=k+1$, $a_{2k+3}=3k-2$, …, $a_{4k-2}=2k-1$, $a_{4k-1}=2k$.

Problem 499. Let ABC be a triangle with circumcenter O and incenter I. Let Γ be the escribed circle of $\triangle ABC$ meeting side BC at L. Let line AB meet Γ at M and line AC meet Γ at N. If the midpoint of line segment MN lies on the circumcircle of $\triangle ABC$, then prove that points O, I, L are collinear.

***Solution*.** **George SHEN.**



Let P be the midpoint of MN. From $AM=AN$, we see $AP\perp MN$. So A,I,P are collinear. Let Q be on MN such that $LQ\perp MN$. Now $\angle BMQ=\angle CNQ$ and

$$\frac{MQ}{NQ}=\frac{ML\cos\angle LMQ}{NL\cos\angle LNQ}=\frac{2MB\cos\angle LMB\cos\angle LNC}{2NC\cos\angle LNC\cos\angle LMB}=\frac{MB}{NC}.$$

This implies $\triangle BMQ$, $\triangle CNQ$ are similar.

Let $a=BC$, $b=CA$, $c=AB$, $s=(a+b+c)/2 = AM = AN$ and $\alpha = \angle BAC$.

We have

$$AP = AM\cos(\alpha/2) = s\cos(\alpha/2).$$

By extended sine law, $BC = a = 2R\sin\alpha$. From $IP=BP=CP$ [see <u>Math Excalibur</u>, *vol.* 11, *no.* 2, *page* 1, *Theorem in middle column−Ed.*], we have

$$a = BC = 2BP\sin\frac{180^\circ-\alpha}{2} = 2BP\cos\frac{\alpha}{2},$$

$$\cos\frac{\alpha}{2} = \frac{a}{2IP} = \frac{a}{2(AP-AI)} = \frac{a}{2(s\cos\frac{\alpha}{2}-AI)}.$$

Applying $AI\cos(\alpha/2)=s-a$ and the last equation, we can get

$$2s\cos^2\frac{\alpha}{2} = 2s-a = b+c,$$

$$2s\sin^2\frac{\alpha}{2} = a.$$

Next $MN = 2AM\sin(\alpha/2) = 2s\sin(\alpha/2)$ and $(MQ+NQ)\sin(\alpha/2) = MB+NC$. Using $MQ/NQ=MB/NC$, we get

$$MQ\sin(\alpha/2)=MB$$

and

$$NQ\sin(\alpha/2)=NC,$$

which says $\angle QBA = 90^\circ = \angle QCA$. Then Q is on Γ and AQ is a diameter of Γ.

Let line LQ meet the circumcircle Γ of $\triangle ABC$ at X as labeled in the figure. Observe that $APQX$ is a rectangle and AQ, XP are diameters of Γ intersecting at O. We claim $LQ=AI$ (then $LI\cap AQ$ at O and so O,I,L are collinear).

Now $BO=CO$, $BJ=CJ$ and $\angle BAP = \angle CAP$ implies $BP=CP$. Hence, O, J, P are collinear. Next $OJ\perp BC$ implies $\angle LJP=90^\circ=\angle LQP$. Then, J,P,Q,L are concyclic. Hence,

$$XL\cdot XQ = XJ\cdot XP$$

Let R be the circumradius of $\triangle ABC$. From

$$XJ = \frac{a}{2}\cot\frac{\alpha}{2},\ XP = 2R,$$

$$IP = 2R\sin\frac{\alpha}{2},\ AP = s\cos\frac{\alpha}{2},$$

We get $XJ\cdot XP = IP\cdot AP$. Then $XL\cdot XQ = IP\cdot AP$. Since $XQ=AP$, so $XL=IP$. Then $QL=XQ-XL=AP-IP=AI$. The conclusion follows.

Other commended solvers: **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 500. Determine all positive integers n such that there exist $k\geq 2$ positive rational numbers such that the sum and the product of these k numbers are both equal to n.

Solution. **Mark LAU** (Pui Ching Middle School), **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Observe that for a composite number n, there exist integer $s,t\geq 2$ such that $n=st$, the sequence $s,t,1,1,\ldots,1$ (with $st-s-t$ 1's) has sum and product equals $st=n$.

For prime numbers $n\geq 11$, the sequence $n/2,1/2,2,2,1,1,\ldots,1$ (with $n-4-(n+1)/2$ 1's) satisfies the condition by a simple checking.

For $n=7$, the sequence 9/2, 4/3, 7/6, satisfies the condition by a simple checking.

Next we claim the cases $n=1,2,3,5$ have no solution. Assume $a_1, a_2,\ldots,a_k$ are positive rational numbers with sum and product equals to n. By the AM-GM inequality, we have

$$\frac{n}{k} = \frac{a_1+\cdots+a_k}{k} \geq \sqrt[k]{a_1\cdots a_k} = \sqrt[k]{n}.$$

Then $n\geq k^{k/(k-1)}>k$. Since $n>k\geq 2$, cases $n=1$ or 2 are impossible.

Finally, for $n=3$ or 5, since $3^{3/(3-1)}=5.1\ldots$ implies $k=2$, so only cases $(n,k)=(3,2)$ and $(5,2)$ remain. Now

$$\begin{aligned}(a_1-a_2)^2 &= (a_1+a_2)^2-4a_1a_2\\ &= n^2-4n = -3 \text{ or } 5,\end{aligned}$$

which have no rational solutions a_1, a_2. Therefore, the answers are all positive integers except 1,2,3,5.

Olympiad Corner

(*Continued from page 1*)

Problem 3 (Cont'd).

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 5. An integer $N\geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

⋮

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6. An ordered pair (x,y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x,y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 20th Hong Kong (China) Mathematical Olympiad held on December 2, 2017. Time allowed is 3 hours.

Problem 1. The sequence $\{x_n\}$ is defined by $x_1=5$ and $x_{k+1}=x_k^2-3x_k+3$ for $k=1,2,3,\ldots$. Prove that $x_k > 3^{2^{k-1}}$ for all positive integer k.

Problem 2. Suppose $ABCD$ is a cyclic quadrilateral. Produce DA and DC to P and Q respectively such that $AP=BC$ and $CQ=AB$. Let M be the midpoint of PQ. Show that $MA\perp MC$.

Problem 3. Let k be a positive integer. Prove that there exists a positive integer ℓ with the following property: if m and n are positive integers relatively prime to ℓ such that $m^m\equiv n^n \pmod{\ell}$, then $m\equiv n \pmod{k}$.

Problem 4. Suppose 2017 points in a plane are given such that no three points are collinear. Among the triangles formed by any three of these 2017 points, those triangles having the largest area are said to be *good*. Prove that there cannot be more than 2017 good triangles.

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 10, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Functional Inequalities

Kin Y. Li

In the volume 8, number 1 issue of Math Excalibur, we provided a number of examples of functional equation problems. In the volume 10, number 5 issue of Math Excalibur, problem 243 in the problem corner section was the first functional inequality problem we posed. That one was from the 1998 Bulgarian Math Olympiad. In this article, we would like to look at some functional inequality problems that appeared in various math Olympiads.

Example 1 (2016 Chinese Taipei Math Olympiad Training Camp. Let function $f:[0,+\infty)\to[0,+\infty)$ satisfy

(1) for arbitrary $x,y\geq 0$, we have

$$f(x)f(y)\leq y^2 f\left(\frac{x}{2}\right)+x^2 f\left(\frac{y}{2}\right);$$

(2) for arbitrary $0\leq x\leq 1$, we have $f(x)\leq 2016$.

Prove that for arbitrary $x\geq 0$ we have $f(x)\leq x^2$.

Solution. In (1), let $x=y=0$, then $f(0)=0$. Assume there is $x_0>0$ such that $f(x_0)>x_0^2$. By (1), we see $f(x_0/2)>x_0^2/2$. By math induction, for all positive integer k, we have

$$f\left(x_0/2^k\right)>2^{2^k-2k-1}x_0^2.$$

As k gets large, eventually we have $x_0/2^k$ is in $[0,1]$, but $f(x_0/2^k)>2016$. This contradicts (2). So for all $x\geq 0$, $f(x)\leq x^2$.

Example 2 (2005 Russian Math Olympiad). Does there exist a *bounded* function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)>0$ and for all $x,y\in\mathbb{R}$, it satisfies the inequality

$$f^2(x+y)\geq f^2(x)+2f(xy)+f^2(y)\ ?$$

Solution. Assume such f exists. Let $a=2f(1)>0$. For $x_1\neq 0$, let $y_1=1/x_1$, then

$$f^2(x_1+y_1)\geq f^2(x_1)+2f(1)+f^2(y_1)\geq f^2(x_1)+a.$$

For $n>1$, let $x_n=x_{n-1}+y_{n-1}$, $y_n=1/x_n$. Then

$$f^2(x_n+y_n)\geq f^2(x_n)+a=f^2(x_{n-1}+y_{n-1})+a\geq f^2(x_{n-1})+2a\geq\cdots\geq f^2(x_1)+na.$$

As $n\to\infty$, f becomes unbounded, which is a contradiction.

Example 3 (2016 Ukranian Math Olympiad). Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that for arbitrary real numbers x, y, we have

$$f(x-f(y))\leq x-yf(x)\ ?$$

Solution. Assume such function exists. Let $y=0$. Then $f(x-f(0))\leq x$. Replacing x by $x+f(0)$, we get $f(x)\leq x+f(0)$. Then setting $x=f(y)$, we get

$$f(0)\leq f(y)-yf(f(y))\leq y+f(0)-yf(f(y)),$$

which implies $yf(f(y))\leq y$. If $y<0$, then

$$1\leq f(f(y))\leq f(y)+f(0)\leq y+2f(0).$$

The last inequality is satisfied for all $y<0$, which is a contradiction.

Example 4 (The Sixth IMAR Math Competition, 2008). Show that for any function $f:(0,+\infty)\to(0,+\infty)$ there exists real numbers $x>0$ and $y>0$ such that $f(x+y)<yf(f(x))$.

Solution. Assume $f(x+y)\geq yf(f(x))$ for all $x,y>0$. Let $a>1$, then $t=f(f(a))>0$. Now for $b\geq a(1+t^{-1}+t^{-2})>a$, we have

$$f(b)=f(a+(b-a))\geq(b-a)f(f(a))=(b-a)t\geq a(1+t^{-1})>a.$$

Then

$$f(f(b))=f(a+(f(b)-a))\geq(f(b)-a)t\geq a.$$

If we take $x\geq(ab+2)/(a-1)>b$, then

$$f(x)=f(b+(x-b))\geq(x-b)f(f(b))\geq(x-b)a\geq x+2.$$

Hence, $f(x)>x+1$ (*). However,

$$f(f(x))=f(x+(f(x)-x))\geq(f(x)-x)f(f(x)).$$

Cancelling $f(f(x))$ on both sides, we get $f(x)\leq x+1$, which contradicts (*).

(continued on page 2)

<u>*Example 5 (2016 Romanian Math Olympiad).*</u> Determine all functions f:ℝ→ℝ satisfying for arbitrary a, b∈ℝ, we have

$$f(a^2)-f(b^2)\le(f(a)+b)(a-f(b)). \quad (1)$$

<u>*Solution.*</u> Let a=b=0, then $f^2(0)\le0$, so $f(0)=0$. Let b=0, then $f(a^2)\le af(a)$. Let a=0, then $f(b^2)\ge bf(b)$. So for all x, we have <u>(2) $f(x^2)=xf(x)$</u>. Using this on the left side of (1), we get <u>(3) $f(a)f(b)\le ab$</u>. Next, by (2), we have

$$-xf(-x)=f((-x)^2)=f(x^2)=xf(x).$$

So f is an odd function. This implies

$$f(a)f(b)=-f(a)f(-b)\ge-(-ab)=ab.$$

Using (3), we have $f(a)f(b)=ab$. Then $f^2(1)=1$. So $f(1)=\pm1$. Hence, for all x, $f(x)f(1)=x$, i.e. either $f(x)=x$ for all x or $f(x)=-x$ for all x. Simple checking shows both of these satisfy (1).

<u>*Example 6 (1994 APMO).*</u> Let f: ℝ→ℝ be a function such that

(i) for all x,y∈ℝ

$$f(x)+f(y)+1\ge f(x+y)\ge f(x)+f(y),$$

(ii) for all $x\in[0,1)$, $f(0)\ge f(x)$,

(iii) $-f(-1)=f(1)=1$.

Find all such functions.

<u>*Solution.*</u> By (iii), $f(-1)=-1$, $f(1)=1$. So $f(0)=f(-1+1)\ge f(1)+f(-1)=0$. By (i), $f(1)=f(1+0)\ge f(1)+f(0)$. So $f(0)\le0$. Then $f(0)=0$.

Next we claim $f(x)=0$ for all x in (0,1). Since $f(0)=0$, by (ii), $f(x)\le0$ for all x in (0,1). By (i) and (ii), $f(x)+f(1-x)+1 f(1)=1$. So $f(x)\ge-f(1-x)$. If $x\in(0,1)$, then $1-x\in(0,1)$. So $f(1-x)\le0$ and $f(x)\ge-f(1-x)\ge0$. Then $f(x)=0$.

Next by (i) and (iii), we have $f(x+1)\ge f(x)+f(1)=f(x)+1$ and $f(x)\ge f(x+1)+f(-1)=f(x+1)-1$. These give $f(x+1)=f(x)+1$.

So $f(x)=0$ for $x\in[0,1)$ and $f(x+1)=f(x)+1$. Hence, $f(x)=[x]$. We can check directly $[x]$ satisfies (i), (ii) and (iii).

<u>*Example 7 (2007 Chinese IMO Team Training Test).*</u> Does there exist any function f:ℝ→ℝ satisfy $f(0)>0$ and

$$f(x+y)\ge f(x)+y\,f(f(x)) \text{ for all } x,y\in ℝ?$$

<u>*Solution.*</u> Assume such function exists. In that case, we claim there would exist real z such that $f(f(z))>0$. (Otherwise, for all x, $f(f(x))\le0$. So for all $y\le0$, we have $f(x+y)\ge f(x)+yf(f(x))\ge f(x)$. Then f is a decreasing function. So for all x∈ℝ, $f(0)>0\ge f(f(x))$, which implies $f(x)>0$. This contradicts $f(f(x))\le0$.)

From the claim, we see as $x\to+\infty$, $f(z+x)\ge f(z)+xf(f(z))\to+\infty$. So we get

$$f(x)\to+\infty \text{ as well as } f(f(x))\to+\infty.$$

Then there are x, $y>0$ such that $f(x)\ge0$, $f(f(x))>1$, $f(x+y)>0$, $f(f(x+y+1))>0$ and <u>(*) $y\ge(x+1)/(f(f(x))-1)$</u>. Define $A=x+y+1$, $B=f(x+y)-(x+y+1)$. Then $f(f(A))>0$ and

$$f(x+y)\ge f(x)+yf(f(x))\ge x+y+1 \text{ by (*)}.$$

So $B\ge0$. Next,

$$\begin{aligned}f(f(x+y)) &= f(A+B)\ge f(A)+Bf(f(A))\\ &\ge f(A)=f((x+y)+1)\\ &= f(x+y)+f(f(x+y))\\ &> f(f(x+y)),\end{aligned}$$

which is a contradiction.

<u>*Example 8 (2015 Greek IMO Team Selection Test).*</u> Determine all functions f:ℝ→ℝ such that for arbitrary x, y∈ℝ, we have

$$f(xy)\le yf(x)+f(y). \quad (1)$$

<u>*Solution.*</u> In (1), using $-y$ to replace y, we get

$$f(-xy)\le-yf(x)+f(-y). \quad (2)$$

Adding (1) and (2), we get

$$f(xy)+f(-xy)\le f(y)+f(-y). \quad (3)$$

Setting y=1, we get

$$f(x)+f(-x)\le f(1)+f(-1). \quad (4)$$

In (3), using $1/y$ with $y\neq0$ to replace x, we get

$$f(1)+f(-1)\le f(y)+f(-y). \quad (5)$$

By (4) and (5), for $y\neq0$, we have

$$f(y)+f(-y)=f(1)+f(-1).$$

Let $c=f(1)+f(-1)$. Then (2) becomes

$$c-f(xy)\le-yf(x)+c-f(y).$$

Then

$$f(xy)\ge yf(x)+f(y). \quad (6)$$

By (1) and (6), for all $x,y\neq0$,

$$f(xy)=yf(x)+f(y). \quad (7)$$

Setting x=y=1, we get $f(1)=0$. In (7), interchanging x and y, we get

$$f(yx)=xf(y)+f(x). \quad (8)$$

Subtracting (7) and (8), we get

$$(y-1)f(x)=(x-1)f(y).$$

Then for x, $y\neq0,1$, we get $\dfrac{f(x)}{x-1}=\dfrac{f(y)}{y-1}$.

Since $f(1)=0$, we see there exists a such that $f(x)=a(x-1)$ for all $x\neq0$. Setting x=0 in (1), we get $f(y)\ge(1-y)f(0)$. Then for $y\neq0$, we get $a(y-1)\ge(1-y)f(0)$, which is $(y-1)(a+f(0))\ge0$. Then $a=-f(0)$ and we get for all real x, $f(x)=f(0)(1-x)$. Setting $f(0)$ to be any real constant, we can check all such functions satisfy (1).

<u>*Example 9 (2013 Croatian IMO Team Selection Test).*</u> Determine all functions f:ℝ→ℝ such that for all real numbers x, y, we have $f(1)\ge0$ and

$$f(x)-f(y)\ge(x-y)f(x-y). \quad (*)$$

<u>*Solution.*</u> Setting y=$x-1$, we get $f(x)-f(x-1)\ge f(1)\ge0$. So

$$f(x)\ge f(x-1). \quad (1)$$

Setting y=0, we get

$$f(x)-f(0)\ge xf(x). \quad (2)$$

Replacing y by x and x by 0, we get

$$f(0)-f(x)\ge-xf(-x). \quad (3)$$

Adding (2) and (3), we get

$$0\ge xf(x)-xf(-x).$$

Then for every x>0, we get

$$f(-x)\ge f(x). \quad (4)$$

Setting x=1, y=0 in (0), we get

$$f(0)\le0. \quad (5)$$

By (5), (1), (4), we get $0\ge f(0)\ge f(-1)\ge f(1)\ge0$. So $f(0)=f(-1)=f(1)=0$. Using (1) repeatedly, we get

$$f(x)\ge f(x-1)\ge f(x-2)\ge\cdots, \quad (6)$$

i.e. $f(x)\ge f(x-k)$ for all real x, positive integer k. Using (6), (1) and replacing x by $x-1$ and y by -1 in (*), we get

$$f(x)\ge f(x-1)=f(x-1)-f(-1)\ge xf(x).$$

Then $f(x)(x-1)\le0$. So if x>1, then $f(x)\le0$. If x<1, then $f(x)\ge0$.

For x>1, there is y<1 such that k=$x-y$ is a positive integer. Then

$$0\ge f(x)\ge f(x-k)=f(y)\ge0.$$

So for x>1, $f(x)$=0. Similarly, for x<1, there is y>1 such that k=$y-x$ is a positive integer. Then as above, all $f(x)$=0. We can check directly $f(x)$=0 satisfies (*).

<u>*Example 10 (2011 IMO Problem 3 proposed by Belarus).*</u> Let f:ℝ→ℝ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y)\le yf(x)+f(f(x)) \quad (1)$$

for all real numbers x and y. Prove that $f(x)=0$ for all $x\le0$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***February 10, 2018.***

Problem 506. Points A and B are on a circle Γ_1. Line AB is tangent to another circle Γ_2 at B and the center O of Γ_2 is on Γ_1. A line through A intersects Γ_1 at points D and E (with D between A and E). Line BD intersects Γ_1 at a point F, different from B. Prove that D is the midpoint of BF if and only if BE is tangent to Γ_1.

Problem 507. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y\in\mathbb{R}$,

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x)).$$

Problem 508. Determine the largest integer k such that for all integers x,y, if $xy+1$ is divisible by k, then $x+y$ is also divisibly by k.

Problem 509. In $\triangle ABC$, the angle bisector of $\angle CAB$ intersects BC at a point L. On sides AC, AB, there are points M, N respectively such that lines AL, BM, CN are concurrent and $\angle AMN=\angle ALB$. Prove that $\angle NML=90°$.

Problem 510. Numbers 1 to 20 are written on a board. A person randomly chooses two of these numbers with a difference of at least 2. He adds 1 to the smaller one and subtracts 1 from the larger one. Then he performs an operation by replacing the original two chosen numbers on the board with the two new numbers. Determine the maximum number of times he can do this operation.

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Solutions

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Problem 501. Let x, y, s, m, n be positive integers such that $x+y=s^m$ and $x^2+y^2=s^n$. Determine the number of digits s^{300} has in base 10.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **Soham GHOSH** (RKMRC Narendrapur, Kalkata, India), **Mark LAU, LEE Jae Woo** (Hamyang High School, South Korea), **Toshihiro SHIMIZU** (Kawasaki, Japan).

Since $s^{2m}=(x+y)^2>x^2+y^2=s^n$, so $2m>n$. Then

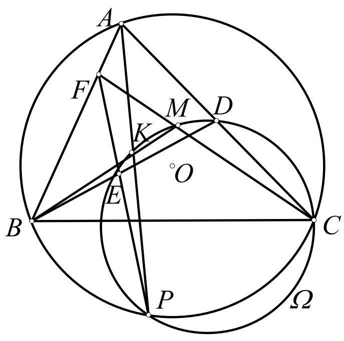
$$0\le (x-y)^2=2(x^2+y^2)-(x-y)^2$$
$$=2s^n-s^{2m}=s^n(2-s^{2m-n}).$$

If $s\ge 3$, then we have $2-s^{2m-n}\le 2-s<0$, a contradiction. If $s=1$, then we have $1+1\le x+y=s^m=1$, a contradiction. So s must be 2. Since $\log_{10}2^{300}=300\log_{10}2=0.3010\ldots\times 300=90.3\ldots$, 2^{300} has 91 digits.

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **George SHEN**.

Problem 502. Let O be the center of the circumcircle of acute $\triangle ABC$. Let P be a point on arc BC so that A, P are on opposite sides of side BC. Point K is on chord AP such that BK bisects $\angle ABC$ and $\angle AKB>90°$. The circle Ω passing through C, K, P intersect side AC at D. Line BD meets Ω at E and line PE meets side AB at F. Prove that $\angle ABC=2\angle FCB$.

Solution. **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Take point M on line KB such that $MB=MC$. Then we have $\triangle BMC$ is isosceles and

$$\angle KPC=\angle APC=\angle ABC$$
$$=\angle MBC+\angle MCB$$
$$=180°-\angle BMC$$
$$=180°-\angle KMC.$$

This implies M is on the circle Ω. Applying Pascal's theorem to the points P, E, D, C, M, K on Ω, we have $PE\cap CM$, $ED\cap MK=B$ and $DC\cap KP=A$ are collinear. Since this line coincides with line AB, so $PE\cap CM=F$. Then

$$2\angle FCB=2\angle MCB=2\angle MBC=\angle ABC.$$

Other commended solvers: **LEE Jae Woo** (Hamyang High School, South Korea), **Vijaya Prasad NALLURI** (Retd Principal APES, Rajahmundry, India) and **Akash Singha ROY** (Hariyana Vidya Mandir High School, India).

Problem 503. Let S be a subset of $\{1,2,\ldots,2015\}$ with 68 elements. Prove that S has three pairwise disjoint subsets A, B, C such that they have the same number of elements and the sums of the elements in A, B, C are the same.

Solution. **Mark LAU** and **George SHEN**.

There are totally $(68\times 67\times 66)/6=50116$ 3-element subsets of S. The possible sums of the three elements in these subsets of S are from $1+2+3=6$ to $2013+2014+2015=6042$. Now $50116>8\times(6042-6+1)$. So by the pigeonhole principle, there are 9 distinct 3-element subsets $A_1, A_2,\ldots, A_9$ of S with the same sum of elements.

Assume $x\in S$ appears in $A_1, A_2,\ldots, A_9$ at least 3 times, say in A_1, A_2, A_3. Then no two of the sets $U=A_1\setminus\{x\}$, $V=A_2\setminus\{x\}$, $W=A_3\setminus\{x\}$ are the same. Otherwise say $U=V$, then $A_1=A_2$, contradiction.

So every $x\in S$ appear at most twice among $A_1, A_2,\ldots, A_9$. Then there can only be at most 3 of $A_2,\ldots, A_9$ (say A_2, A_3, A_4) having an element in common with A_1 (as every element of A_1 can only appear in at most one of $A_2,\ldots, A_9$). Without loss of generality, say each of $A_5,\ldots, A_9$ is disjoint with A_1. Similarly, among $A_6,\ldots, A_9$, there are at most three of them (say A_6, A_7, A_8) have a common element with A_5. Then A_9 and A_5 are disjoint. So the pairwise disjoint sets $A=A_1$, $B=A_5$, $C=A_9$ have the same sum of elements.

Other commended solvers: **LEE Jae Woo** (Hamyang High School, South Korea), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India), and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 504. Let $p>3$ be a prime number. Prove that there are infinitely many positive integers n such that the sum of k^n for $k=1,2,\ldots,p-1$ is divisible by p^3.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **DBS Maths Solving Team**

(Diocesan Boys' School), **Mark LAU**, **LEE Jae Woo** (Hamyang High School, South Korea), **LEUNG Hei Chun** (SKH Tang Shiu Kin Secondary School), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

As $\varphi(p^3)=p^2(p-1)$, by Euler's theorem, for all positive integers r,s, we have

$$\sum_{k=1}^{p-1} k^{r+p^2(p-1)s} \equiv \sum_{k=1}^{p-1} k^r \pmod{p^3}.$$

In the case $r=p^2$, we have

$$\sum_{k=1}^{p-1} k^{p^2} = \sum_{k=1}^{(p-1)/2} (k^{p^2}+(p-k)^{p^2})$$
$$= \sum_{k=1}^{(p-1)/2}\left(k^{p^2}+\sum_{t=0}^{p^2}\binom{p^2}{t}p^t(-k)^{p^2-t}\right)$$
$$\equiv \sum_{k=1}^{(p-1)/2}\left(p^3k^{p^2-1}-\frac{p^4(p^2-1)}{2}k^{p^2-2}\right)$$
$$\equiv 0 \pmod{p^3}.$$

So all cases $n=p^2+p^2(p-1)s$ works.

Other commended solvers: **Soham GHOSH** (RKMRC Narendrapur, Kalkata, India) and **George SHEN.**

Problem 505. Determine (with proof) the least positive real number r such that if z_1, z_2, z_3 are complex numbers having absolute values less than 1 and sum 0, then

$$|z_1z_2+z_2z_3+z_3z_1|^2+|z_1z_2z_3|^2<r.$$

Solution. **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **George SHEN**.

For i=1,2,3, let $a_i=|z_i|^2$, then $0\le a_i<1$. Since $z_1+z_2+z_3=0$, we have

$$z_2\overline{z_3}+\overline{z_2}z_3$$
$$=(z_2+z_3)(\overline{z_2}+\overline{z_3})-|z_2|^2-|z_3|^2$$
$$=(-z_1)(-\overline{z_1})-a_2-a_3$$
$$=a_1-a_2-a_3.$$

Let $b=z_1z_2+z_2z_3+z_3z_1$ and $c=z_1z_2z_3$. Let the notation $\Sigma f(u,v,w)$ denote the sum of $f(u,v,w)$, $f(v,w,u)$ and $f(w,u,v)$. We have

$$|z_1z_2+z_2z_3+z_3z_1|^2+|z_1z_2z_3|^2$$
$$=b\overline{b}+c\overline{c}$$
$$=\sum|z_1z_2|^2+\sum|z_1|^2(z_2\overline{z_3}+\overline{z_2}z_3)+|z_1z_2z_3|^2$$
$$=\sum a_1a_2+\sum a_1(z_2\overline{z_3}+\overline{z_2}z_3)+a_1a_2a_3$$
$$=\sum a_1a_2+\sum a_1(a_1-a_2-a_3)+a_1a_2a_3$$
$$=a_1^2+a_2^2+a_3^2-a_1a_2-a_2a_3-a_3a_1+a_1a_2a_3$$
$$\le a_1+a_2+a_3-a_1a_2-a_2a_3-a_3a_1+a_1a_2a_3$$
$$=1-(1-a_1)(1-a_2)(1-a_3)<1.$$

Next, for $0<x<1$, consider $z_1=x$, $z_2=-x$ and $z_3=0$. Then $|z_1z_2+z_2z_3+z_3z_1|^2+|z_1z_2z_3|^2=x^4<r$. Letting x tend to 1, we get $1\le r$. Therefore, the least positive r is 1.

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **LEE Jae Woo** (Hamyang High School, South Korea) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Functional Inequalities

(Continued from page 2)

Solution. In (1), let $y=t-x$, then

$$f(t)\le tf(x)-xf(x)+f(f(x)). \qquad (2)$$

Consider $a,b\in\mathbb{R}$. Using (2) to $t=f(a)$, $x=b$ and $t=f(b)$, $x=a$, we get

$$f(f(a))-f(f(b)) \le f(a)f(b)-bf(b),$$
$$f(f(b))-f(f(a)) \le f(b)f(a)-af(a).$$

Adding these, we get

$$2f(a)f(b)\ge af(a)+bf(b).$$

Setting $b=2f(a)$, we get

$$2f(a)f(b)\ge af(a)+2f(a)f(b) \text{ or } af(a)\le 0.$$

Then for $a<0, f(a)\ge 0$. (3)

Now suppose $f(x)>0$ for some x. By (2), we see for every $t<(xf(x)-f(f(x)))/f(x)$, we have $f(t)<0$. This contradicts (3). So

$$f(x)\le 0 \quad \text{for all real } x. \qquad (4)$$

By (3) again, we get $f(x)=0$ for all $x<0$. Finally setting $t=x<0$ in (2), we get $f(x)\le f(f(x))$. As $f(x)=0$, this implies $0\le f(0)$. This together with (4) give $f(0)=0$.

Example 11 (2009 IMO Shortlisted Problem proposed by Belarus). Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x-f(y))>yf(x)+x. \qquad (1)$$

Solution. Assume the contrary, i.e. $f(x-f(y))\le yf(x)+x$ for all real x and y. Let $a=f(0)$. Setting $y=0$ in (1) gives $f(x-a)\le x$ for all real x. This is equivalent to

$$f(y)\le y+a \quad \text{for all real } y. \qquad (2)$$

Setting $x=f(y)$ in (1) and using (2), we get

$$a=f(0)\le yf(f(y))+f(y)\le yf(f(y))+y+a.$$

This implies $0\le y(f(f(y))+1)$ and so

$$f(f(y))\ge -1 \text{ for all } y>0. \qquad (3)$$

By (2) and (3), we get $-1\le f(f(y))\le f(y)+a$ for all $y>0$. So

$$f(y)\ge -a-1 \text{ for all } y>0. \qquad (4)$$

Next, we claim $f(x)\le 0$ for all real x. (5) Assume the contrary, i.e. there is some $f(x)>0$. Now take y such that $y<x-a$ and

$$y<(-a-x-1)/f(x). \qquad (6)$$

By (2), we get $x-f(y)\ge x-(y+a)>0$. By (1) and (4), we get

$$yf(x)+x\ge f(x-f(y))\ge -a-1.$$

Then $y\ge(-a-x-1)/f(x)$, contradicting (6). So (5) is true.

Now setting $y=0$ in (5) leads to $a=f(0)\le 0$ and using (2), we get

$$f(x)\le x \quad \text{for all real } x. \qquad (7)$$

Now choose $y>0$, $y>-f(-1)-1$ and set $x=f(y)-1$. By (1),(5) and (7), we get

$$f(-1)=f(x-f(y))$$
$$\le yf(x)+x=yf(f(y)-1)+f(y)-1$$
$$\le y(f(y)-1)-1\le -y-1.$$

Then $y\le -f(-1)-1$, which contradicts the choice of y.

Example 12 (64th Bulgarian Math Olympiad in 2015). Determine all functions $f:(0,+\infty)\to(0,+\infty)$ such that for arbitrary positive real numbers x, y, we have

(1) $f(x+y)\ge f(x)+y$;
(2) $f(f(x))\le x$.

Solution. As $y>0$, (1) implies f is strictly increasing on $(0,+\infty)$. By (2) and (1), we have

$$x+y\ge f(f(x+y))\ge f(f(x)+y). \qquad (*)$$

Using (*) and in (1), replacing x by y and y by $f(x)$, we get

$$x+y\ge f(f(x)+y)\ge f(x)+f(y). \qquad (**)$$

Since f is strictly increasing and $f(x)>0$, so the limit of $f(x)$ as $x\to 0^+$ is a nonnegative number c. By (2), the limit of $f(f(x))$ as $x\to 0^+$ is 0.

If $c>0$, then since f is strictly increasing, $f(f(x))\ge f(c)>0$. Taking the limit of $f(f(x))$ as $x\to 0^+$ leads to $0\ge f(c)>0$, contradiction. So $c=0$.

Now taking limit as $y\to 0^+$ in (**), we get $x\ge f(x)$ for all $x>0$. This and (1) lead to

$$x+y\ge f(x+y)\ge f(x)+y. \qquad (***)$$

Subtracting $f(x)+y$ in (***), we get $x-f(x)\ge f(x+y)-f(x)-y\ge 0$. Letting $w=x+y$ in (***) and taking limit of $w\ge f(w)\ge f(x)+w-x$ as $x\to 0^+$, we get $w=f(w)$. So $f(x+y)=f(w)=w=x+y$. Then f is the identity function on $(0,+\infty)$, which certainly satisfy (1) and (2).

Mathematical Excalibur

Volume 21, Number 4 *February 2018 – March 2018*

Olympiad Corner

Below were the problems of the 2017 Serbian Mathematical Olympiad for high school students. The event was held in Belgrade on March 31 and April 1, 2017.

Time allowed was 270 minutes.

First Day

Problem 1. (*Nikola Petrović*) Let a, b and c be positive real numbers with $a+b+c=1$. Prove the inequality

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} \le \sqrt{2-(a^2+b^2+c^2)}.$$

Problem 2. (*Dušan Djukić*) A convex quadrilateral $ABCD$ is inscribed in a circle. The lines AD and BC meet at point E. Points M and N are taken on the sides AD, BC respectively, so that $AM{:}MD=BN{:}NC$. Let the circum-circles of triangle EMN and quadrilateral $ABCD$ intersect at points X and Y. Prove that either the lines AB, CD and XY have a common point or they are all parallel.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 21, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Perfect Squares

Kin Y. Li

In this article, we will be looking at one particular type of number theory problems, namely problems on integers that have to do with the set of perfect squares 1, 4, 9, 16, 25, 36, …. This kind of problems have appeared in many Mathematical Olympiads from different countries for over 50 years. Here are some examples.

Example 1 (1953 Kürschák Math Competition Problems). Let n be a positive integer and let d be a positive divisor of $2n^2$. Prove that n^2+d is not a perfect square.

Solution. We have $2n^2=kd$ for some positive integer k. Suppose $n^2+d=m^2$ for some positive integer m. Then $m^2 = n^2+2n^2/k$ so that $(mk)^2=n^2(k^2+2k)$. Then k^2+2k must also be the square of a positive integer, but $k^2<k^2+2k<(k+1)^2$ leads to a contradiction.

Example 2 (1980 Leningrad Math Olympiad). Find all prime numbers p such that $2p^4-p^2+16$ is a perfect square.

Solution. For $p=2$, $2p^4-p^2+16=44$ is not a perfect square. For $p=3$, $2p^4-p^2+16 =169=13^2$. For prime $p>3$, $p \equiv 1$ or $2 \pmod 3$ and $2p^4-p^2+16\equiv 2 \pmod 3$. Assume $2p^4-p^2+16=k^2$. Then $k^2 \equiv 0^2,1^2$ or $2^2 \equiv 0$ or $1 \pmod 3$. So $2p^4-p^2+16 \ne k^2$. Then $p=3$ is the only solution.

Example 3 (2008 Singapore Math Olympiad). Find all prime numbers p satisfying 5^p+4p^4 is a perfect square.

Solution. Suppose $5^p+4p^4=q^2$ for some integer q. Then

$$5^p = q^2-4p^4 = (q-2p^2)(q+2p^2).$$

Since 5 is a prime number, we have

$$q-2p^2 = 5^s \text{ and } q+2p^2 = 5^t$$

for some integers s, t with $t > s \ge 0$ and $s+t = p$. Eliminating q, we have

$$4p^2 = 5^s(5^{t-s}-1).$$

If $s>0$, then from 5 divides $4p^2$, we get $p=5$. So $5^p+4p^4=5625=75^2$ and $q=75$ is a solution. If $s=0$, then $t=p$. So $5^p=4p^2+1$. Now, for integer $k\ge 2$, we claim $5^k>4k^2+1$. The case $k=2$ is clear. Suppose the case k is true. Then

$$\frac{4(k+1)^2+1}{4k^2+1} = 1+\frac{8k}{4k^2+1}+\frac{4}{4k^2+1} < 1+1+1<5.$$

So $5^{k+1}=5\times 5^k > 5(4k^2+1) > 4(k+1)^2+1$. By mathematical induction, the claim is true. Therefore, $5^p = 4p^2+1$ has no prime solution p.

Example 4 (2009 Croatian Math Olympiad). Find all positive integers m, n such that 6^m+2^n+2 is a perfect square.

Solution. If

$$6^m+2^n+2=2(3^m\times 2^{m-1}+2^{n-1}+1)$$

is a perfect square, then $3^m\times 2^{m-1}+2^{n-1}+1$ is even. So one of the integers $3^m\times 2^{m-1}$ and 2^{n-1} is odd and the other is even.

Suppose $3^m\times 2^{m-1}$ is odd, then $m = 1$ and $6^m+2^n+2 = 8+2^n = 4(2^{n-2}+2)$. So $2^{n-2}+2$ is a perfect square. Since every perfect square dived by 4 has remainder 0 or 1, so $2^{n-2}+2$ cannot be of the form 4k+2. Hence, $n-2=1$, i.e. $n=3$. So $(m,n)=(1,3)$ is a solution.

If 2^{n-1} is odd, then $n=1$ and

$$6^m+2^n+2 = 6^m+4 \equiv (-1)^m+4 \pmod 7.$$

This means 6^m+2^n+2 divided by 7 has remainder 3 or 5. However,

$(7k)^2 \equiv 0 \pmod 7$, $(7k\pm 1)^2 \equiv 1 \pmod 7$, $(7k\pm 2)^2\equiv 4 \pmod 7$, $(7k\pm 3)^2\equiv 2 \pmod 7$.

So every perfect square divided by 7 cannot have remainder 3, 5 or 6. Therefore, $(m,n) = (1,3)$ is the only solution.

(continued on page 2)

Example 5 (2008 German Math Olympiad). Determine all real numbers x such that $4x^5-7$ and $4x^{13}-7$ are perfect squares.

Solution. Suppose there are positive integers a and b such that

$$4x^5-7=a^2 \quad\text{and}\quad 4x^{13}-7=b^2.$$

Then $x^5=(a^2+7)/4>1$ is rational and $x^{13}=(b^2+7)/4>1$ is rational. So $x=(x^5)^8/(x^{13})^3$ is rational. Suppose $x=p/q$ with p and q positive relatively prime integers. Then from $(p/q)^5=(a^2+7)/4$, it follows q^5 divides $4p^5$ and so $q=1$. So x must be a positive integer and $x\geq 2$.

In the case x is an odd integer, we have $a^2\equiv 0, 1, 4 \pmod 8$, but $a^2=4x^5-7\equiv 5 \pmod 8$, contradiction. So x is even. In the case $x=2$, we have $4x^5-7=11^2$ and $4x^{13}-7=181^2$. For an even $x\geq 4$, $(ab)^2=(4x^5-7)(4x^{13}-7)=16x^{18}-28x^{13}-28x^7+49$. However, expanding $(4x^9-7x^4/2-1)^2$ and $(4x^9-7x^4/2)^2$ and using $x^9\geq 4x^8\geq 4^2x^7\geq 4^5x^4$, we see $(ab)^2$ is strictly between them. Then $x=2$ is the only solution.

Example 6 (2011 Iranian Math Olympiad). Integers a, b satisfy $a>b$. Also $ab-1$, $a+b$ are relatively prime and $ab+1$, $a-b$ are relatively prime. Prove that $(a+b)^2+(ab-1)^2$ is not a perfect square.

Solution. Assume $(a+b)^2+(ab-1)^2=c^2$ for some integer c. Then

$$c^2=a^2+b^2+a^2b^2+1=(a^2+1)(b^2+1).$$

Assume (*) there is a prime p such that $p\mid a^2+1$ and $p\mid b^2+1$, then $p\mid a^2+1-b^2+1=a^2-b^2$. So (**) $p\mid a-b$ or $p\mid a+b$.

Assume $p\mid a-b$. Then $p\mid ab-b^2$. Since $p\mid b^2+1$, so $p\mid ab-b^2+b^2+1=ab+1$, which contradicts $ab+1$, $a-b$ are relatively prime. Similarly, assume $p\mid a+b$. Then $p\mid ab+b^2$. Since $p\mid a^2+1$, so $p\mid ab+b^2+b^2-1=ab-1$, which contradicts $ab-1$, $a+b$ are relatively prime. So (**) as well as (*) are wrong.

Then a^2+1, b^2+1 are relatively prime. Since $a>b$, not both of them are 0. So $(a+b)^2+(ab-1)^2$ equals a^2+1 (if $b=0$) or b^2+1 (if $a=0$) or $(a^2+1)(b^2+1)$. Then $(a+b)^2+(ab-1)^2$ is not a perfect square.

Example 7 (2000 Polish Math Olympiad). Let m, n be positive integers such that m^2+n^2+m is divisible by mn. Prove that m is a perfect square.

Solution. Since m^2+n^2+m is divisible by mn, so for some positive integer k, $m^2+n^2+m=kmn$. Then $n^2-kmn+(m^2+m)=0$, which can be viewed as a quadratic equation in n. Then the discriminant $\Delta=k^2m^2-4m^2-4m$ is a perfect square. Suppose d is $\gcd(m, k^2m-4m-4)=1$. If $d=1$, then m (and $k^2m-4m-4$) are both perfect squares. If $d>1$, then

$$d=\gcd(m, k^2m-4m-4)=\gcd(m,4).$$

Since $d>1$ divides 4, so d is even. Then m is even. Also, $n^2\equiv m^2+n^2+m \pmod 2$. So n is even. Then mn, m^2+n^2 are divisible by 4.

As m^2+n^2+m is given to be divisible by mn, so m^2+n^2+m is divisible by 4. Then $m=m^2+n^2+m-(m^2+n^2)$ is divisible by 4. So we get $d=4$. Then

$$1=\gcd(m/4,\ k^2(m/4)-m-1).$$

Now $\Delta/16=(m/4)(k^2(m/4)-m-1)$ is a perfect square. So $m/4$ and $k^2(m/4)-m-1$ are perfect squares. Therefore, m is a perfect square.

Example 8 (2006 British Math Olympiad). Let n be an integer If $2+2\sqrt{1+12n^2}$ is an integer, then it is a perfect square.

Solution. If $2+2\sqrt{1+12n^2}$ is an integer, then $1+12n^2$ is a perfect square. Suppose $1+12n^2=m^2$ for some odd positive integer m. Then $12n^2=(m+1)(m-1)$. Let t be the integer $(m+1)/2$ and we have (*) $t(t-1)=3n^2$.

Now we claim $2+2\sqrt{1+12n^2}=2+2m=4t$ is a perfect square. By (*), we see $t-1$ or t is divisible by 3. Now $\gcd(t-1, t)=1$. Assume t is divisible by 3, then $(t/3)(t-1)=n^2$ and both $t/3$ and $t-1$ are perfect squares. Let $t/3=k^2$ for some integer k, Then $t-1=3k^2-1\equiv 2\not\equiv 0^2, 1^2$ or $2^2 \pmod 3$, contradiction. So $t-1$ is divisible by 3. Then we have $\gcd(t, (t-1)/3)=1$. From $t\times(t-1)/3=n^2$, we see t is a perfect squares. So the claim is true.

Example 9 (2002 Australian Math Olympiad). Find all prime numbers p, q, r such that p^q+p^r is a perfect square.

Solution. If $q=r$, then $p^q+p^r=2p^q$. So $p=2$ and q is an odd prime at least 3. All prime triples $(p,q,r)=(2,q,q)$ are solutions.

If $q\neq r$, then without loss of generality, let $q<r$ and so $p^q+p^r=p^q(1+p^s)$, where $s=r-q$ is at least 1. Since p^q and $1+p^s$ are relatively prime, so they are both perfect squares. Then, the prime q is 2. Also, since $1+p^s$ is a perfect square, $1+p^s=u^2$ for some positive integer u. Then

$$p^s=u^2-1=(u+1)(u-1).$$

Since $\gcd(u+1,u-1)=1$ or 2, so if it is 2, then u is odd and p is even. Hence, $p=2$ and both $u+1$ and $u-1$ are powers of 2. Then u can only be 3 and $1+p^s=3^2$ so that $p=2$, $s=3$, $r=q+s=2+3=5$. These lead to the solutions $(p,q,r)=(2,2,5)$ or $(2,5,2)$.

If $\gcd(u+1,u-1)=1$, then u is even and $u-1$ must be 1 (otherwise $u+1$ and $u-1$ have different odd prime factors and cannot be powers of the same prime). Then $u=2$, $p^s=(u-1)(u+1)=3$, $p=3$, $s=1$, $r=q+s=3$. The only such prime triples are $(p,q,r)=(3,2,3)$ or $(3,3,2)$.

Then all the solutions are $(p,q,r)=(2,2,5)$, $(2,5,2)$, $(3,2,3)$, $(3,3,2)$ and $(2,q,q)$ with q being a prime at least 3.

Example 10 (2008 USA Team Selection Test). Let n be a positive integer. Prove that n^7+7 is not a perfect square.

Solution. Assume $n^7+7=x^2$ for some positive integer x. Then

(1) n is odd (for otherwise $x^2\equiv 3 \pmod 4$, which is false).
(2) $n\equiv 1 \pmod 4$ (due to n odd and $x^2\not\equiv 2 \pmod 4$).

(3) $x^2+11^2=n^7+128=(n+2)N$, where N is $n^6-2n^5+4n^4-8n^3+16n^2-32n+64$.

(4) If $11\nmid x$, then every prime factor p of x^2+11^2 must be odd and $p\equiv 1 \pmod 4$ (for if $p=4k+3$, then $x^2\equiv -11^2 \pmod p$ and by Fermat's little theorem, $x^{p-1}\equiv -11^{p-1}\equiv -1 \pmod p$), contradiction).

From (3), we get $n+2\mid x^2+11^2$, $n+2\equiv 3 \pmod 4$ implies x^2+11^2 has a prime factor congruent 3 (mod 4), which contradicts (4).

If $x=11y$ for some integer y, then (3) becomes $121(y^2+1)=(n+2)N$, but checking $n\equiv -5$ to $5 \pmod{11}$, we see N is not a multiple of 11. So $n+2$ is a multiple of 121, say $M=(n+2)/121$. Then $y^2+1=MN$. Similarly, it can be checked that every prime factor of y^2+1 is congruent to 1 (mod 4). Hence, every odd factor of y^2+1 is congruent to 1 (mod 4). However, $M\equiv 3 \pmod 4$, so $y^2+1=MN$ cannot be true. Therefore, n^7+7 is not a perfect square.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 21, 2018.***

Problem 511. Let $x_1,x_2,\ldots,x_{40}$ be positive integers with sum equal to 58. Find the maximum and minimum possible value of $x_1^2+x_2^2+\cdots+x_{40}^2$.

Problem 512. Let AD, BE, CF be the altitudes of acute ΔABC. Points P and Q are on segments DF and EF respectively. If $\angle PAQ=\angle DAC$, then prove that AP bisects $\angle FPQ$.

Problem 513. Let $a_0,a_1,a_2,\ldots$ be a sequence of nonnegative integers satisfying the conditions:

(1) $a_{n+1}=3a_n-3a_{n-1}+a_{n-2}$ for $n>1$,
(2) $2a_1=a_0+a_2-2$,
(3) for every positive integer m, in the sequence $a_0,a_1,a_2,\ldots$, there exist m terms $a_k,a_{k+1},\ldots,a_{k+m-1}$, which are perfect squares.

Prove that every term in $a_0,a_1,a_2,\ldots$ is a perfect square.

Problem 514. Let n be a positive integer and let $p(x)$ be a polynomial with real coefficients on the interval $[0,n]$ such that $p(0)=p(n)$. Prove that there are n distinct ordered pairs (a_i, b_i) with $i=1,2,\ldots,n$ such that $0\le a_i<b_i\le n$, b_i-a_i is an integer and $p(a_i)=p(b_i)$.

Problem 515. There are ten distinct nonzero real numbers. It is known that for every two of the numbers, either the sum or the product of them is rational. Prove that the square of each of the ten numbers is rational.

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Solutions

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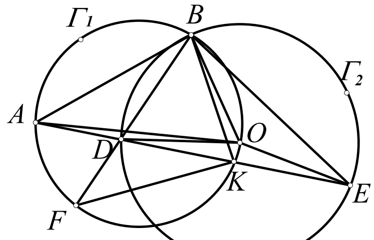
Problem 506. Points A and B are on a circle Γ_1. Line AB is tangent to another circle Γ_2 at B and the center O of Γ_2 is on Γ_1. A line through A intersects Γ_2 at points D and E (with D between A and E). Line BD intersects Γ_1 at a point F, different from B. Prove that D is the midpoint of BF if and only if BE is tangent to Γ_1.

Solution. **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School) and **George SHEN**.



Let point K be the intersection of Γ_1 with line DE. Then $\Delta KFD \sim \Delta ABD$. Since $\angle ABD = \angle AEB$, so $\Delta ABD \sim \Delta AEB$. Then $\Delta KFD \sim \Delta AEB$. Hence, $FD/DK = AB/BE$.

Let O be the center of Γ_2. Since $OB\perp AB$, AO is a diameter of Γ_1. So $AK\perp OK$. Then $\angle DKO =\angle AKO = 90^\circ$. So $DK=EK$. Now BE is tangent to $\Gamma_1 \Leftrightarrow \angle EBK = \angle BAD \Leftrightarrow \Delta EBK \sim \Delta BAD \Leftrightarrow AB/BE=DB/KE \Leftrightarrow FD/DK=DB/KE \Leftrightarrow FD=DB$ (i.e. D is the midpoint of BF).

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **Jae Woo LEE** (Hamyang High School, South Korea), **LIN Meng Fei**, **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 507. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y \in\mathbb{R}$,

$$(x-2)f(y) + f(y+2f(x)) = f(x+yf(x)). \quad (*)$$

Solution. **DBS Maths Solving Team** (Diocesan Boys' School), **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Jae Woo LEE** (Hamyang High School, South Korea) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If $f(0)=0$, then setting $x=0$ in (*) yields $f(y)=0$ for all $y \in\mathbb{R}$, i.e. f is the zero function, which is a solution of (*).

If $f(0)\neq 0$, then setting $y=0$ in (*) yields $(x-2)f(0) + f(2f(x)) = f(x)$ for all $x \in\mathbb{R}$. Now $f(x) = f(y)$ implies $(x-2)f(0) + f(2f(x)) = f(x) = f(y) = (y-2)f(0) + f(2f(y)) = (y-2)f(0) + f(2f(x))$ yielding $x=y$. So f is injective.

Setting $x=2$ in (*) yields $f(y+2f(2)) = f(2+yf(2))$ for all $y\in\mathbb{R}$. Since f is injective, $y+2f(2)=2+yf(2)$ for all $y\in\mathbb{R}$. Setting $y=0$, we get $f(2)=1$. Since f is injective, $f(3)\neq 1$. Setting $x=3$, $y=3/(1-f(3))$ in (*), we get $f(3/(1-f(3))+2f(3))=0$. Thus, f has a root at $a =3/(1-f(3))+2f(3)$. Setting $y=a$ in (*), we get $f(a+2f(x))=f(x+af(x))$ for all $x\in\mathbb{R}$. Since f is injective, we get $a+2f(x) = x+af(x)$. Now $a\neq 2$. So $f(x) = (x-a)/(2-a)$. Putting this in (*), we get $a=1$. Then the function can only be (1) $f(x)=0$ for all $x\in\mathbb{R}$ or (2) $f(x)=x-1$ for all $x\in\mathbb{R}$. Putting these in (*) show they are in fact solutions of (*).

Other commended solvers: **Yagub N. ALIYEV** (Problem Solving Group of ADA University, Baku, Azerbaijan) and **Akash Singha ROY** (West Bengal, India).

Problem 508. Determine the largest integer k such that for all integers x,y, if $xy+1$ is divisible by k, then $x+y$ is also divisibly by k.

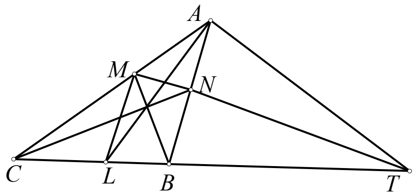
Solution. **George SHEN.**

Let k be such an integer. Let S be the set of all integers x such that $\gcd(x,k)=1$. For x in S, choose integer m in $[1,k-1]$ such that $mx^2\equiv -1 \pmod k$. Let $y=mx$, then $k\,|\,xy+1$. So $k\,|\,x+y$ and $k\,|\,(x+y)x-(xy+1) = x^2-1$. Then for every x in S, every prime factor p of k satisfies $x^2\equiv 1 \pmod p$. If all prime factors p of k are at least 5, then $x=2$, 3 are in S, but $x^2\equiv 1 \pmod p$ fails. So the prime factors of k can only be 2 or 3. So k is of the form 2^r3^s and $S=\{x\colon \gcd(x,2)=1=\gcd(x,3)\}$ Then for $x=5$ in S, $x^2\equiv 1 \pmod{2^r}$ implies $2^r\,|\,24$ and so $r\le 3$. Also, for $x=5$ in S, $x^2\equiv 1 \pmod{3^s}$ implies $3^s\,|\,24$ and so $s\le 1$. Then $k\le 2^33=24$.

Finally, for $k=24$, $xy\equiv -1 \pmod{24}$ implies $\gcd(x,24) = 1 = \gcd(y,24)$. Then $x,y \equiv 1, 5, 7, 11, 13, 17, 19$ or $23 \pmod{24}$. The only possible cases for $xy\equiv -1 \pmod{24}$ are $\{x,y\} = \{1,23\}$, $\{5,19\}$, $\{7,17\}$, $\{11,23\}$. Then $24\,|\,x+y$. So $k=24$ is the required largest integer.

Other commended solvers: **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4) and **DBS Maths Solving Team** (Diocesan Boys' School), **Jae Woo LEE** (Hamyang High School, South Korea), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 509. In ΔABC, the angle bisector of $\angle CAB$ intersects BC at a point L. On sides AC, AB, there are points M, N respectively such that lines AL, BM, CN are concurrent and $\angle AMN =\angle ALB$. Prove that $\angle NML= 90^\circ$.



Solution 1. **Apostolis MANOLOUDIS** and **George SHEN**.

Let $T=MN\cap BC$. From $\angle AMT=\angle AMN=\angle ALB=\angle ALT$, we get A, M, L, T are concyclic. So $\angle NML=\angle TML=\angle TAL$. To get $\angle TAL=90°$, it suffices to show AT is the exterior bisector of $\angle CAB$.

By Menelaos' theorem, as M,N,T are collinear, $(AM/MC)(CT/TB)(BN/NA)=1$. By Ceva's theorem, as AL, BM, CN concur, $(AM/MC)(CL/LB)(BN/NA)=1$. Then $CL/LB=CT/TB$. By the angle bisector theorem, $CA/AB=CL/LB=CT/TB$. So AT is the external bisector of $\angle CAB$.

Solution 2. **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

AL, BM, CN concurrent implies *T, B, L, C* is a harmonic range of points. Then $\angle AMT=\angle AMN=\angle ALB=\angle ALT$ led to *T, A, M, L* concyclic. By Apollonius' Theorem, $90°=\angle TAL=\angle NML$.

Other commended solvers: **Jae Woo LEE** (Hamyang High School, South Korea), **LEUNG Hei Chun** (SKH Tang Shiu Kin Secondary School), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" School, Buzău, Romania).

Problem 510. Numbers 1 to 20 are written on a board. A person randomly chooses two of these numbers with a difference of at least 2. He adds 1 to the smaller one and subtracts 1 from the larger one. Then he performs an operation by replacing the original two chosen numbers on the board with the two new numbers. Determine the maximum number of times he can do this operation.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Note after each operation, the sum of the numbers is always 210. Suppose the person chooses m,n with $m-n\ge 2$, then $(m-1)^2+(n+1)^2=n^2+m^2+2-2(m-n)\le n^2+m^2-2$ with equality only for $m-n=2$. If the absolute value of the difference of the two numbers is 1, then the operation does not change anything. At the end, the board has ten 10's and ten 11's.

In the beginning, the sum of the squares is $1^2+2^2+\cdots+20^2=2870$ and at the end, it is $10\times(10^2+11^2)=2210$. After each operation, the sum of squares reduces by at least 2, so the number of operation that can be done is at most $(2870-2210)/2=330$. Below we will show the person can do 330 operations with the absolute values of the difference of the two numbers is 2.

The plan is to eliminate the minimum and the maximum of the remaining numbers until we get only 10's and 11's. In round 1, we eliminate 1's and 20's by operating on pairs (1,3), (2,4), …, (18,20) one time for every pair. In round 2, we eliminate 2's and 19's by operating on pairs (2,4), (3,5), …, (17,19) two times for every pair. Keep on eliminating in this way until we have only 9's, 10's, 11's and 12's. In round 9, we eliminate 9's and 12's by operating on pairs (9,11) and (10,12) nine times. The total number of operations is $18\times1+16\times2+\cdots+2\times9=330$.

Olympiad Corner

(*Continued from page 2*)

Problem 3. (*Dušan Djukić*) There are $2n-1$ bulbs in a line. Initially, the central (n-th) bulb is on, whereas all others are off. A step consists of choosing a string of at least three (consecutive) bulbs, the leftmost and rightmost ones being off and all between them being on, and changing the states of all bulbs in the string (for instance, the configuration ●○○○● will turn into ○●●●○). At most how many steps can be performed?

__Second Day__

Problem 4. (*Dušan Djukić*) Suppose that a positive integer a is such that, for any positive integer n, the number n^2a-1 has a divisor greater than 1 and congruent to 1 modulo n. Prove that a is a perfect square.

Problem 5. (*Bojan Bašić* and *PSC*) Determine the maximum number of queens that can be placed on a 2017×2017 chessboard so that each queen attacks at most one of the others.

Problem 6. (*Dušan Djukić*) Let k be the circumcircle of triangle ABC, and let k_a be its excircle opposite to A. The two common tangents of k and k_a meet the line BC at points P and Q. Prove that $\angle PAB=\angle QAC$.

Perfect Squares

(*Continued from page 2*)

__*Example 11 (2006 Thai Math Olympiad).*__ Determine all prime numbers p such that $(2^{p-1}-1)/p$ are perfect squares.

__*Solution.*__ For every prime number p, let $f(p)=(2^{p-1}-1)/p$. We will show for $p>7$, $f(p)$ is not a perfect square.

Assume there is a prime $p>7$ such that $2^{p-1}-1=pm^2$ for some positive integer m. Then m must be odd. Now there are two cases, (1) p is of the form $4k+1$ with $k>1$ or (2) p is of the form $4k+3$ with $k>1$.

In case (1), we have $2^{p-1}-1=pm^2=(4k+1)m^2\equiv 1 \pmod 4$, but also $2^{p-1}-1=2^{4k}-1\equiv 3 \pmod 4$, which is a contradiction.

In case (2), we have $2^{p-1}-1=2^{4k+2}-1=(2^{2k+1}-1)(2^{2k+1}+1)=pm^2$.

Since $\gcd(2^{2k+1}-1,2^{2k+1}+1)=1$, again we have two subcases:

(a) $2^{2k+1}-1=u^2$, $2^{2k+1}+1=pv^2$ for some positive integers u, v;

(b) $2^{2k+1}-1=pu^2$, $2^{2k+1}+1=v^2$ for some positive integers u, v.

In subcase (a), since $k>1$, $2^{2k+1}+1\equiv1 \pmod 4$, but $pv^2\equiv3\times1=4 \pmod 4$, which is a contradiction.

In subcase (b), we have $2^{2k+1}=v^2-1=(v-1)(v+1)$. Then $v-1=2^s$, $v+1=2^t$ for some positive integers $s<t$. Observe that $2^{t-s}=(v+1)/(v-1)=2/(v-1)+1$. Then $v=2$ or 3. If $v=2$, then $2^{2k+1}+1=v^2=4$, which is a contradiction. If $v=3$, then $2^{2k+1}=v^2-1=8$ leads to $k=1$, which is a contradiction as $k>1$.

Finally, checking the cases $p=2,3,5,7$, we see only cases $p=3$ and 7 have solutions $(2^{3-1}-1)/3=1^2$ and $(2^{7-1}-1)/7=3^2$.

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2017 Serbian IMO Team Selection Competition for high school students. The event was held in Belgrade on May 21 and 22, 2017.

Time allowed was 270 minutes per day.

First Day

Problem 1. (*Dušan Djukić*) Let D be the midpoint of side BC of a triangle ABC. Points E and F are taken on the respective sides AC and AB such that $DE=DF$ and $\angle EDF=\angle BAC$. Prove that

$$DE \geq \frac{AB+AC}{4}.$$

Problem 2. (*Bojan Bašić*) Given an ordered pair of positive integers (x,y) with exactly one even coordinate, a *step* maps this pair to $(x/2, y+x/2)$ if $2|x$, and to $(x+y/2, y/2)$ if $2|y$. Prove that for every odd positive integer $n>1$ there exists an even positive integer b, $b<n$, such that after finitely many steps the pair (n,b) maps to the pair (b,n).

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 31, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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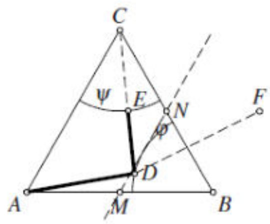
Strategies and Plans

Kin Y. Li

In this article, we will be looking at some Math Olympiad problems from different countries and regions. Some require strategies or plans to perform certain tasks. We hope these arouse your interest. Here are the examples.

<u>*Example 1 (1973 IMO).*</u> A soldier has to investigate whether there are mines in an area that has the form of an equilateral triangle. The radius of his detector is equal to one-half of an altitude of the triangle. The soldier starts from one vertex of the triangle. Determine the shortest path that the soldier has to traverse in order to check the whole region.

<u>*Solution.*</u> Suppose that the soldier starts at the vertex A of the equilateral triangle ABC of side length a. Let φ and ψ be the arcs of circles with centers B and C and radii $a\sqrt{3}/4$ respectively, that lie inside the triangle. In order to check the vertices B and C he must visit some point D in φ and E in ψ.



Thus his path cannot be shorter than the path ADE (or AED) itself. The length of the path ADE is $AD+DE\geq AD+DC-a\sqrt{3}/4$. Let F be the reflection of C across the line MN, where M and N are the midpoints of AB and BC respectively. Then $DC\geq DF$ and hence $AD+DC\geq AD+DF\geq AF$. So

$$AD+DE \geq AF-\frac{a\sqrt{3}}{4}=a\left(\frac{\sqrt{7}}{2}-\frac{\sqrt{3}}{4}\right)$$

with equality if and only if D is the midpoint of arc φ and E is the intersection point of CD and arc ψ. In following the path ADE, the soldier will check the whole region. Therefore, this path (as well as the one symmetric to it) is the shortest path the soldier can check the whole field.

<u>*Example 2 (2011 Saudi Arabia Math Competition).*</u> A Geostationary Earth Orbit is situated directly above the equator and has a period equal to the Earth's rotational period. It is at the precise distance of 22,236 miles above the Earth that a satellite can maintain an orbit with a period of rotation around the Earth exactly equal to 24 hours. Because the satellites revolve at the same rotational speed of the Earth, they appear stationary from the Earth surface. That is why most stationary antennas (satellite dishes) do not need to move once they have been properly aimed at a target satellite in the sky. In an international project, a total of ten stations were equally spaced on this orbit (at the precise distance of 22,236 miles above the equator). Given that the radius of the Earth is 3960 miles, find the exact straight distance between two neighboring stations. Write your answer in the form $a+b\sqrt{c}$, where a, b, c are integers and $c>0$ is square-free.

<u>*Solution.*</u> Let A and B be neighboring stations and O be the center of the Earth. Now $\angle AOB=36°$. Let $\theta=18°$. Then $AB=2R\sin\theta$, where $R = 22236 + 3960 =26196$. Since we have $\sin 36°=\cos 54°$, so $\sin 2\theta=\cos 3\theta$. That is, $2\cos\theta\sin\theta = 4\cos^3\theta-3\cos\theta$. Dividing by $\cos\theta$ and expressing in terms of $\sin\theta$, we get $4\sin^2\theta+2\sin\theta-1=0$. Using the quadratic formula, we have $\sin\theta=(\sqrt{5}-1)/4$. Then $AB=2R\sin\theta=13098(\sqrt{5}-1)$. So $a = -13098$, $b = 13098$ and $c = 5$.

<u>*Example 3 (2008 German National Math Competition).*</u> On a bookshelf, there are n books ($n\geq3$) from different authors standing side by side. A librarian inspects the two leftmost books and changes their places if and

(continued on page 2)

only if they are not in alphabetical order. Afterward, he does the same to the second and the third book from the left and so on. Acting this way, he passes the whole row of books three times in total. Determine the number of different starting arrangements for which the books will finally be ordered alphabetically.

<u>*Solution.*</u> There are exactly $6\cdot 4^{n-3}$ arrangements for which the books are in order after 3 runs. For a proof, we number the positions and the books in alphabetical order from 1 to n. Obviously, for the position of $p(k)$ of book number k at the beginning it is necessary that $p(k)-k\le 3$. Now this condition is also sufficient: At every ordering run, all of the books standing right to their correct place are shifted one place to the left. On the other hand, no book can be shifted to the right beyond its correct place because if there is a book at position $p(k)$ with $p(k)>k$, there must be at least one book on the left side of $p(k)$ with its number larger than $p(k)$. Such a book takes over any book with a number smaller than $p(k)$.

The number given in the answer is then calculated by regarding that each of the books with numbers $1,2,\ldots, n-4$ that is not occupied by a book with a smaller number. For the last three books there are only 3, 2 and 1 places left. Hence the result follows.

<u>*Example 4 (2000 Russian Math Olympiad).*</u> Two pirates divide their loot, consisting of two sacks of coins and one diamond. They decide to use the following rules. On each turn, one pirate chooses a sack and takes $2m$ coin from it, keeping m for himself and putting the rest into the other sack. The pirates alternatively taking turns until no more moves are possible; the first pirate unable to make a move loses the diamond, and the other pirate takes it. For which initial numbers of coins can the first pirate guarantee that he will obtain the diamond?

<u>*Solution.*</u> We claim that if there are x and y coins left in the two sacks, respectively, then the next player P_1 to move has a winning strategy if and only if $|x-y|>1$. Otherwise, the other player P_2 has a winning strategy.

We prove the claim by induction on the total numbers of coins, $x+y$. If $x+y=0$, then no moves are possible and the next player does not have a winning strategy. Now assuming that the claim is true when $x+y\le n$ for some nonnegative n, we prove that it is true when $x+y=n+1$.

First consider the case $|x-y|\le 1$. Assume that a move is possible. Otherwise, the next player P_1 automatically loses, in accordance with our claim. The next player must take $2m$ coins from one sack, say the one containing x coins, and put m coins into the sack containing y coins. Hence the new difference between the number of coins in the sacks is

$$|(x-2m)-(y+m)| \ge |-3m| - |y-x| \ge 3-1=2.$$

At this point, there are now a total of $x+y-m$ coins in the sacks, and the difference between the numbers of coins in the two sacks is at least 2. Thus, by induction hypothesis, P_2 has a winning strategy. This proves the claim when $|x-y|\le 1$.

Now consider the case $|x-y|\ge 2$. Without loss of generality, let $x>y$. P_1 would like to find a m such that $2m\le x$, $m\ge 1$ and

$$|(x-2m)-(y+m)|\le 1.$$

The number $m=[(x-y-1)/3]$ satisfies the last two inequalities above and we claim $2m\le x$ as well. Indeed, $x-2m$ is nonnegative because it differs by at most 1 from the positive number $y+m$. After taking $2m$ coins from the sack with x coins, P_1 leaves a total of $x+y-m$ coins, where the difference between the numbers of coins in the sacks is at most 1. Hence, by the induction hypothesis, the other player P_2 has no winning strategy. It follows that P_1 has a winning strategy, as desired.

This completes the proof of the induction and of the claim. It follows that the first pirate can guarantee that he will obtain the diamond if and only if the number of coins initially in the sacks differs by at least 2.

<u>*Example 5 (2015 Croatian National Math Competition).*</u> In a country between every two cities there is a direct bus or a direct train line (all lines are two-way and they don't pass through any other city). Prove that all cities in that country can be arranged in two disjoint sets so that all cities in one set can be visited using only train so that no city is visited twice, and all cities in the other set can be visited using only bus so that no city is visited twice.

<u>*Solution.*</u> Let G be the set of all cities in the country. For disjoint subsets A, Z of G, we call a pair (A,Z) *good* if all cities in the set A can be visited using only bus such that no city is visited twice and all cities in the set Z can be visited using only train such that no city is visited twice.

Let (A,Z) be a good pair such that $A\cup Z$ has the maximum number of elements. If we prove $A\cup Z=G$, then the statement of the problem will follow.

Let us assume the opposite, i.e. there is a city g which is not from A nor Z. Without loss of generality we can assume that A and Z are non-empty because otherwise we can transfer any city from a non-empty set to an empty one.

Let n be the number of cities in the set A and m be the number of cities in the set Z. Let us arrange the cities from A in the series $a_1,\ldots,a_n$ such that every two consecutive cities in that series are connected by a direct bus line. Also, let us arrange the cities from Z in the series $z_1,\ldots,z_m$ such that every two consecutive cities in that series are connected by a direct train line.

Since we assumed that the pair (A,Z) is maximum, the cities g and a_1 have to be connected by train (otherwise the pair $(A\cup\{g\},Z)$ would be a good pair whose union would have more elements than $A\cup Z$, and g and z_1 have to be connected by bus (otherwise the pair $(A,Z\cup\{g\})$ would be a good pair whose union would have more element than $A\cup Z$).

The cities a_1 and z_1 have to be connected by bus or by train. If a_1 and z_1 are connected by bus, let us put $A'=\{z_1,g,a_1,\ldots,a_n\}$ and $Z'=\{z_2,\ldots,z_m\}$. Then (A',Z') is a good pair and the number of elements of $A'\cup Z'$ is greater than the number of elements of $A\cup Z$, which contradicts the assumption.

If a_1 and z_1 are connected by train, let us put $A''=\{a_2,\ldots,a_n\}$ and $Z''=\{a_1,g,z_1,\ldots,z_m\}$. Then (A'',Z'') is a good pair and the number of elements of $A''\cup Z''$ is greater than the number of elements of $A\cup Z$, which contradicts the assumption.

Since all cases lead to contradiction, we conclude that the assumption was wrong and that every city is either in the set A or in the set Z.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 31, 2018.***

Problem 516. Determine all triples (p,m,n) of positive integers such that p is prime and $2^m p^2+1=n^5$ holds.

Problem 517. For all positive x and y, prove that

$$x^2y^2(x^2+y^2-2) \ge (xy-1)(x+y).$$

Problem 518. Let I be the incenter and AD be a diameter of the circumcircle of ΔABC. Let point E be on the ray BA and point F be on the ray CA. If the lengths of BE and CF are both equal to the semiperimeter of ΔABC, then prove that lines EF and DI are perpendicular.

Problem 519. Let A and B be subsets of the positive integers with 10 and 9 elements respectively. Suppose for every $x,y,u,v\in A$ satisfying $x+y=u+v$, we have $\{x,y\}=\{u,v\}$. Prove that the set $A+B=\{a+b: a\in A, b\in B\}$ has at least 50 elements.

Problem 520. Let P be the set of all polynomials $f(x)=ax^2+bx$, where a,b are nonnegative integers less than 2010^{18}. Find the number of polynomials f in P for which there is a polynomial g in P such that $g(f(k))\equiv k \pmod{2010^{18}}$ for all integers k.

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Solutions

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Problem 511. Let $x_1,x_2,\ldots,x_{40}$ be positive integers with sum equal to 58. Find the maximum and minimum possible value of $x_1^2+x_2^2+\cdots+x_{40}^2$.

Solution. **Arpon BASU** (AECS-4, Mumbai, India), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **William KAHN** (Sidney, Australia), **LAI Wai Lok** (La Salle Primary School), **LEUNG Hei Chun**, **LUI On Ki**, **George SHEN**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

If there exist x_m, $x_n\ge 2$, then we can replace them by x_m+x_n-1, 1 due to

$$(x_m+x_n-1)^2+1^2-(x_m^2+x_n^2) = 2(x_m-1)(x_n-1) \ge 0.$$

So the maximum case can be attained by one 19 and thirty-nine 1's. This gives the maximum value $39\times1^2+1\times19^2=400$.

For the minimum case, there exists at least one 1, otherwise $58=x_1+x_2+\cdots+x_{40}\ge 2\times40=80$, contradiction. Let x_k be a largest term. If $x_k\ge3$, then we can replace x_k and 1 by x_k-1 and 2 to lower the square sums since

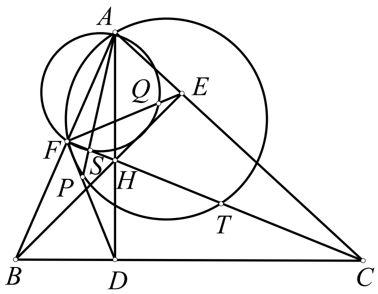
$$(x_k^2+1^2)-[(x_k-1)^2+2^2] = 2(x_k-2) > 0.$$

So in the minimum case, there are twenty-two 1's and eighteen 2's yielding $22\times1^2+18\times2^2=94$.

Other commended solvers: **George SHEN** and **Nicușor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 512. Let AD, BE, CF be the altitudes of acute ΔABC. Points P and Q are on segments DF and EF respectively. If $\angle PAQ=\angle DAC$, then prove that AP bisects $\angle FPQ$.

Solution. **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Let H be the orthocenter of ΔABC. Let S be the intersection of AP and CF. Let T be the intersection of AQ and CF. Now $\angle AFC=90°=\angle ADC$. As $AFDC$ is cyclic,

$$\angle PAT=\angle PAQ=\angle DAC=\angle DFC=\angle PFT,$$

points A, T, F, P are concyclic. Also, since

$$\angle SFQ=\angle HFE=\angle HAE =\angle DAC=\angle PAQ=\angle SAQ,$$

points A, F, S, Q are concyclic. Then since

$$\angle SQT=\angle SFA=90^\circ =\angle AFT=\angle APT=\angle SPT=\angle SAQ,$$

points S,P,T,Q are concyclic. Therefore, we have

$$\angle FPA=\angle FTA=\angle STQ=\angle SPQ,$$

which implies AP bisects $\angle FPQ$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **William KAHN** (Sidney, Australia), **LEUNG Hei Chun**, **George SHEN** and **ZHANG Yupei** (HKUST).

Problem 513. Let a_0, a_1, a_2, … be a sequence of nonnegative integers satisfying the conditions:

(1) $a_{n+1}=3a_n-3a_{n-1}+a_{n-2}$ for $n>1$,

(2) $2a_1=a_0+a_2-2$,

(3) for every positive integer m, in the sequence a_0, a_1, a_2, …, there exist m terms a_k, a_{k+1}, … , a_{k+m-1}, which are perfect squares.

Prove that every term in $a_0,a_1,a_2,\ldots$ is a perfect square.

Solution. **William KAHN** (Sidney, Australia), **LEUNG Hei Chun**, **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We show we can select integers α, β, γ such that $a_n=n(n-1)\alpha/2+n\beta+\gamma$. For $n=0$, we must have $\gamma=a_0$. For $n=1$, we must have $a_1=\beta+\gamma$ and we can set integer β as $a_1-\gamma = a_1-a_0$. Finally for $n=2$, we must have $a_2=\alpha+2\beta+\gamma$ and we can set integer $\alpha = a_2-2\beta-\gamma = a_2-2(a_1-a_0) - a_0$. Then since all three sequences $b_n=n^2$, $b_n=n$ and $b_n=1$ satisfy the relation $b_{n+1}=3b_n-3b_{n-1}+b_{n-2}$, we also have $a_n= n(n-1)\alpha/2 + n\beta + \gamma = n^2\alpha/2+n(\beta-\alpha/2)+\gamma$ satisfies the relation.

From (2), we get $2(\beta+\gamma) = \gamma + \alpha + 2\beta + \gamma - 2$ or $\alpha = 2$. Therefore, we have $a_n= n(n-1)+n\beta+\gamma$, which can be put in the form $[(2n+t)^2+s]/4$ for some integers s and t.

Assume $s\ne0$. If

$$(2n+t-1)^2<(2n+t)^2+s<(2n+t+1)^2 \quad (*),$$

then a_n cannot be a perfect square. However, (*) is equivalent to

$$-(4n+2t-1) < s < 4n+2t+1$$

or $-2t+1-s<4n$ and $s-2t-1<4n$, which is valid for sufficiently large n. Therefore, (3) would lead to $s=0$.

Since $a_0 = t^2/4$ must be an integer, so t must be even. Let $t=2t'$, then

$$a_n = \frac{(2n+2t')^2}{4} = (n+t')^2,$$

which implies that every term in a_n is a perfect square.

Other commended solvers: **Arpon BASU** (AECS-4, Mumbai, India), **George SHEN** and **ZHANG Yupei** (HKUST).

Problem 514. Let n be a positive integer and let $p(x)$ be a polynomial with real coefficients on the interval $[0,n]$ such that $p(0)=p(n)$. Prove that there are n distinct ordered pairs (a_i, b_i) with $i=1,2,\ldots,n$ such that $0\le a_i<b_i\le n$, b_i-a_i is an integer and $p(a_i)=p(b_i)$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

We can solve the problem with continuous functions in place of polynomials. We will prove this by using mathematical induction. The case $n=1$ is trivial. Suppose the case $n-1$ is true. Define $f(x)=p(x+1)-p(x)$. Then

$f(0)+f(1)+\cdots+f(n-1)=p(n)-p(0)=0.$ (*)

First we show there exists $w\in[0,n-1]$ such that $p(w)=p(w+1)$. In fact, if there exists $k\in\{0,1,2,\ldots,n-1\}$ such that $f(k)=0$, then taking $w=k$, we are done. Otherwise, from (*), we know there exists $j\in\{0,1,2,\ldots,n-1\}$ such that $f(j)f(j+1)<0$. Then there is $w\in(j, j+1)$ such that $f(w)=0$. So $p(w) = p(w+1)$.

Next, define $g(x)=p(x)$ for $x\in[0,w]$ and $g(x)=p(x+1)$ for $x\in[w,n-1]$. Then $g(x)$ is continuous on $[0,n-1]$ and $g(0) = g(n-1)$. From induction hypothesis, there exist x_i and y_i with $y_i-x_i\in\mathbb{N}$ satisfying $g(x_i)=g(y_i)$ for $i=1,2,\ldots,n-1$. Then there are three cases:

(1) for $y_i<w$, $0 = g(y_i) - g(x_i) = p(y_i) - p(x_i)$,

(2) for $x_i \le w \le y_i$, $0 = g(y_i) - g(x_i) = p(y_i+1) - p(x_i)$ and

(3) for $w < x_i$, $0 = g(y_i) - g(x_i) = p(y_i+1) - p(x_i+1)$.

Together with $p(0) = p(n)$, we get the case n completing the induction step.

Other commended solvers: **William KAHN** (Sidney, Australia) and **George SHEN**.

Problem 515. There are ten distinct nonzero real numbers. It is known that for every two of the numbers, either the sum or the product of them is rational. Prove that the square of each of the ten numbers is rational.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Pick six of the nonzero distinct real numbers, say A_1, A_2, $\cdots$, A_6 (with the property that for $i\neq j$, either $A_iA_j\in\mathbb{Q}$ or $A_i+A_j\in\mathbb{Q}$). Consider a graph with A_1, A_2, $\cdots$, A_6 as vertices and color the edge with vertices A_i, A_j blue if $A_i+A_j\in\mathbb{Q}$, otherwise red for $A_iA_j\in\mathbb{Q}$. By Ramsay's Theorem, there is a red or a blue triangle in the complete graph with A_1, A_2, $\cdots$, A_6 as vertices.

There are two cases. In case 1, there is a blue triangle with vertices, say A_1, A_2 and A_3. Then A_1+A_2, A_2+A_3, $A_3+A_1\in\mathbb{Q}$. So $2A_1=(A_1+A_2)+(A_3+A_1)-(A_2+A_3)\in\mathbb{Q}$. Then $A_1\in\mathbb{Q}$ and similarly $A_2,A_3\in\mathbb{Q}$.

Next, for any $B\in\{A_4, A_5, \ldots, A_{10}\}$, we see $A_1+B\in\mathbb{Q}$ or $A_1B\in\mathbb{Q}$. So $B=(A_1+B)-A_1\in\mathbb{Q}$ or $B=(A_1B)/A_1\in\mathbb{Q}$. Then all ten $A_i\in\mathbb{Q}$.

In case 2, there is a red triangle with vertices, say A_1, A_2 and A_3. Then A_1A_2, A_2A_3, $A_3A_1\in\mathbb{Q}$. Now

$$A_1^2=(A_1A_2)(A_3A_1)/(A_2A_3)\in\mathbb{Q}$$

and similarly A_2^2, $A_3^2\in\mathbb{Q}$. If at least one of A_1, A_2, $A_3\in\mathbb{Q}$, say $A_1\in\mathbb{Q}$, then pick any $C\in\{A_2, A_3, \ldots, A_{10}\}$. Observe that $A_1+C\in\mathbb{Q}$ or $A_1C\in\mathbb{Q}$. It follows that we get $C=(A_1+C)-A_1\in\mathbb{Q}$ or $C=(A_1C)/A_1\in\mathbb{Q}$. Then all ten $A_i\in\mathbb{Q}$.

Otherwise, if $A_1^2\in\mathbb{Q}$, but $A_1\notin\mathbb{Q}$, then $A_1=m\sqrt{x}$, where $m=1$ or $m=-1$ and $x\in\mathbb{Q}$. Since $A_1A_2\in\mathbb{Q}$, we get $A_1A_2=(m\sqrt{x})A_2=b$ for some $b\in\mathbb{Q}$. Then we get $A_2=b/(m\sqrt{x})=r\sqrt{x}$, where $r=b/(mx)\in\mathbb{Q}$ and $m\neq r$ due to $A_1\neq A_2$. For $A_i\neq A_1$, A_2, if $A_1+A_i\in\mathbb{Q}$ and $A_2+A_i\in\mathbb{Q}$, then $(A_1+A_i)-(A_2+A_i)\in\mathbb{Q}$, but $(A_1+A_i)-(A_2+A_i)= A_1- A_2 = (m-r)\sqrt{x}\notin\mathbb{Q}$. Finally, if $A_1A_i\in\mathbb{Q}$ or $A_2A_i\in\mathbb{Q}$, then as above we get $A_i= s_i\sqrt{x}$ for some $s_i\in\mathbb{Q}$ with $s_i\neq m,r$. Then we have $A_i^2=s_i^2x\in\mathbb{Q}$.

Other commended solvers: **Arpon BASU** (AECS-4, Mumbai, India), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **William KAHN** (Sidney, Australia), **LUO On Ki** and **George SHEN**.

Olympiad Corner

(*Continued from page 2*)

Problem 3. (*Marko Radovanović*) Call a function $f:\mathbb{N}\to\mathbb{N}$ *lively* if

$f(a+b-1)=f(f(\cdots f(b)\cdots))$ for all $a,b\in\mathbb{N}$, where f appears a times on the right side.

Suppose that g is a lively function such that $g(A+2018)=g(A)+1$ holds for some $A\ge2$.

(a) Prove that $g(n+2017^{2017})=g(n)$ for all $n\ge A+2$.

(b) If $g(A+2017^{2017})\neq g(A)$, determine $g(n)$ for $n\le A-1$.

Second Day

Problem 4. (*Dušan Djukić*) An $n\times n$ square is divided into unit squares. One needs to place a number of isosceles right triangles with hypotenuse 2, with vertices at grid points, in such a way that every side of every unit square belongs to exactly one triangle (i.e. lies inside it or on its boundary). Determine all numbers n for which this is possible.

Problem 5. (*Dušan Djukić*) For a positive integer $n\ge2$, let $C(n)$ be the smallest positive real constant such that there is a sequence of n real numbers $x_1,x_2,\ldots,x_n$, not all zero, satisfying the following conditions:

(i) $x_1+x_2+\cdots+x_n=0$;

(ii) for each $i=1,2,\ldots,n$, it holds that $x_i\le x_{i+1}$ or $x_i\le x_{i+1}+C(n)x_{i+2}$ (the indices are taken modulo n).

Prove that:

(a) $C(n)\ge2$ for all n;

(b) $C(n)=2$ if and only if n is even.

Problem 6. (*Bojan Bašić*) Let k be a positive integer and let n be the smallest positive integer having exactly k divisors. If n is a perfect cube, can the number k have a prime divisor of the form $3j+2$?

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the Balkan Mathematical Olympiad which took place in Belgrade, Serbia on May 9, 2018.

Time allowed was 270 minutes. Each problem was worth 10 points

Problem 1. A quadrilateral $ABCD$ is inscribed in a circle k, where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and the perpendicular from M to AB intersects the segment AB at the point E. If EM bisects the angle CED, prove that AB is a diameter of the circle k. (*Bulgaria*)

Problem 2. Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute (n=1,2,…) each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q. (*United Kingdom*)

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 1, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Miscellaneous Inequalities

Kin Y. Li

There are many kinds of inequality problems in mathematical Olympiad competitions. Some of these can be solved by applying certain powerful inequalities such as rearrangement or majorization or Muirhead's inequalities. Some can be solved by techniques like tangent line methods using a bit of differential calculus.

In this article, we will be looking at some inequality problems that are not solved by these kinds of powerful tools and techniques.

Example 1. (*1983 IMO Shortlisted Problem proposed by Finland*) Let p and q be integers with q>0. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left|P(x)-\frac{p}{q}\right|<\frac{1}{q^2}$$

for all $x\in I$.

Solution. Pick $P(x) = p((qx-1)^{2n+1}+1)/q$ and $I = [1/(2q),\ 3/(2q)]$. Then all the coefficients of P are integers and

$$\left|P(x)-\frac{p}{q}\right|=\left|\frac{p}{q}(qx-1)^{2n+1}\right|\le\left|\frac{p}{q}\right|\frac{1}{2^{2n+1}}$$

for all $x\in I$. Choose n large so that $2^{2n+1} > |pq|$. Then we are done.

Example 2 (*1994 IMO*) Let m and n be positive integers. The set $A=\{a_1, a_2, \ldots, a_m\}$ is a subset of $1,2,\ldots,n$. Whenever $a_i+a_j\le n$, $1\le i\le j\le m$, a_i+a_j also belong to A. Prove that

$$\frac{a_1+a_2+\cdots+a_m}{m}\ge\frac{n+1}{2}.$$

Solution. We may assume that $a_1>a_2>\cdots>a_m$. We claim that for $i=1,2,\ldots,m$,

$$a_i+a_{m+1-i}\ge n+1. \qquad (*)$$

If not, then $a_i+a_{m+1-i}, \ldots, a_i+a_{m-1}, a_i+a_m$ are i different elements of A greater than a_i, which is impossible. By adding the cases $i=1,2,\ldots,m$ of (*), we get

$$2(a_1+\cdots+a_m)\ge m(n+1).$$

The result follows.

Example 3 (2001 IMO Shortlisted Problem proposed by Bulgaria). Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100}=\frac{a_0}{a_1}+\frac{a_1}{a_2}+\cdots+\frac{a_{n-1}}{a_n},$$

where $a_0=1$ and $(a_{k+1}-1)a_{k-1}\ge a_k^2(a_k-1)$ for $k=1,2,\ldots,n-1$.

Solution. Let $a_1, a_2, \ldots, a_n$ satisfy the conditions of the problem. Then $a_k>a_{k-1}$ and hence $a_k\ge 2$ for $k=1,2,\ldots,n$. The inequality $(a_{k+1}-1)a_{k-1}\ge a_k^2(a_k-1)$ can be rewritten as

$$\frac{a_{k-1}}{a_k}+\frac{a_k}{a_{k+1}-1}\le\frac{a_{k-1}}{a_k-1}.$$

Adding these inequalities for $k = i+1,\ldots, n-1$ and using $a_{n-1}/a_n< a_{n-1}/(a_n-1)$, we obtain

$$\frac{a_i}{a_{i+1}}+\cdots+\frac{a_{n-1}}{a_n}<\frac{a_i}{a_{i+1}-1}.$$

Then

$$\frac{a_i}{a_{i+1}}\le\frac{99}{100}-\frac{a_0}{a_1}-\cdots-\frac{a_{i-1}}{a_i}<\frac{a_i}{a_{i+1}-1} \qquad (*)$$

for $i=1,2,\cdots,n-1$. Now given $a_0,a_1,\ldots, a_i$, there is at most one possibility for a_{i+1}. By (*), this yields $a_1=2$, $a_2=5$, $a_3=56$, $a_4=78400$. These values satisfy the condition of the problem. So this is a unique solution.

Example 4 (1999 Polish Math Olympiad). Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be integers. Prove that

$$\sum_{1\le i<j\le n}(|a_i-a_j|+|b_i-b_j|)\le\sum_{1\le i<j\le n}|a_i-b_j|.$$

(continued on page 2)

Solution. For integer x, let $f_{\{a,b\}}(x)=1$ if either $a\le x<b$ or $b\le x<a$ and $f_{\{a,b\}}(x)=0$ otherwise. Observe that when a,b are integers, $|a-b|$ equals the sum of $f_{\{a,b\}}(x)$ over all integers x. Now fix an integer x and suppose $a_{\le}$ is the number of values of i for which $a_i\le x$.

Define $a_>$, $b_\le$, $b_<$ analogously. We have

$$\begin{aligned}&(a_\le - b_\le) + (a_> - b_>)\\ &= (a_\le + a_<) - (b_\le + b_>)\\ &= n - n = 0,\end{aligned}$$

which implies $(a_\le - b_\le)(a_> - b_>)\le 0$. Thus

$$a_\le a_> + b_\le b_> \le a_\le b_> + a_> b_\le.$$

Now

$$a_\le a_> = \sum_{1\le i<j\le n} f_{\{a_i,b_j\}}(x).$$

because both sides count the same set of pairs and the other terms reduce similarly, yielding

$$\sum_{1\le i<j\le n} f_{\{a_i,a_j\}}(x) + f_{\{b_i,b_j\}}(x) \le \sum_{1\le i<j\le n} f_{\{a_i,b_j\}}(x).$$

Because x was an arbitrary integer, this last inequality holds for all integers x. Summing over all integers x and using our first observation, we get the desired inequality. Equality holds if and only if the above inequality is an inequality for all x, which is true precisely when the a_i equal the b_i in some order.

Example 5 (2007 Chinese Math Olympiad). Let a,b,c be complex numbers. Let $|a+b|=m$, $|a-b|=n$ and $mn\ne 0$. Prove that

$$\max\{|ac+b|,|a+bc|\} \ge \frac{mn}{\sqrt{m^2+n^2}}.$$

Solution. Since

$$\begin{aligned}&\max\{|ac+b|,|a+bc|\}\\ \ge\ & \frac{|b|\cdot|ac+b|+|a|\cdot|a+bc|}{|b|+|a|}\\ \ge\ & \frac{|b(ac+b)-a(a+bc)|}{|a|+|b|} = \frac{|b^2-a^2|}{|a|+|b|}\\ \ge\ & \frac{|b+a|\cdot|b-a|}{\sqrt{2(|a|^2+|b|^2)}}\end{aligned}$$

and $m^2+n^2=|a-b|^2+|a+b|^2=2(|a|^2+|b|^2)$, so

$$\max\{|ac+b|,|a+bc|\} \ge \frac{mn}{\sqrt{m^2+n^2}}.$$

Example 6 (1999 Balkan Math Olympiad). Let $x_0,x_1,x_2,\ldots$ be a non-decreasing sequence of nonnegative integers such that for every $k\ge 0$, the number of terms of the sequence which are less than or equal to k is finite; let this number be y_k. Prove that for all positive integers m and n,

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j \ge (n+1)(m+1).$$

Solution. Under the given construction, $y_s\le t$ if and only if $x_t>s$. Thus the sequences $x_0,x_1,x_2,\ldots$ and $y_0,y_1,y_2,\ldots$ are dual, meaning that applying the given algorithm to $y_0,y_1,y_2,\ldots$ will restore the original $x_0,x_1,x_2,\ldots$.

To find $x_0+x_1+\cdots+x_n$, observe that among the numbers $x_0,x_1,\cdots,x_n$, there are exactly y_0 terms equal to 0, y_1-y_0 terms equal to 1, … and $y_{x_{n-1}}-y_{x_{n-2}}$ terms equal to x_{n-1}, while the remaining $n+1-x_{n-1}$ terms equal to x_n. Hence, $x_0+x_1+\cdots+x_n$ equals

$$\begin{aligned}&\sum_{i=1}^{x_n-1} i(y_i-y_{i-1}) + x_n(n+1-y_{x_n-1})\\ &= -y_0-y_1-\cdots-y_{x_{n-1}}+(n+1)x_n.\end{aligned}$$

First suppose that $x_n-1\ge m$. Write $x_n-1=m+k$ for $k\ge 0$. Because $x_n>m+k$, from our initial observations we have $y_{m+k}\le n$. Then

$$n+1 \ge y_{m+k} \ge y_{m+k-1} \ge \cdots \ge y_m.$$

So

$$\begin{aligned}\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j &= (n+1)x_n - \left(\sum_{j=0}^{x_{n-1}} y_j - \sum_{i=0}^{m} y_i\right)\\ &= (n+1)x_n - \sum_{i=m+1}^{m+k} y_i\\ &\ge (n+1)(m+k+1) - k(n+1)\\ &= (n+1)(m+1).\end{aligned}$$

Next suppose that $x_n-1<m$. Then $x_n\le m$ implies $y_m>n$, which implies $y_m-1\ge n$. Because $x_0,x_1,x_2,\ldots$ and $y_0,y_1,y_2,\ldots$ are dual, we may apply the same argument with the roles of the two sequences reversed. This completes the proof.

Example 7 (2007 Chinese Girls' Math Olympiad). Let m,n be integers, $m>n\ge 2$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1,a_2,\ldots,a_n\}$ be a subset of S. Suppose every two elements of T are not both the divisors of any element of S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} < \frac{m+n}{m}.$$

Solution. For $i=1,2,\ldots,n$, let k_i be the integer such that $k_i\le m/a_i<k_i+1$. Let $T_i=\{ka_i : k=1,\ldots,k_i\}$. Then $|T_i|=k_i$ Since every two elements of T are not both the divisors of any element of S, so if $i\ne i'$, then $T_i\cap T_{i'}$ is empty. Hence,

$$\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} |T_i| = |T| \le |S| = m.$$

Since $m/a_i < k_i+1$, we have

$$m\sum_{i=1}^{n} 1/a_i \le \sum_{i=1}^{n}(k_i+1) \le m+n.$$

Dividing by m, we get the desired conclusion.

Example 8 (1987 IMO Shortlisted Problem proposed by Netherland). Given five real numbers u_0, u_1, u_2, u_3, u_4, prove that it is always possible to find five real numbers v_0, v_1, v_2, v_3, v_4 that satisfy the following conditions:

(i) $u_i-v_i\in\mathbb{N}$.
(ii) $\Sigma_{0\le i<j\le 4}(v_i-v_j)^2<4$.

Solution. Observe that

$$\begin{aligned}\sum_{0\le i<j\le 4}(v_i-v_j)^2 &= \sum_{0\le i<j\le 4}[(v_i-v)-(v_j-v)]^2\\ &= 5\sum_{i=0}^{4}(v_i-v)^2 - \left(\sum_{i=0}^{4}(v_i-v)\right)^2\\ &\le 5\sum_{i=0}^{4}(v_i-v)^2.\end{aligned}$$

Let us take v_i's satisfying the last line with $v_0\le v_1\le v_2\le v_3\le v_4\le 1+v_0$. Define $v_5=1+v_0$. We see that one of the differences $v_{i+1}-v_i$, $i=0,\ldots,4$, is at most 1/5. Let $v=(v_{i+1}+v_i)/2$. Then place the other three v_i's in $[v-1/2,v+1/2]$. Now we have $|v-v_i|\le 1/10$, $|v-v_{i+1}|\le 1/10$ and $|v-v_k|\le 1/2$ for any k other than i and $i+1$. Finally, we have

$$\sum_{0\le i<j\le 4}(v_i-v_j)^2 \le 5(2(1/10)^2+3(1/2)^2) < 4.$$

Example 9 (2000 Romanian Math Olympiad). Let $n\ge 1$ be an odd positive integer and x_1, x_2, …, x_n be real numbers such that $|x_{k+1}-x_k|\le 1$ for $k=1,2,\ldots,n-1$. Show that

$$\sum_{k=1}^{n}|x_k| - \left|\sum_{k=1}^{n} x_k\right| \le \frac{n^2-1}{4}.$$

Solution. Let P, N be the sets of positive, negative numbers among x_1, x_2, …, x_n respectively. Without loss of generality, assume that there are more k such that x_k is negative than there are k such that x_k is positive. Let $(a_1,\ldots,a_n)$ be a permutation of $(x_1,\ldots,x_n)$ such that $a_1,\ldots,a_n$ is a nondecreasing sequence. By construction, $|P|\le(n-1)/2$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 1, 2018.***

Problem 521. Given 20 points in space so that no three of them are collinear, prove that the number of planes determined by these points is not equal to 1111.

Problem 522. Determine all functions f:ℝ→ℝ such that for all real x and y,

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x)).$$

Problem 523. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that $P(d)=(n/d)^2$ for each positive divisor d of n.

Problem 524. (proposed by *Andrew WU*, St. Albans School, Mc Lean, VA, USA) In ΔABC with centroid G, M and N are the midpoints of AB and AC, and the tangents from M and N to the circumcircle of ΔAMN meet BC at R and S, respectively. Point X lies on side BC satisfying $\angle CAG=\angle BAX$. Show that GX is the radical axis of the circumcircles of ΔBMS and ΔCNR.

Problem 525. Find all positive integer n such that $n(n+2)(n+4)$ has at most 15 positive divisors.

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Solutions

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Problem 516. Determine all triples (p,m,n) of positive integers such that p is prime and $2^mp^2+1=n^5$ holds.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School) and **ZHANG Yupei** (HKUST).

Let $q=n^4+n^3+n^2+n+1$. Then $2^mp^2=(n-1)q$ and $\gcd(n-1,q)=\gcd(n-1,5)=1$ or 5. Now $q>1$ is odd and so p is an odd prime. Let $p=2k+1$. Then $\gcd(2^m,p^2)=1$. So $n-1=2^m$, $q=p^2$. Then $n=2^m+1$. So $n^4+n^3+n^2+n=p^2-1$ can be expressed as

$$(2^{2m}+2^{m+1}+2)(2^{2m}+3\cdot 2^m+2)=4k(k+1).$$

If $m\geq 2$, then the left side is 4 (mod 8) and the right side is 0 (mod 8). Hence, $m=1$. Then $p=11$ and $n=3$. So $(p,m,n)=(11,1,3)$ only.

Other commended solvers: **Ioan Viorel CODREANU** (Satulung, Maramures, Romania), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 517. For all positive x and y, prove that

$$x^2y^2(x^2+y^2-2)\geq(xy-1)(x+y).$$

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School).

Let $k=xy$. We have

$$2\sqrt{k}-\frac{2k+2}{2\sqrt{k}}-\frac{k-1}{k^2}$$

$$=\frac{(\sqrt{k}-1)^2(k\sqrt{k}+2k+2\sqrt{k}+1)}{k^2}\geq 0$$

Since $2\sqrt{k}=2\sqrt{xy}\leq x+y$, so

$$\frac{x^2+y^2-2}{x+y}=x+y-\frac{2xy+2}{x+y}$$

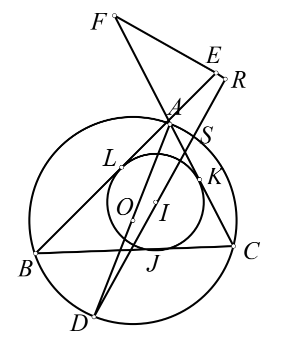
$$\geq 2\sqrt{k}-\frac{2k+2}{2\sqrt{k}}\geq\frac{k-1}{k^2}=\frac{xy-1}{x^2y^2}.$$

Then $x^2y^2(x^2+y^2-2)\geq(xy-1)(x+y)$.

Other commended solvers: **LEUNG Hei Chun**, **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ioannis D. SFIKAS** (Athens, Greece), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 518. Let I be the incenter and AD be a diameter of the circumcircle of ΔABC. Let point E be on the ray BA and point F be on the ray CA. If the lengths of BE and CF are both equal to the semiperimeter of ΔABC, then prove that lines EF and DI are perpendicular.

Solution. **ZHANG Yupei** (HKUST).



Let circle ABC intersect line DI at S. Let K, J, L be the feet of the perpendiculars from I to sides AC, CB, BA of ΔABC respectively. Since AD is a diameter of the circumcircle of ΔABC, we get $\angle ASD=\angle AKI=\angle ALI=90°$. So A,S,K,I,L are concyclic.

Next, $\angle BLS=180°-\angle ALS=180°-\angle AKS=\angle CKS$ and $\angle LBS=\angle KCS$. So ΔBLS, ΔCKS are similar. Since $BE=CF$, $AF/AE=BL/CK=SB/SC$. We get $\angle EAF=\angle CAB=\angle CSB$. So $\Delta EAF\cong\Delta CSB$. Then $\angle SBC=\angle SAC=\angle EFA$. We get $EF\parallel AS$. Then $DI\perp EF$.

Other commended solvers: **William KAHN** (Sidney, Australia), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 519. Let A and B be subsets of the positive integers with 10 and 9 elements respectively. Suppose for every $x,y,u,v\in A$ satisfying $x+y=u+v$, we have $\{x,y\}=\{u,v\}$. Prove that the set $A+B=\{a+b: a\in A, b\in B\}$ has at least 50 elements.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School).

If a_1, $a_2\in A$ and b_1, $b_2\in B$ such that $a_1+b_1=a_2+b_2$, then $a_1-a_2=b_2-b_1$ (with $a_1\neq a_2$ and $b_1\neq b_2$). Assume the equation $x+b_1=y+b_2$ has two distinct solutions $(x,y)=(a_3,a_4)$ and (a_5,a_6) such that a_3, a_4, a_5, $a_6\in A$. Then we have $a_3-a_4=b_2-b_1=a_5-a_6$, which implies $a_3+a_6=a_4+a_5$. By the condition of A, we have $\{a_3,a_6\}=\{a_4,a_5\}$. Then we have 2 cases.

Case 1: $a_3=a_4$ and $a_5=a_6$. From $a_3+b_1=a_4+b_2$, we get $b_1=b_2$. Then $|a_3-a_4|+|b_1-b_2|=0$, contradiction.

Case 2: $a_3=a_5$ and $a_4=a_6$. Then $(a_3,a_4)=(a_5,a_6)$, contradiction.

So $x+b_1=y+b_2$ has at most one solution. Since there are 36 choices of $b_1\neq b_2\in B$, so there must be 36 solutions of (a_1,a_2,b_1,b_2) such that $a_1\neq a_2\in A$, $b_1\neq b_2\in B$ and $a_1+b_1=a_2+b_2$.

However, we have a_1+b_1, $a_2+b_2\in A+B$. Since $A+B$ has 90 not necessary distinct elements, so $A+B$ has at least 54 distinct elements. In particular, $A+B$ has at least 50 distinct elements.

Other commended solvers: **William KAHN** (Sidney, Australia), **Akash Singha ROY** (West Bengal, India), **George SHEN, Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Problem 520. Let P be the set of all polynomials $f(x)=ax^2+bx$, where a,b are nonnegative integers less than 2010^{18}. Find the number of polynomials f in P for which there is a polynomial g in P such that $g(f(k))\equiv k \pmod{2010^{18}}$ for all integers k.

Solution. **William KAHN** (Sidney, Australia) and **George SHEN**.

We will show that there exists $Q(x) = cx^2+dx$ for $P(x) = ax^2+bx$ if and only if $2^8 1005^9|a$ and $\gcd(2010,b)=1$. Then it follows that the answer is $2\cdot 2010^9\cdot 2010^{18}(1-1/2)(1-1/3)(1-1/5)(1-1/67) = 2^5 3\cdot 11\cdot 2010^{26}$.

Assume that $Q(P(n))\equiv n \pmod{2010^{18}}$ for all n. Then $n\to P(n)$ is one-to-one $\pmod{2010^{18}}$ and using the Chinese remainder theorem we deduce that $n\to P(n)$ is one-to-one $\pmod{p^{18}}$ for p in $\{2,3,5,67\}$.

Let $p\in\{2,3,5,67\}$. If $p|b$, then $P(p^{17})\equiv P(0) \pmod{p^{18}}$ gives a contradiction. Hence, $p\nmid b$. If $p\nmid a$, then $P(-a^{-1}b)\equiv P(0) \pmod{p^{18}}$ gives a contradiction. So $p\mid a$. Hence $2010\mid a$ and $\gcd(2010,b) = 1$. In particular, $(b(a^2-b^2))^{-1} \pmod{2010^{18}}$ exists. Since

$$\begin{aligned} &Q(P(1)) \equiv 1 \pmod{2010^{18}}\\ \Rightarrow\ & c(a+b)^2+d(a+b)\equiv 1 \pmod{2010^{18}}\\ \Rightarrow\ & 2b(a^2-b^2)c \equiv 2a \pmod{2010^{18}}\end{aligned}$$

and

$$\begin{aligned} &Q(P(-1)) \equiv -1 \pmod{2010^{18}}\\ \Rightarrow\ & c(a-b)^2+d(a-b) \equiv -1 \pmod{2010^{18}}\\ \Rightarrow\ & 2b(a^2-b^2)d\equiv -2(a^2+b^2) \pmod{2010^{18}}\end{aligned}$$

we have

$c\equiv(b(a^2-b^2))^{-1}a+2010^{18}e \pmod{2010^{18}}$ and $d \equiv -(b(a^2-b^2))^{-1}(a^2+b^2) + 2010^{18}e \pmod{2010^{18}}$, where $e = 0$ or ½.

Therefore,

$$\begin{aligned} &Q(P(x))-x\\ \equiv\ & -(b(a^2-b^2))^{-1}a^2x(x-1)(x+1)(ax+b)\\ &+2010^{18}ex(x-1)\\ \equiv\ & -(b(a^2-b^2))^{-1}a^2x(x-1)(x+1)(ax+2b)\\ &\pmod{2010^{18}}.\end{aligned}$$

Now if $x=2$, we get $2010^{18} \mid 2^2 3a^2$, hence $2^8 1005^9 \mid a$.

Conversely, if $2^8 1005^9 \mid a$ and $\gcd(2010,b)=1$, then we can define c and d as above. Since $2 \mid n(n-1)$ and $2 \mid an+2b$ for all n, $Q(P(n))\equiv n \pmod{2010^{18}}$ follows.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Olympiad Corner

(*Continued from page 1*)

Problem 3. Alice and Bob play the following game: They start with two non-empty piles of coins. Taking turns, with Alice playing first, each player choose a pile with an even number of coins and moves half of the coins of this pile to the other piles. The game ends if a player cannot move, in which case the other player wins. (*Cyprus*)

Problem 4. Find all primes p and q such that $3p^{q-1}+1$ divides 11^p+17^p. (*Bulgaria*)

Miscellaneous Inequalities

(*Continued from page 2*)

Suppose that $1\le i\le n-1$. In the sequence $x_1,\ldots,x_n$, there must be two adjacent terms x_k and x_{k+1} which are separated by the interval (a_i, a_{i+1}), i.e. such that either $x_k\le a_i\le a_{i+1}\le x_{k+1}$ or $x_{k+1}\le a_i\le a_{i+1}\le x_k$. So $a_{i+1}-a_i\le|x_k-x_{k+1}|\le 1$. That is $a_1,\ldots,a_n$ is a nondecreasing sequence of terms, such that any two adjacent terms differ by at most 1.

Let σ_P denote the sum of the numbers in P. We claim that $\sigma_P\le(n^2-1)/8$. This is certainly true if P is empty.

If P is nonempty, then the elements of P are $a_i\le a_{i+1}\le\cdots\le a_n$ for some $2\le i\le n$. Because $a_{i-1}\le 0$ by assumption and $a_i\le a_{i-1}+1$ from the previous paragraph, we have $a_i\le 1$.

Similarly, $a_{i+1}\le a_i+1\le 2$ and so on up to $a_n\le|P|$. Hence, $\sigma_P\le 1+2+\cdots+|P|$. From $|P|\le(n-1)/2$, we get $\sigma_P\le(n^2-1)/8$, as claimed.

Let σ_N denote the sum of the numbers in N. The left-hand side of the required inequality then equals

$$\begin{aligned} &|\sigma_P-\sigma_N|-|-\sigma_P-\sigma_N|\\ \le\ & |2\sigma_P|\\ \le\ & 2\left(\frac{n^2-1}{8}\right)=\frac{n^2-1}{4}\end{aligned}$$

as needed.

<u>*Example 10 (2000 Asia Pacific Math Olympiad).*</u> Let n, k be positive integers with $n > k$. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}}<\frac{n!}{k!(n-k)!}<\frac{n^n}{k^k(n-k)^{n-k}}.$$

<u>*Solution.*</u> By the binomial theorem, we have $n^n=(k+(n-k))^n=a_0+\cdots+a_n$, where for $i=0,1,\ldots,n$,

$$a_i=\binom{n}{i}k^i(n-k)^{n-i}>0.$$

We claim that

$$\frac{n^n}{n+1}<a_i<n^n.$$

The right inequality holds because $n^n = a_0+\cdots+a_n>a_i$. To prove the left inequality, it suffices to prove that a_i is larger than $a_0,\ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ because then

$$n^n=\sum_{m=0}^n a_m<\sum_{m=0}^n a_i=(n+1)a_i.$$

Next, we will show a_i is increasing for $i\le k$ and decreasing for $i\ge k$. Observe that

$$\binom{n}{i}=\frac{i+1}{n-i}\binom{n}{i+1}.$$

Hence

$$\frac{a_i}{a_{i+1}}=\frac{\binom{n}{i}k^i(n-k)^{n-i}}{\binom{n}{i+1}k^{i+1}(n-k)^{n-i-1}}=\frac{n-k}{n-i}\cdot\frac{i+1}{k}.$$

This expression is less than 1 when $i<k$ and it is greater than 1 when $i\ge k$. In other words, $a_0<\cdots<a_k$ and $a_k>\cdots>a_n$ as desired.

Mathematical Excalibur

Volume 22, Number 2 *November 2018 – January 2019*

Olympiad Corner

Below were the Day 1 problems of the Croatian Mathematical Olympiad which took place on May 5, 2018.

Problem A1. Let *a, b* and *c* be positive real numbers such that $a+b+c=2$. Prove that

$$\frac{(a-1)^2}{b}+\frac{(b-1)^2}{c}+\frac{(c-1)^2}{a}$$
$$\geq\frac{1}{4}\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right).$$

Problem C1. Let n be a positive integer. A *good word* is a sequence of $3n$ letters, in which each of the letters *A, B* and *C* appears exactly n times. Prove that for every good word X there exists a good word Y such that Y cannot be obtained from X by swapping neighbouring letters fewer than $3n^2/2$ times.

Problem G1. Let k be a circle centered at O. Let $\overline{AB}$ be a chord of that circle and M its midpoint. Tangent on k at points A and B intersect at T. The line ℓ goes through T, intersects the shorter arc AB at the point C and the longer arc AB at the point D, so that $|BC|=|BM|$.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 15, 2019***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Austrian Math Problems

Kin Y. Li

In this article, we would like to look at some of the Austrian Math Olympiad problems. This competition is going into its 50th year. For the young math students, the Austrian math problems are treasures that are everlasting, especially the problems appeared in the recent decades. Below are some examples that we hope you will enjoy.

<u>*Example 1.*</u> (*Beginners Competition: June 7th, 2001*) Prove that the number n^n-1 is divisible by 24 for all odd positive integer values of n.

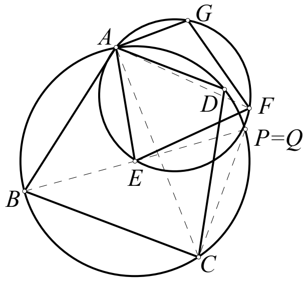
<u>*Solution.*</u> Since n is an odd positive integer, we can write $n=2k+1$ with $k=0,1,2,\ldots$. Substituting yields

$$n^n-1=n(n^{n-1}-1)=n(n^{2k}-1).$$

Since $1^2\equiv3^2\equiv5^2\equiv7^2\equiv1 \pmod 8$, we see that $n^{2k}\equiv1 \pmod 8$ certainly holds, and $n^{2k}-1$ is therefore divisible by 8.

If n is divisible by 3, we see that $n(n^{2k}-1)$ is certainly divisible by $3\cdot8=24$ as required. If n is not divisible by 3, we note that $1^2\equiv2^2\equiv1 \pmod 3$, and $n^{2k}\equiv1 \pmod 3$ holds, so that $n^{2k}-1$ is not only divisible by 8, but also by 3. It follows that $n^{2k}-1$ is therefore divisble by $3\cdot8=24$, and therefore so is $n(n^{2k}-1)$ as required.

<u>*Example 2*</u> (*National Competition: June 6th, 2002*) Let $ABCD$ and $AEFG$ be similar inscribed quadrilaterals, whose vertices are labeled counter-clockwise. Let P be the second common point of the circumcircles of the quadrilaterals beside A. Show that P must lie on the line connecting B and E.



<u>*Solution.*</u> Rotation and stretching with center A, $\angle BAC$ and factor $AB{:}AC$ maps B onto C and E onto F. This mapping therefore transforms the line $BE=BQ$ onto the line $FC=FQ$, whereby we let Q denote the point of intersection of lines BE and FC. Since this mapping rotates by $\angle BAC$, this is also the angle between the lines BQ and FQ, and since this is equal to $\angle BAC$ (or its supplement), Q must lie on the circumcircle of $\triangle ABC$, which is also the circumcircle of $ABCD$. By analogous reasoning, it must also lie on the circumcircle of $AEFG$, and we see that $P=Q$ must hold, which proves that P must lie on the line BE, as required.

<u>*Example 3*</u> (*National Competition: May 26th, 2004*). Prove without the use of calculus:

a) If *a, b, c* and *d* are real numbers, then

$$a^6+b^6+c^6+d^6-6abcd \geq -2$$

holds. When does equality hold?

b) For which positive integers k does there exist an inequality of the form

$$a^k+b^k+c^k+d^k-kabcd \geq M_k$$

that holds for all real values of *a, b, c* and *d*? Determine the largest possible values of M_k and determine when equality holds.

<u>*Solution.*</u> a) The given inequality can be proved by applying the AM-GM inequality as

$$\frac{a^6+b^6+c^6+d^6+1^6+1^6}{6}\geq|abcd|\geq abcd.$$

Equality holds for $|a|=|b|=|c|=|d|=1$, more precisely when (a,b,c,d) equals one of

$(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(-1,1,1,-1)$, $(1,-1,-1,1)$, $(-1,1,-1,1)$, $(-1,-1,1,1)$ or $(-1,-1,-1,-1)$.

(continued on page 2)

b) First of all, we note that no such number M_k can possibly exist if k is odd, since a choice of negative values for a, b, c and d with sufficiently large absolute value yields negative values with arbitrary large absolute value for the expression $a^k+b^k+c^k+d^k-kabcd$.

Similarly, no such number exists for k=2, since a choice of $a=b=c=d=r$ yields $a^2+b^2+c^2+d^2-2abcd = 4r^2-2r^4$, for which a choice of sufficiently large values of r again yields negative values with arbitrarily large absolute value.

This leaves even values of k with $k\geq 4$ to consider. In this case, choosing $a=b=c=d=1$ yield $a^k+b^k+c^k+d^k-kabcd = 4-k$, and as in a), we can apply AM-GM inequality to get

$$\frac{a^k+b^k+c^k+d^k+(k-4)1^k}{k} \geq |abcd| \geq abcd$$

with equality for the same values of (a,b,c,d) as in a).

Example 4 (*National Competition: June 6th, 2007*) We are given a convex n-gon with a triangulation, i.e. a division into triangles by non-intersecting diagonals. Prove that the n corners of the n-gon can each be labeled by the digits of 2007 such that any quadrilateral composed of two triangles in the triangulation with a common side has corners labeled by digits with the sum 9.

Solution. We shall prove this by induction on n. If n=4, we label the vertices 2, 0, 0, 7 and the claim holds. (Note that this is the only possible combination of digits summing to 9, since $4\cdot 2<9$ and $2\cdot 7>9$ hold. Also note that the three corners of any triangle must be labeled with three of the digits 2, 0, 0, 7.)

We now assume that the claim holds as stated for any convex n-gon, and consider a convex $(n+1)$-gon. Any triangulation of such an $(n+1)$-gon certainly contains at least one triangle (in fact, at least two), two of whose sides are consecutive sides of the $(n+1)$-gon with common vertex V. The n-gon obtained by removing this one triangle from the triangulation with the implied triangulation in the remaining n-gon as given can certainly be labeled as required.

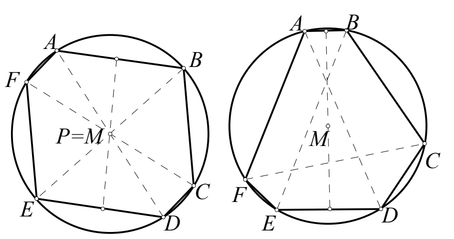
We now note that the triangle with vertex V only has a side in common with one other triangle of the triangulation, the corners of which are already labeled with three of the four required digits. Labelling V with the fourth digit results in a labeling of the $(n+1)$-gon with the required property.

Example 5 (*National Competition: June 3rd, 2010*) A diagonal in a hexagon is considered a *long* diagonal if it divides the hexagon into two quadrilaterals. Any two long diagonals divide the hexagon into two triangles and two quadrilaterals.

We are given a convex hexagon with the property that the division into pieces by any two long diagonals always yields two isosceles triangles with sides of the hexagon as bases. Show that such a hexagon must have a circumcircle.

Solution. Since any two opposing isosceles triangles (such as ABP and DEP) have a common angle at their vertices, they must be similar, and their bases therefore parallel. The angle bisector in their common vertex is therefore also the common altitude.

If all three diagonals of the hexagon intersect at M, this point is also a common point of all angle bisectors. It must therefore be the same distance from A to B, as it lies on the bisector of AB, but the same holds for B and C, C and D, and so on. This point is therefore equidistant from all corners of the hexagon, and is therefore the mid-point of the circumcircle of the hexagon.



If the diagonals of the hexagon do not have a common point, they form a triangle. The angle bisectors have a common point, namely the incenter of this triangle, which we again call M. The same holds for this point M as in the previous situation, and we once again have established the existence of a circumcircle of the hexagon, as claimed.

Example 6 (*National Competition: May 1st, 2015*) A *police emergency number* is a positive integer that ends with the digits 133 in decimal representation. Prove that every police emergency number has a prime factor larger than 7.

(*In Austria, 133 is the emergency number of the police.*)

Solution. Let $n=1000k+133$ be a police emergency number and assume that all its prime divisors are at most 7. It is clear from the last digit that n is odd and that n is not divisible by 5, so $1000k+133 = 3^a7^b$ for suitable integers $a,b\geq 0$. Thus, $3^a7^b\equiv 133 \pmod{1000}$.

This also implies $3^a7^b\equiv 133\equiv 5 \pmod 8$. We know that 3^a is congruent to 1 or 3 modulo 8 and 7^b is congruent to 1 or 7 modulo 8. In order for the product 3^a7^b to be congruent to 5 modulo 8, 3^a must therefore be congruent to 3 and 7^b must be congruent to 7. Therefore, we can conclude that a and b are both odd.

We also have $3^a7^b\equiv 133\equiv 3 \pmod 5$. As a and b are odd, 3^a and 7^b are each congruent to 3 or 2 modulo 5. Neither 3^2, nor $3\cdot 2$ is congruent to 3 modulo 5, a contradiction.

Example 7 (*National Competition: April 30th, 2016*) Consider 2016 points arranged on a circle. We are allowed to jump ahead by 2 or 3 points in clockwise direction. What is the minimum number of jumps required to visit all points and return to the starting point?

Solution. Clearly it takes at least 2016 jumps to visit all points. It is impossible to use only jumps of length 2 or only jumps of length 3 because this would confine us to a single residue class modulo 2 or 3 respectively.

If the problem could be solved with 2016 jumps, the total distance covered by these jumps would be strictly between $2\cdot 2016$ and $3\cdot 2016$ which makes a return to the original point impossible. Therefore, at least 2017 jumped are required.

This is indeed possible, for example with the following sequence of points on the circle

0,3,6,…,2013,2015, 2,5,…,2012,2014, 1,4,…, 2011, 2013,0.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***February 15, 2019.***

Problem 526. Let $a_1=b_1=c_1=1$, $a_2=b_2=c_2=3$ and for $n\ge 3$, $a_n=4a_{n-1}-a_{n-2}$,

$$b_n=\frac{b_{n-1}^2+2}{b_{n-2}},\, c_n=2c_{n-1}+\sqrt{3c_{n-1}^2-2}.$$

Prove that $a_n=b_n=c_n$ for all $n=1,2,3,\ldots$.

Problem 527. Let points O and H be the circumcenter and orthocenter of acute ΔABC. Let D be the midpoint of side BC. Let E be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let F be the point such that $AEHF$ is a rectangle. Prove that points D, E, F are collinear.

Problem 528. Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $n+m$ congruent squares.

Problem 529. Determine all ordered triples (x,y,n) of positive integers satisfying the equation $x^n+2^{n+1}=y^{n+1}$ with x is odd and the greatest common divisor of x and $n+1$ is 1.

Problem 530. A square can be decomposed into 4 rectangles with 12 edges. If square $ABCD$ is decomposed into 2005 convex polygons with degrees of A, B, C, D at least 2 and degrees of all other vertices at least 3, then determine the maximum number of edges in the decomposition.

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Solutions

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Problem 521. Given 20 points in space so that no three of them are collinear, prove that the number of planes determined by these points is not equal to 1111.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Assume the number of planes is 1111. The 20 points would define $(20\cdot 19\cdot 18)/3!$ = 1140 planes so that $1140-1111=29$ triplets of points lie in the planes already determined by other triplets. If one of the planes contain 7 or more points, then there are $(7\cdot 6\cdot 5)/3!$ = 35 triplets of points in this plane and the number of triplets is greater than the number of planes by at least $35-1=34$. So the greatest possible number of planes is $1140-34=1105$. Clearly, this cannot happen if there are 1111 planes.

So each plane can contain at most 6 of the points. Let a, b, c be the number of planes containing 4, 5, 6 points respectively. When counting triplets, in cases k=4,5,6, we consider each plane containing k points $k(k-1)(k-2)/3!$ = 4, 10, 20 times, which are 3, 9, 19 times too many, respectively. So the number of planes satisfies $1140-3a-9b-19c$ = 1111. Hence $3a+9b+19c=29$. However, there are no nonnegative integers a,b,c satisfying $3a+9b+19c=29$. So we arrive at a contradiction.

Other commended solvers: **ZHANG Yupei** (HKUST).

Problem 522. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x and y,

$$(x-2)\,f(y)+f\,(y+2f(x)) = f\,(x+y\,f(x)).$$

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We will refer to the given equation as (*). In case $f(0)=0$, setting x=0 in (*), we get $f(y)=0$ for all y. In case $f(0)\ne 0$, setting y=0, (*) becomes $(x-2)f(0)+f(2f(x)) = f(x)$ for all real x. If $f(x)=f(x')$, then $x=x'$ and so f is injective.

Next, putting x=2 into (*), we get $f(y+2f(2)) = f(2+yf(2))$ for all real y. Since f is injective, we get $y+2f(2) = 2+yf(2)$ for all real y. Setting y=0, we get $f(2)$=1. Since f is injective, $f(3)\ne 1$. Setting x=3 and $y=3/(1-f(3))$ (which is $y=3+yf(3)$) into (*), we get $f(y+2f(3))=0$. So f has a root at $r=y+f(3)$. Next, setting $y=r$ in (*), we get $f(r+2f(x))=f(x+rf(x))$ for all real x. Since f is injective, we get $r+2f(x) = x+rf(x)$ for all real x.

Now due to $f(2)=1\ne 0$, $r\ne 2$. So $f(x)=(x-r)/(2-r)$. Finally, substituting $f(x)$ by $(x-r)/(2-r)$ we get r=1 so that $f(x)=x-1$. As a result, it is easy to check (*) has the two solutions $f(x)$=0 and $f(x)=x-1$.

Other commended solvers: **Alex Kin Chit O** (G.T. (Ellen Yeung) College).

Problem 523. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that $P(d) = (n/d)^2$ for each positive divisor d of n.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

For n=1, let $P(x)=x$, then $P(1)$=1 satisfies the condition. If n is a prime, then its only positive divisors are 1 and n and the conditions on P is $P(1)=n^2$ and $P(n)$=1. We can satisfy this with $P(x)=n^2+(n+1)(1-x)$.

Next we consider $n=km$ is not prime with $k,m>1$. We have conditions $P(1)=n^2$, $P(k)=m^2$, $P(m)=k^2$ and $P(n)$=1. For arbitrary integers a, b, by factoring, we see $P(a)-P(b)$ is divisible by $a-b$. So $n-k=k(m-1)$ divides $P(n)-P(k)$ = $1-m^2$ = $(1-m)(1+m)$. This leads to k divides $m+1$. Similarly, $n-m$ divides $P(n)-P(m)$ and so $m(k-1)$ divides $(1-k)(1+k)$ and m divides $k+1$. Hence, km divides $(k+1)(m+1)$ and it also divides $(k+1)(m+1)-km = k+m+1$. We must have $km\le k+m+1$, which implies that $km-k-m+1\le 2$ or $(k-1)(m-1)\le 2$. We may assume $k\le m$. Then the only possible case is k=2 and m=3 so that n=6.

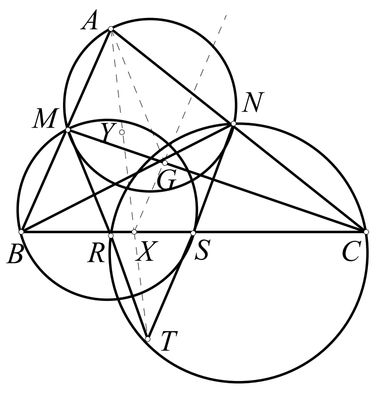
For n=6, we will find a polynomial P such that $P(1)$=36, $P(2)$=9, $P(3)$=4 and $P(6)$=1. We can apply the Lagrange interpolation formula to get $P(x)$ = $1-(x-6)(1+(x-3)(2x-5))$, which can be easily checked to satisfy $P(1)$=36, $P(2)$=9, $P(3)$=4 and $P(6)$=1.

Other commended solvers: **Akash Singha ROY** (West Bengal, India).

Problem 524. (proposed by *Andrew WU*, St. Albans School, Mc Lean, VA, USA) In ΔABC with centroid G, M

and N are the midpoints of AB and AC, and the tangents from M and N to the circumcircle of ΔAMN meet BC at R and S, respectively. Point X lies on side BC satisfying $\angle CAG = \angle BAX$. Show that GX is the radical axis of the circumcircles of ΔBMS and ΔCNR.

Solution. By Proposer.



Observe that BN is the radical axis of the circumcircles of ΔANM and ΔCNR. To prove this, we will show $BM{\cdot}BA=BR{\cdot}BC$ or equivalently that $AMRC$ is a cyclic quadrilateral. By the tangency condition, we have $\angle AMR = 180°-\angle ANM=180°-\angle ACR$, so $AMRC$ is cyclic, as desired. Similarly, we have CM is the radical axis of the circumcircles of ΔANM and ΔBMS. Thus, by the radical center theorem, BN, CM and the radical axis of the circumcircles of ΔBMS and ΔCNR concur. This implies the centroid G lies on the radical axis.

Next, by properties of symmedians, we get lines MR, AX, NS concur at some point T. Suppose lines AX and MN meet at Y. Then by similar triangles, we have $RX/XS=MY/YN=BX/XC$ due to the facts that $\Delta TRS\sim\Delta TMN$ and $\Delta AMN\sim\Delta ABC$.

Thus, it follows that $XR{\cdot}XC=XS{\cdot}XB$. So X has equal power with respect to the circumcircles of ΔBMS and ΔCNR. Then line GX is the radical axis of ΔBMS and ΔCNR.

Other commended solvers: **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **LEUNG Hei Chun** and **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Problem 525. Find all positive integer n such that $n(n+2)(n+4)$ has at most 15 positive divisors.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Ioan Viorel CODREANU** (Satulung, Maramures, Romania), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun**, **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Let $a_n=n(n+2)(n+4)$ and let b_n be the number of positive divisors of a_n. The values of b_1 to b_{10} are 4, 10, 8, 14, 12, 24, 12, 28, 12, 40. Next, we recall if a positive integer m has prime factorization $p_1^{e_1}\cdots p_j^{e_j}$, then m has $(e_1+1)\cdots(e_j+1)$ positive divisors. If m divides a positive integer M, then M has at least as many divisors as m.

Let $n\geq 11$. If n is even, say $n=2k$, then $a_n=2^3k(k+1)(k+2)$. At least one of the numbers k, $k+1$, $k+2$ is divisible by 2 and exactly one of them is divisible by 3. Since $k\geq 6$, the numbers k, $k+1$, $k+2$ cannot all be powers of 2 or 3. So $k(k+1)(k+2)$ has a prime divisor p not equal to 2 or 3. Hence, $2^4 3p$ divides a_n and this implies that a_n has at least $5{\cdot}2{\cdot}2 =20$ positive divisors.

Let $n\geq 11$ be odd. Then the numbers n and $n+2$ are relativity prime, as are $n+2$ and $n+4$ and also n and $n+4$. One of these three numbers is divisible by 3. This number has at least one other prime divisor p or else is a power of 3. In the latter case it is divisible by 3^3 since $n\geq 11$. Let q and r be prime divisors of the other two numbers. In the first case the number a_n is divisible by $3pqr$. The number n, $n+2$, $n+4$ are relatively prime, so 3, p, q, r are relatively prime. This implies that a_n has at least $2{\cdot}2{\cdot}2{\cdot}2 =16$ divisors. In the second case a_n is divisible by 3^3qr. The primes 3, q, r are again distinct. So a_n has at least $4{\cdot}2{\cdot}2=16$ divisors.

The number a_n has at most 15 positive divisors only for n=1, 2, 3, 4, 5, 7, 9.

Other commended solvers: **Christos ALVANOS** (Mandoulides, Thessaloniki, Greece), **Alex Kin Chit O** (G.T. (Ellen Yeung) College) and **Akash Singha ROY** (West Bengal, India).

Olympiad Corner

(*Continued from page 1*)

Problem G1. (*cont.*) Prove that the circumcenter of the triangle ADM is the reflection of O across the line AD.

Problem N1. Determine all pairs (m,n) of positive integers such that

$$2^m = 7n^2+1.$$

Austrian Math Problems

(*Continued from page 2*)

<u>*Example 8*</u> (*National Competition: April 30th, 2017*) Anna and Berta play a game in which they take turns in removing marbles from a table. Anna takes the first turn. When at the beginning of a turn there are $n\geq 1$ marbles on the table, then the player whose turn it is removes k marbles, where $k\geq 1$ either is an even number with $k\leq n/2$ or an odd number with $n/2\leq k\leq n$. A player wins the game if she removes the last marble from the table. Find the smallest $N\geq 100{,}000$ such that Berta can enforce a victory if there are exactly N marbles on the table in the beginning.

<u>*Solution.*</u> We claim that the losing situations are those with exactly $n=2^a-2$ marbles left on the table for all integers $a\geq 2$. All other situation are winning situations.

For n=1, the player wins by taking the single remaining marble. For n=2, the only possible move is to take k=1 marbles and the opponent wins in the next move. For $n\geq 3$, (1) if n is odd, the player takes all n marbles and wins; (2) if n is even, but not of the form 2^a-2, then n lies between two other numbers of that form, so there is a unique b with $2^b-2<n<2^{b+1}-2$. From $n\geq 3$, we get $b\geq 2$. So all 3 parts of the inequalities are even and so $2^b\leq n\leq 2^{b+1}-4$. By the induction hypothesis, we know 2^b-2 is a losing situation. Taking $k = n-(2^b-2) \leq n/2$ marbles, we leave it to the opponent; (3) if n is even of the form 2^a-2, the player cannot leave a losing situation with 2^b-2 marbles to the opponent (where $b<a$ holds due to at least 1 marble must be removed and $b\geq 2$ holds as after a legal move starting from an even n, at least 1 marble remains). The player would then remove $k=2^a-2^b$ marbles. As $b\geq 2$, k is even and greater than $n/2$ due to $k\geq 2^{a-1}>2^{a-1}-1= n/2$, which is impossible. This means Berta can enforce a victory if and only if N is of the form 2^a-2. The smallest number $N\geq 100{,}000$ of this form is $N = 2^{17}-2 = 131{,}070$.

Mathematical Excalibur

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Olympiad Corner

Below were the Day 2 problems of the Croatian Mathematical Olympiad which took place on May 6, 2018.

Problem A2. determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(y))=(1-y)f(xy)+x^2y^2f(y)$$

holds for all real numbers x and y.

Problem C2. Let n be a positive integer. Points $A_1, A_2, \ldots, A_n$ are located on the inside of a circle, and points $B_1, B_2, \ldots, B_n$ are on the circle, so that the lines $A_1B_1, A_2B_2, \ldots, A_nB_n$ are mutually disjoint. A grasshopper can jump from point A_i to point A_j (for $i,j\in\{1,\ldots,n\}$, $i\neq j$) if and only if the lines A_iA_j does not go through any of the inner points of the lines $A_1B_1, A_2B_2, \ldots, A_nB_n$.

Problem G2. Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Point D is the midpoint of the shorter arc BC of the circumcircle of the triangle ABC. Point I is the incenter of the triangle ABC, and point J is the reflection of I across the line BC.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 25, 2019***.

For individual subscription for the next five issues for the 18-19 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Sum of Digits of Positive Integers

Pedro Pantoja, Natal/RN, Brazil

In this short article we will explore some types of problems in number theory about the sum of digits of a positive integer.

Throughout this article, $S(a)$ will denote the sum of the digits of a positive integer a. For example $S(12)=1+2=3$, $S(349)=3+4+9=16$. Let $c(n,m)$ denote the total number of carries, which arises when adding a and b, for example $c(100,4)=0$, $c(23,17)=1$, $c(88,99)=2$.

Proposition 1. For positive integer a, we have

i) $S(a)\le a$;
ii) $S(a)\equiv a \pmod 9$;
iii) if a is even, then $S(a+1)-S(a)=1$;
iv) $S(a+b)=S(a)+S(b)-9c(a,b)$,
in particular, $S(a+b)\le S(a)+S(b)$;
v) $S(ab)\le \min\{aS(b),bS(a)\}$;
vi) $S(ab)\le S(a)S(b)$;
vii) $S(a)\le 9([\log a]+1)$.

Proof. i) and ii) are obvious.

iii) If a is even, then $S(a+1)-S(a)=1$. In fact, a and $a+1$ differ only in the unit digit, which for a will be 0, 2, 4, 6 or 8 and for $a+1$ will be, respectively, 1, 3, 5, 7 or 9.

iv) We proceed by induction on the maximal number of digits k of b and a. If both b and a are single digit numbers, then we have just two cases. If $b+a<10$, then we have nocarries and clearly $S(b+a)=b+a=S(b)+S(a)$. If on the other hand, $b+a=10+k\ge 10$, then

$$\begin{aligned}S(b+a) &= 1+k = 1+(b+a-10)\\ &= S(b)+S(a)-9.\end{aligned}$$

Assume that the claim holds for all pairs with at most k digits each. Let

$$b = b_1+n\cdot 10^{k+1} \text{ and } a = a_1+n\cdot 10^{k+1},$$

where b_1 and a_1 are at most k digit numbers. If there is no carry at the $k+1^{\text{st}}$ digit, then $c(b,a)=c(b_1,a_1)$ and thus

$$\begin{aligned}S(b+a) &= S(b_1+a_1) + m + n\\ &= S(b_1)+m+S(a_1)+n-9c(n_1,m_1)\\ &= S(b)+S(a)-9c(b,a).\end{aligned}$$

If there is a carry, then $c(n,a) = 1 + c(n_1,ma_1)$ and thus

$$\begin{aligned}S(b+a) &= S(b_1+a_1)+m+n-9\\ &= S(b_1)+m+S(a_1)+n-9(c(b_1,a_1)+1)\\ &= S(b)+S(a)-9c(b,a).\end{aligned}$$

This finishes the induction and we are done.

v) Because of symmetry, in order to prove v), it suffices to prove that $S(ab) \le aS(b)$. The last inequality follows by applying the subadditivity (iv) property repeatedly. Indeed, $S(2b)=S(b+b)\le S(b) +S(b) = 2S(b)$. After a steps we obtain

$$\begin{aligned}S(ab) &= S(b+\cdots+b)\\ &\le S(b)+\cdots+S(b) = aS(b).\end{aligned}$$

vi) and vii) Left as exercises for the reader.

For applications, we provide

Example 1: Find all positive integers with $n \le 1000$ such that $n = (S(n))^3$.

Solution: The perfect cube numbers smaller than 1000 are 1, 8, 27, 64, 125, 216, 343, 512, 729. From these numbers the only one that satifies the conditions of the problem is $n = 512$.

Example 2: (*MAIO-2012*) Evaluate

$$S(1) - S(2) + S(3) - S(4) + \cdots + S(2011) - S(2012).$$

Solution: The problem becomes trivial using Proposition 1, item iii). We have $S(3)-S(2)=1$, $S(5)-S(4) = 1$, …, $S(2011) - S(2010) = 1$ and $S(1) = 1$, $S(2012) = 5$. Therefore, $S(1) - S(2) + S(3) - S(4) + \cdots + S(2011) - S(2012) =1+1005-5 =1001$.

(continued on page 2)

Example 3: (*Nordic Contest 1996*) Show that there exists an integer divisible by 1996 such that the sum of its decimal digits is 1996.

Solution. We affirm that the number m = 199619961996…199639923992 satisfies the conditions of the statement. Note that $S(m)$=25·78 +2·23=1996. On the other hand, m is divisible by 1996, since m equals

$$1996\cdot 100010001000\ldots 1000200002.$$

Example 4: Find $S(S(S(S(2018^{2018}))))$.

Solution: Using proposition 1, item vii) several times we have

$$S(2018^{2018})\le 9([2018\log 2018]+1) < 60030,$$

$$S(S(2018^{2018}))\le 9([\log 60030]+1) < 45,$$

$$S(S(S(2018^{2018})))\le 9([\log 45]+1) < 18.$$

On the other hand, $2018^{2018}\equiv 2^{2018} = (2^3)^{672}\cdot 2^2 \equiv 4 \pmod 9$. Hence,

$$S(S(S(2018^{2018}))) = 4 \text{ or } 13.$$

So $S(S(S(S(2018^{2018})))) = 4$.

Example 5: Prove that $S(n)+S(n^2)+S(n^3)$ is a perfect square for infinitely many positive integers n that are not divisible by 10.

Solution: Let us prove that the numbers of the form $n=10^{m^2}-1$ satisfy the problem. The result follows immediately because there are infinitely many number of this form. Firstly, $S(n)=9m^2$ and

$$n^2 = 10^{2m^2} - 2\cdot 10^{m^2} + 1 = 99..9800..01$$

where there are m^2-1 9's and 0's. Then $S(n^2)=9m^2$. Similarly,

$$S(n^3)=99\ldots 9700\ldots 0299\ldots 9,$$

where there are m^2-1 9's and 0's and m^2 9's at the end. Then $S(n^3)=18m^2$. Finally, $S(n)+S(n^2)+S(n^3)=36m^2$.

Remark 1: The numbers of the previous problem are registered in On-Line Encyclopedia of Integer Sequences (OEIS) A153185. Some examples of such numbers: 9, 18, 45, 90, 171, 180, 207, 279, 297, 396, 414, 450, 459, ….

Remark 2: Notice that sometimes mathematical intuition deceives us. That is, the nine numbers 1, 11, 111, …, 111…1 satisfy $S(n^2) = (S(n))^2$. Unfortunately, the next number in this family is

$$1111111111^2 = 1234567900987654321.$$

So $S(1111111111) = 10$, but $S(1111111111^2) = 82$. The smallest positive integer such that $S(n) = 10$ and $S(n^2) = 100$ is n = 1101111211.

Example 6: We say that a superstitious number is equal to 13 times a sum of its digits. Find all superstitious numbers.

Solution: Obviously there is no superstitious number with one digit. If a two digit number ab is superstitious, then $10a+b=13(a+b)$, that is $3a+12b=0$, which is impossible.

If a three-digit number abc is superstitious, we would have $100a+10b+c=13(a+b+c)$. that is $29a=b+4c$. The maximum possible value for $b+4c$ is 45 (for $b=c=9$). So a must be 1 and the equation $29=b+4c$ has solutions (b,c) = (1,7),(5,6), and (9,5). The numbers 117, 156 and 195 are the only superstitious numbers with three digits.

If a four-digit number abcd is superstitious, it would result in $1000a+100b+10c+d$ $=13(a+b+c+d)$. As the number on the left is at least 1000 and the number on the right is at most 13·36=468, there is no superstitious numbers of four digits. Finally, there is no superstitious number with more than four digits, since each added digit contributes at least 1,000 to the number on the left, while the one on the right contributes at most 13·9=117. So the only superstitious numbers are 117, 156 and 195.

Example 7: (*Romanian Team Selection Test 2002*) Let a, b > 0. Prove that the sequence $S([an+b])$ contains a constant subsequence.

Solution. For any positive integer k, let n_k equals $[(10^k+a-b)/b]$. Then

$$10^k = a\left(\frac{10^k+a-b}{a}-1\right)+b$$

$$< an_k + b = a\left[\frac{10^k+a-b}{a}\right]+b$$

$$\le 10^k + b.$$

It follows that $10^k=[an_k+b]\le 10^k+b$.

If k is sufficiently large, that is 10^{k-1}>b, it follows from above that S_{n_k} is one plus the sum of the digits of one of the numbers t in the set $\{0,1,\ldots,[b]\}$. Since k takes infinitely many values and the set of the numbers t is finite, it follows that for infinitely many k, the sum of digits of numbers $[an_k+b]$ is the same.

Example 8: (*2016 IMO Shortlisted Problem*) Find all polynomials $P(x)$ with integer coefficients such that for any positive integer n≥2016, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)). \qquad (*)$$

Solution: Let

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0.$$

Clearly $a_d > 0$. There exists an integer m>1 such that $|a_i|<10^m$ for all $0 \le i \le d$. Consider $n=9\cdot 10^k$ for a sufficiently large integer k in (*). If there exists an index $0 \le i \le d-1$ such that $a_i < 0$, then all digits of $P(n)$ in positions from 10^{ik+m+1} to $10^{(i+1)k-1}$ are all 9's Hence, we have $S(P(n))>9(k-m-1)$. On the other hand, $P(S(n))=P(9)$ is a fixed constant. Therefore, (*) cannot hold for large k. This shows a_i>0 and for all $0\le i \le d-1$. Hence, $P(n)$ is an integer formed by the nonnegative integers a_d9^d, $a_{d-1}9^{d-1}$,…, a_0 by inserting some zeros in between.

This yields

$$S(P(n))=S(a_d9^d)+S(a_{d-1}9^{d-1})+\cdots+S(a_0).$$

Combining with (*), we have

$$S(a_d9^d)+S(a_{d-1}9^{d-1})+\cdots+S(a_0) = P(9) = a_d9^d+a_{d-1}9^{d-1}+\cdots+a_0.$$

As $S(m)\le m$ for any positive integer m, with equality when $1\le m\le 9$, this forces each a_i9^i to be a positive integer between 1 and 9. In particular, this shows a_i=0 for i>2 and hence $d\le 1$. Also, we have $a_1\le 1$ and $a_0\le 9$. If a_1=1 and $1\le a_0\le 9$, we take $n=10^k+(10-a_0)$ for sufficiently large k in (*). This yields a contradiction. Since

$$S(P(n)) = S(10^k+10) = 2 \ne 11 = P(11-a_0) = P(S(n)).$$

The zero polynomial is also rejected since $P(n)$ is positive for large n. The remaining candidates are $P(x)=x$ or $P(x)=a_0$ where $1\le a_0\le 9$, all of which satisfy (*), and hence are the only solutions.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 25, 2019.***

Problem 531. *BCED* is a convex quadrilateral such that $\angle BDC = \angle CEB = 90°$ and *BE* intersects *CD* at *A*. Let *F, G* be the midpoints of sides *DE, BC* respectively. Let *O* be the circumcenter of ΔBAC. Prove that lines *AO* and *FG* are parallel.

Problem 532. Prove that there does not exist a function f:(0,+∞)→(0,+ ∞) such that for all $x,y>0$,

$$f^2(x) \ge f(x+y)(f(x)+y).$$

Problem 533. Let ℤ and ℕ be the sets of all integers and all positive integers respectively. Let $r, s \in \mathbb{N}$. Prove that there are exactly $(r+1)^{s+1}-r^{s+1}$ functions $g:[1,s]\cap\mathbb{N}\to[-r,r]\cap\mathbb{Z}$ such that for all $x,y\in[1,s]\cap\mathbb{N}$, we have $|g(x)-g(y)|\le r$.

Problem 534. Prove that for any two positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct prime divisors.

Problem 535. Determine all integers $n>4$ such that it is possible to color the vertices of a regular n-sided polygon using at most 6 colors such that any 5 consecutive vertices have distinct colors.

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Solutions

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Problem 526. Let $a_1=b_1=c_1=1$, $a_2=b_2=c_2=3$ and for $n\ge3$, $a_n=4a_{n-1}-a_{n-2}$,

$$b_n = \frac{b_{n-1}^2+2}{b_{n-2}}, c_n = 2c_{n-1}+\sqrt{3c_{n-1}^2-2}.$$

Prove that $a_n=b_n=c_n$ for all n =1,2,3,….

Solution. **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School), **Prithwijit DE** (HBCSE, Mumbai, India), **O Long Kin Oscar** (St. Joseph's College), **TAM Choi Nang Julian** (Yan Chai Hospital Law Chan Chor Si College), **Duy Quan TRAN** (University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam) and **Bruce XU** (West Island School).

The cases n = 1,2 can easily be checked. For $n\ge3$, $b_nb_{n-2} = b_{n-1}^2+2$ implies $b_{n+1}b_{n-1} = b_n^2+2$. Subtracting these and factoring, we get $(b_{n+1}-b_{n-1})/b_n=(b_n-b_{n-2})/b_{n-1}$. Then

$$(b_n-b_{n-2})/b_{n-1}= (b_{n-1}-b_{n-3})/b_{n-2}$$
$$= \cdots = (b_3-b_1)/b_2 = 4.$$

Hence, $b_n = 4b_{n-1}- b_{n-2}$ for $n \ge 3$. Since $a_1=b_1$ and $a_2=b_2$, $a_n=b_n$ for all n = 1,2,3,…. Next, from

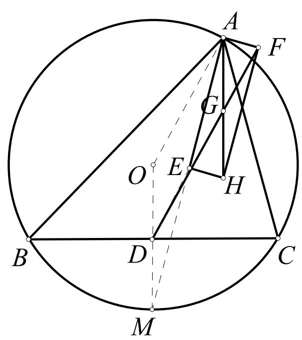
$$c_n = 2c_{n-1}+\sqrt{3c_{n-1}^2-2},$$

we can see c_n is strictly increasing and for $n\ge2$, $(c_n-2c_{n-1})^2 = 3c_{n-1}^2-2$. Then $c_n^2 - 4c_nc_{n-1} + c_{n-1}^2 = -2$ and $c_{n+1}^2-4c_{n+1}c_n + c_n^2 = -2$. Subtracting these and factoring, we get $(c_{n+1}-c_{n-1})(c_{n+1}-4c_n+c_{n-1}) = 0$. As $c_{n+1} > c_{n-1}$, we get $c_{n+1}=4c_n-c_{n-1}$ for $n\ge2$. So $a_n=b_n=c_n$ for all n = 1,2,3,….

Other commended solvers: **AISINGIUR To To**, **Alvin LUKE** (Portland, Oregon, USA), **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan), **SO Tsz To** (S.K.H. Lam Woo Memorial Secondary School), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** (Buzău, Romania).

Problem 527. Let points *O* and *H* be the circumcenter and orthocenter of acute ΔABC. Let *D* be the midpoint of side *BC*. Let *E* be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let *F* be the point such that *AEHF* is a rectangle. Prove that points *D, E, F* are collinear.

Solution. **Alvin LUKE** (Portland, Oregon, USA).



Connect *AO, OD* and extend *OD* to meet the circumcircle of ΔABC at *M*. Then $OD\perp BC$ and *M* bisects arc *BC*. Also, *A, E, M* are collinear. Observe *AE, AF* are internal and external bisectors of $\angle BAC$. So $AE\perp AF$.

Since $HE\perp AE$ and $HF\perp AF$, so *AEHF* is a rectangle. Hence, segments *AH* and *EF* bisect each other. Let *AH* and *EF* meet at *G*. Then $AG=½AH= ½EF=EG$.

Also, $OA=OM$ and $OD \parallel AH$. So

$$\angle OAE=\angle OME=\angle EAG=\angle GEA.$$

So (*) $EG \parallel OA$.

Next, observe *O* and *H* are the circumcenter and the orthocenter of of ΔABC respectively. Since $OD\perp BC$, so $OD=½AH=AG$. Finally, connect *DG*. We see *AODG* is a parallelogram. So (**) $DG \parallel OA$. Therefore, by (*) and (**), *D, E, G, F* are collinear.

Other commended solvers: **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School), **Prithwijit DE** (HBCSE, Mumbai, India), **Andrea FANCHINI** (Cantú, Italy), **Jon GLIMMS, Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Apostolos MANOLOUDIS**, **George SHEN, Toshihiro SHIMIZU** (Kawasaki, Japan), **Mihai STOENESCU** (Bischwiller, France), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** (Buzău, Romania).

Problem 528. Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $n+m$ congruent squares.

Solution. **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School) , and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Suppose rectangle *ABCD* can be decomposed into $n+m$ unit squares and also into n squares with sides equal x. Let $x = a/b$ with gcd(a,b) = 1. Then the area of rectangle *ABCD* is $n+m$ as well as $n(a/b)^2$. Then from $n+m = n(a/b)^2$, we can solve for n to get

$$n=\frac{mb^2}{a^2-b^2}=\frac{mb^2}{(a-b)(a+b)}.$$

Since $\gcd(b,a+b) = \gcd(b,a-b) = \gcd(a,b) = 1$, so $(a-b)(a+b) \mid m$. Now $a+b$, $a-b$ are of the same parity. If m is the product of positive integers i, j, k with j, k odd and greater than 1, then $(a+b,a-b) = (j,k)$ or $(jk,1)$ leading to $n=i(j-k)^2/4$ or $i(jk-1)^2/4$, contradicting the uniqueness of n. So m can have at most one odd factor greater than 1, i.e. $m=2^c$ or 2^cp with p an odd prime.

In case $m=2^c$, for $c=1,2$, there is no n; for $c=3$, $m=8$ and $(a,b)=(2,4)$, $n=1$; for $c\geq4$, $(a+b,a-b)=(4,2)$ or $(8,2)$ resulting in $n = 2^{c-3}$ or 2^{c-4} contradicting the uniqueness of n.

In case $m=2^cp$, for $c=0$, $m=p$ and $(a+b,a-b) = (p,1)$, $n = (p-1)^2/4$; for $c = 1$, $(a+b,a-b) = (p,1)$, $n = (p-1)^2/2$; for $c = 2$, $(a+b,a-b) = (p,1)$, $n = (p-1)^2$; for $c\geq3$, $(a+b,a-b) = (4,2)$ or $(p,1)$ contradict the uniqueness of n.

So the only solutions are $m = 8$, p, $2p$, $4p$, where p is an odd prime.

Other commended solvers: **Victor LEUNG Chi Shing** and **Charles POON Tsz Chung**.

Problem 529. Determine all ordered triples (x,y,n) of positive integers satisfying the equation $x^n+2^{n+1} = y^{n+1}$ with x is odd and the greatest common divisor of x and $n+1$ is 1.

Solution. **Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

When $n=1$, let $y=t$ be an integer at least 3 and $x=t^2-4$ are solutions. When $n\geq2$,

$$x^n = y^{n+1}-2^{n+1}=(y-2)\sum_{k=0}^{n}2^k y^{n-k}.$$

For any prime factor p of y−2, from above, we see x must be a multiple of p. As x is odd, p is also odd. As $\gcd(x,n+1) = 1$, we see $\gcd(x,(n+1)2^n) = 1$. Then p is not a factor of $(n+1)2^n$. Now

$$S=\sum_{k=0}^{n}2^k y^{n-k}\equiv\sum_{k=0}^{n}2^n=(n+1)2^n \pmod{y-2}.$$

Hence, p is not a factor of S. So we have $\gcd(y-2,S) = 1$. So $S=T^n$ for some positive integer T. Since y is positive, y is at least 3.

When $n\geq2$, we have

$$y^n < S=T^n < (y+2)^n. \qquad (*)$$

So $T = y+1$. However, when y is even, $S \equiv y^n \pmod 2$ is even, but then $S = (y+1)^n$ is odd by (*). Similarly, when y is odd, $S \equiv y^n \pmod 2$ is odd, but then $S=(y+1)^n$ is even by (*). Again this leads to a contradiction.

In conclusion, when integer n is at least 2, there are no solutions. So the only solution are $x=t^2-4$, $y=t$, $n=1$, where integer $t\geq3$.

Other commended solvers: **Ioannis D. SFIKAS** (Athens, Greece).

Problem 530. A square can be decomposed into 4 rectangles with 12 edges. If square $ABCD$ is decomposed into 2005 convex polygons with degrees of A, B, C, D at least 2 and degrees of all other vertices at least 3, then determine the maximum number of edges in the decomposition.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let v, e, f be the number of vertices, edges and faces used in decomposing the square respectively. By Euler's formula, we have $v-e+f = 1$ (omitting the exterior of the square).

Let $d(V)$ be the number of edges connected to V. Let V be a vertex on the square other than A,B,C,D. Then $d(V)\geq3$, which is the same as $d(V)\leq3d(V)-6$.

Now there are $v-4$ vertices not equal to A, B, C, D. The sum of the degrees of the $v-4$ vertices other than A, B, C, D is $2e-[d(A)+d(B)+d(C)+d(D)]$, which is at least $3(v-4)$. Since $d(A)$, $d(B)$, $d(C)$, $d(D)$ ≥2, we get

$$\begin{aligned}2e-8\geq 2e-[d(A)+d(B)+d(C)+d(D)]\\ \geq3(v-4) = 3v-12.\end{aligned}$$

Since $v-e+f = 1$, $3e = 3v+3f-3 \leq 2e+1+3f$, which simplies to $e \leq 3f+1$.

For equality case, we can decompose the unit square into rectangles of size 1 by 1/2005, which has $3\times2005+1=6016$ edges.

Olympiad Corner

(*Continued from page 1*)

Problem G2. (*cont.*) Line DJ intersects the circumcircle of the triangle ABC at the point E which lies on the shorter arc AB. Prove that $|AI|=|IE|$ holds.

Problem N2. Let n be a positive integer. Prove that there exists a positive integer k such that

$$51^k - 17$$

is divisible by 2^n.

Sums of Digits …

(*Continued from page 2*)

Next, we will provide some exercises for the readers.

Problem 1: (*Mexico 2018*) Find all pairs of positive integers (a,b) with $a>b$ which simultaneously satisfy the following two conditions

$$a \mid b+S(a) \quad\text{and}\quad b \mid a+S(b).$$

Problem 2: (*Lusophon 2018*) Determine the smallest positive integer a such that there are infinitely many positive integer n for which you have $S(n)-S(n+a) = 2018$.

Problem 3: (*Cono Sur 2016*) Find all n such that $S(n)(S(n)-1) = n-1$.

Problem 4: (*Iberoamerican 2014*) Find the smallest positive integer k such that

$$\begin{aligned}S(k) = S(2k) = S(3k) = \cdots\\ = S(2013k) = S(2014k).\end{aligned}$$

Problem 5: (*OMCC 2010*) Find all solutions of the equation $n(S(n)-1) = 2010$.

Problem 6: (*Iberoamerican 2012*) Show that for all positive integers n there are n consecutive positive integers such that none is divisible by the sum of their respective digits.

Mathematical Excalibur

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Olympiad Corner

Below were the Hong Kong (China) Mathematical Olympiad on December 1, 2018.

Problem 1. Given that a, b and c are positive real numbers such that $ab+bc+ca\geq 1$, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{\sqrt{3}}{abc}.$$

Problem 2. Find the number of nonnegative integers k, $0\leq k\leq 2188$, and such that $2188!/(k!(2188-k)!)$ is divisible by 2188.

Problem 3. The incircle of ΔABC, with incenter I, meets BC, CA and AB at D, E, F respectively. The line EF cuts the lines BI, CI, BC and DI at points K, L, M and Q respectively. The line through the midpoint of CL and M meets CK at P.

(a) Determine $\angle BKC$.
(b) Show that the lines PQ and CL are parallel.

Problem 4. Find all integers $n\geq 3$ with the following property: there exist n distinct points on the plane such that each point is the circumcenter of a triangle formed by 3 of the points.

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 2, 2019***.

For individual subscription for the next five issues for the 18-19 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Notes on IMO 2019

Tat Wing LEUNG

Despite all its sham, drudgery and broken dreams, the Gifted Section of the Education Department (EDB), the Hong Kong Academy of Gifted Education (HKAGE), and our Committee (International Mathematical Olympiad Hong Kong Committee, IMOHKC) managed to send a team to the 60th International Mathematical Olympiad (IMO 2019). The competition was held from July 11 to July 22, 2019, in Bath, United Kingdom.

The team was composed as follows: Leader: Leung Tat Wing, Deputy Leader: Cesar Jose C. Jr. Alaban (CJ), Members: Bruce Changlong Xu, Daniel Weili Sheremeta, Harris Leung, Wan Lee, Nok To Omega Tong, Sui Kei Ho. A lady from EDB (Miriam Cheung) also went with us as an observer.

Let me briefly discuss the problems of the two contests.

Problem 1 was very interesting. It was initially selected as the easy algebra problem and later selected as the easy pair. Although it was most liked, it was also most hated. I supposed it was because some leaders thought the problem was simply too easy. By substituting suitable values (say a by 0 and b by n+1 one quickly comes to the conclusion that the function is linear (or by Cauchy), and hence by using some initial values to get the answers. Some leaders first tried to replace the easy algebra by another easy problem (which was actually classified as a combinatorial problem), and later tried to add alternate option pairs to the option pairs that contained the easy algebra problem. I myself could not say if it was right or wrong, I just found it funny. Indeed the problem was selected using the approach as agreed, why tried to change it in the middle of the process? At the end of the day, totally 73 students did not get anything in this problem, and only slightly more than half (382 out of 621) scored full mark.

Problem 4 was an easy Diophantine equation. By putting small values of n, one quickly comes up with the solutions (1,1) and (3,2), the hard part is to show that there are no more. Many students lost partial marks while trying to compare values (or 2-adic valuations) of the two sides of the equation. As learned from leaders of stronger teams, I found they considered Legendre's formula and/or the lifting exponent lemma rather common tools, although the lemma was not really necessary. So yes, do we need to ask our students to further enhance their toolkit?

Problem 5 was an *ouroboros*-type problem, namely part of the problem is relating to other part of itself. In this case we are given a sequence of heads and tails of n coins, the k^{th} coin is flipped if there are exactly k heads in the sequence. The problem is not too hard, and given its "natural" condition, it is probably known. Indeed if the first coin is head, then basically we need to deal with the remaining sequence of length $n-1$, and the final step is to flip the first coin. If the last coin is a tail, then it will never be flipped, and we are basically dealing with the first $n-1$ coins.

If the first coin is a tail, and the last coin is a head, then we first deal with the middle $n-2$ coins. After that only one head remaining (at the end), then the first $n-1$ coins are flipped successively and all become heads, then starting from the end, each coin is flipped, until the first one and every coin becomes tail. Using these, we can make up recursive relations and get the answer relatively easy. Our team members, using their own ingenuity and persistence, managed to do the problem well.

(continued on page 4)

Wilson's Theorem

Kin Y. Li

In solving number theory problems, Fermat's or Euler's theorems as well as the Chinese remainder theorem are often applied. In this article, we will look at examples of number theory problems involving factorials. For this type of problems, Wilson's theorem asserts that for every prime number p, we have $(p-1)! \equiv -1 \pmod{p}$. Below are problems using Wilson's theorem.

Problem 1. Let p be an odd integer greater than 1. Prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Solution. By Wilson's theorem, $(p-1)! \equiv -1 \pmod{p}$ when p is an odd prime. Also, we have $i \equiv -(p-i) \pmod{p}$. Multiplying the cases $i = 1, 3, \cdots, p-2$, we get

$$1\cdot 3\cdots(p-2) \equiv (-1)^{(p-1)/2}(p-1)(p-3)\cdots 2 \pmod{p}.$$

Multiplying both sides by $1\cdot 3\cdots(p-2)$, we get

$$1^2\cdot 3^2\cdot 5^2\cdots(p-2)^2 \equiv (-1)^{(p-1)/2}(p-1)! \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Problem 2. Let p be a prime number and $N = 1+2+3+\cdots+(p-1) = (p-1)p/2$. Prove that $(p-1)! \equiv p-1 \pmod{N}$.

Solution. Since p is prime, by Wilson's theorem, $(p-1)! \equiv -1 \pmod{p}$. Then there exists an integer m such that

(*) $(p-1)! = mp-1 = (m-1)p+(p-1)$.

So $(m-1)p = (p-1)!-(p-1) = (p-1)k$, where $k=(p-2)!-1$ and $p|(p-1)k$. Since $\gcd(p,p-1)=1$, so $p|k$. Let $k=np$, then

(**) $(m-1)p=(p-1)pn$,

so $m-1=n(p-1)$. Putting (**) into (*), we get

$$(p-1)! = [n(p-1)+1]\,p-1 = n(p-1)p+p-1 = 2n[(p-1)p/2]+p-1 = 2nN+p-1.$$

So $(p-1)! \equiv p-1 \pmod{N}$.

Problem 3. Determine all positive integers n having the property that there exists a permutation $a_1, a_2, \ldots, a_n$ of $0,1,2,\ldots,n-1$ such that when divided by n, the remainders of $a_1, a_1a_2, \ldots, a_1a_2\cdots a_n$ are distinct.

Solution. When n is a prime number p, let $a_1=1$ and other integers a_i satisfy $0 \le a_i \le p-1$ and $ia_{i+1} \equiv i+1 \pmod{p}$ for $i = 2,\cdots, p$.

Then $a_1, a_1a_2, \ldots, a_1a_2\cdots a_n$ when divided by n have remainders $1,2,\cdots, p$. Also, from $ia_{i+1} \equiv i+1 \pmod{p}$, we see $a_{i+1}-1$ is the inverse of i. So $a_1, a_2, \ldots, a_n$ are distinct.

When $n = 1$ or 4, the permutations (0), (1,3,2,0) satisfy the condition. When $n>4$ is composite, if $n = p^2$, let $q = 2p < n$. Otherwise $n=pq$ with $1<p<q<n$ so that $pq|(n-1)!$.

If the required permutation exists, then $a_n=0$ and $a_1a_2\cdots a_{n-1}=(n-1)!\equiv 0 \pmod{n}$, which is a contradiction. (In fact, when $n>4$ is composite, $n \mid (n-1)!$ and $3! \equiv -2 \pmod{4}$ so that the converse of Wilson's theorem also hold.

Problem 4. For integers n, q satisfying $n\ge 5$ and $n \ge q \ge 2$, prove that $[(n-1)!/q]$ is divisible by $q-1$.

Solution. (1) If $n>q$, then $(q-1)q \mid (n-1)!$. Hence, $(q-1) \mid [(n-1)!/q]$.

(2) If $q=n$ and q is composite, then $[(n-1)!/q]=(n-1)!/n$. Since $\gcd(n-1,n)=1$ and $q-1=(n-1) \mid (n-1)!$. So $q-1$ divides $[(n-1)!/q]$.

(3) If $q=n$ is prime, then by Wilson's theorem, $(n-1)! \equiv -1 \pmod{n}$ so that $(n-1)!+1=kn$ for some integer k. Then $[(n-1)!/q]=k-1$ and $(k-1)n=(n-1)!+1-n$ so that $k-1=((n-2)!-1)(n-1)/n$ is an integer. Since $\gcd(n-1,n)=1$, so n divides $(n-2)!-1$. Therefore, $[(n-1)!/q]=k-1$ is a multiple of $n-1$.

Problem 5. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $a_0, a_1, \cdots, a_n$ are integers, $a_n>0$ and $n\ge 2$. Then prove that there exists a positive integer m such that $P(m!)$ is a composite number.

Solution. If $a_0=0$, then $m! \mid P(m!)$ and the conclusion follows.

Next let $S(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. Suppose $a_0\ne 0$. By Wilson's theorem, for every prime p and positive even integer $k < p$, we have

$$\begin{aligned}&(k-1)!(p-k)!\\ &\equiv (-1)^{k-1}(p-k)!(p-k+1)(p-k+2)(p-1)\\ &= -(p-1)! \equiv 1 \pmod{p}.\end{aligned}$$

So $(p-1)! \equiv -1 \pmod{p}$ and

$$((k-1)!)^n\, P((p-k)!) \equiv S((k-1)!) \pmod{p}.$$

So $p \mid P((p-k)!)$ if and only if $p \mid S((k-1)!)$. Take $k > 2a_n+1$. Then $u = (k-1)!/a_n$ is an integer divisible by all primes not greater than k.

Problem 6. If p and $p+2$ are both prime numbers, then we say they are twin primes. Show that if p and $p+2$ are twin primes, then $4(p-1)!+4+p$ is divisible by $p(p+2)$.

Solution. If p and $p+2$ are prime, then $p>2$ so that p and $p+2$ are odd. By Wilson's theorem, $(p-1)! \equiv -1 \pmod{p}$ and also $(p+1)! \equiv -1 \pmod{p+2}$. Then we have

$$4(p-1)!+4+p \equiv 0 \pmod{p}.$$

Also

$$\begin{aligned}&4(p-1)!+4 \equiv -p(p+1)p[(p-1)!+1]\\ &\equiv -p[(p+1)!+2] \equiv -p \pmod{p+2},\end{aligned}$$

which is $4(p-1)!+4+p \equiv 0 \pmod{p+2}$. As $\gcd(p,p+2)=1$, we get $4(p-1)!+4+p \equiv 0 \pmod{p(p+2)}$.

Problem 6. (Wolstenholme's Theorem) Let p be a prime greater than or equal to 5. For positive integers m and n that are relatively prime and

$$\frac{m}{n} = \frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}.$$

Prove that p is a divisor of m and p^2 is a divisor of

$$(p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right).$$

Solution. If integer k is not divisible by p, then there are integers a, b such that $ak+bp = \gcd(k,p) = 1$. We say a is the inverse of k in mod p and denote a as k^{-1}. We have

$$\begin{aligned}((p-1)!)^2\frac{m}{n} &= \sum_{k=1}^{p-1}\frac{((p-1)!)^2}{k^2}\\ &\equiv (-1)^2(1^2+2^2+\cdots+(p-1)^2)\\ &\equiv \frac{(p-1)p(2p-3)}{6} \equiv 0 \pmod{p}.\end{aligned}$$

Since $\gcd((p-1)!, p) = 1$, so $p \mid m$. Next, let $S=(p-1)!(1+1/2+\cdots+1/(p-1))$. Then

$$\begin{aligned}2S &= (p-1)!\sum_{i=1}^{p-1}\left(\frac{1}{i}+\frac{1}{p-i}\right)\\ &= p\sum_{i=1}^{p-1}\frac{(p-1)!}{i(p-i)} = pT,\end{aligned}$$

where $2S$, p and T are integers. Since $\gcd(p,2)=1$, so p divides S. Due to $p|m$,

$$T = \sum_{i=1}^{p-1}\frac{(p-1)!}{i(p-i)} \equiv (p-1)!\frac{m}{n} \equiv 0 \pmod{p}.$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 2, 2019.***

Problem 536. Determine whether there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that for all real x, we have $f(x^3+x)\le x\le (f(x))^3+f(x)$.

Problem 537. Distinct points A, B, C are on the unit circle Γ with center O inside ΔABC. Suppose the feet of the perpendiculars from O to sides BC, CA, AB are D, E, F. Determine the largest value of $OD+OE+OF$.

Problem 538. Determine all prime numbers p such that there exist integers a and b satisfying $p=a^2+b^2$ and a^3+b^3-4 is divisible by p.

Problem 539. In an exam, there are 5 multiple choice problems, each with 4 distinct choices. For every problem, every one of the 2000 students is required to choose exactly 1 of the 4 choices. Among the 2000 exam papers received, it is discovered that there exists a positive integer n such that among any n exam papers, there exist 4 such that for every 2 of the exam papers, there are at most 3 problems having the same choices. Determine the least such n.

Problem 540. Do there exist a positive integer k and a non-constant sequence $a_1, a_2, a_3, \ldots$ of positive integers such that $a_n=\gcd(a_{n+k}, a_{n+k+1})$ for all positive integer n?

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Solutions

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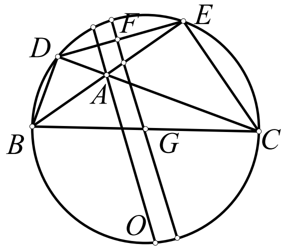
Problem 531. $BCED$ is a convex quadrilateral such that $\angle BDC=\angle CEB=90°$ and BE intersects CD at A. Let F, G be the midpoints of sides DE, BC respectively. Let O be the circumcenter of ΔBAC. Prove that lines AO and FG are parallel.

Solution 1. **Jon GLIMMS, Hei Chun LEUNG** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Since $\angle CAO=(180°-\angle COA)/2=90°-\angle COA/2=90°-\angle CBA=90°-\angle CBE=90°-\angle CDE=90°-\angle ADE$, we have OA and DE are perpendicular. Also, since FG passes through the center G of the circle $CEDG$ and midpoint F of chord DE, FG is perpendicular to DE. Thus, both AO, FG are perpendicular to DE. So lines AO and FG are parallel.

Solution 2. **Prithwijit DE** (HBCSE, Mumbai, India).

Let R be the radius of the circumcircle of triangle BAC. As $\angle BAC>90°$, BC is not the diameter of the circle ABC and therefore D and E are outside the circle ABC. Observe that $EA\cdot EB=EO^2-R^2$ and $DA\cdot DC=DO^2-R^2$. Thus

$$\begin{aligned}EO^2-DO^2&=EA\cdot EB-DA\cdot DC\\&=EA^2-DA^2+EA\cdot AB-DA\cdot DC\\&=EA^2-DA^2.\end{aligned}$$

This implies $OA\perp DE$. Now $FG\perp DE$ because G is the centre of the circle passing through B, C, E and D, and F is the midpoint of chord DE of this circle. Therefore, lines AO and FG are parallel.

Other commended solvers: **CHUI Tsz Fung, Andrea FANCHINI** (Cantù, Italy), **Panagiotis N. KOUMANTOS** (Athens, Greece), **LAU Chung Man** (Lee Kau Yan Memorial School), **LW Maths Solving Team** (SKH Lam Woo Memorial Secondary School), **Jim MAN**, **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania) and **Apostolis MANOLOUDIS**.

Problem 532. Prove that there does not exist a function $f:(0,+\infty)\to(0,+\infty)$ such that for all $x,y>0$,

$$f^2(x)\ge f(x+y)(f(x)+y).$$

Solution. **Jon GLIMMS, Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan),

Assume such function exists. We have

$$-y\,f(x+y)/f(y)\ge f(x+y)-f(x).$$

Since the left hand side is negative, f must be strictly monotone decreasing. Also, for any positive integer n and positive real number a, taking the sum for $x=a+i/n$, $y=1/n$, where $1\le i\le n-1$, we get

$$-\frac{1}{n}\sum_{i=1}^{n}\frac{f(a+\frac{i+1}{n})}{f(a+\frac{i}{n})}\ge f(a+1)-f(a).$$

By the AM-GM inequality, we have

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f(a+\frac{i+1}{n})}{f(a+\frac{i}{n})}\ge\sqrt[n]{\prod_{i=1}^{n}\frac{f(a+\frac{i+1}{n})}{f(a+\frac{i}{n})}}\ge\sqrt[n]{\frac{f(a+1)}{f(a)}}.$$

Since $f(a+1)/f(a)>1$, the right hand side will converge to 1 when $n\to\infty$. Thus, $f(a+1)-f(a)\le-1$ for all $a>0$. Then, from $f(1)\ge f(2)+1\ge f(3)+2\ge\cdots$, we have $f(1)\ge f(n+1)+n$ for all positive integer n. This shows that $f(1)$ cannot be finite, a contradiction.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Apostolos MANOLOUDIS**, **George SHEN** and **Thomas WOO.**

Problem 533. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of all integers and all positive integers respectively. Let $r, s\in\mathbb{N}$. Prove that there are exactly $(r+1)^{s+1}-r^{s+1}$ functions $g:[1,s]\cap\mathbb{N}\to[-r,r]\cap\mathbb{Z}$ such that for all $x,y\in[1,s]\cap\mathbb{N}$, we have $|g(x)-g(y)|\le r$.

Solution. **LAU Chung Man** (Lee Kau Yan Memorial School), **George SHEN** and **Thomas WOO.**

If integer k is in $[-r,r]\cap\mathbb{Z}$, then there are $(\min\{r+1,r-k+1\})^s$ functions satisfying the given conditions which attain values only in $\{k,\ldots,k+r\}$. Of these, $(\min\{r,r-k\})^s$ functions attain values only in $\{k+1,\ldots,k+r\}$. Hence, exactly

$$(\min\{r+1,r+1-k\})^s-(\min\{r,r-k\})^s$$

functions satisfying the given conditions have minimum value k.

This expression equals $(r+1)^s-r^s$ for each of the $r+1$ values $k\le0$, and it equals $(r+1-k)^s-(r-k)^s$ when $k>0$. Thus, the sum of the expression over all $k\le0$ is $(r+1)((r+1)^s-r^s)$, while the sumof the expression over all $k>0$ is the telescoping sum

$$\sum_{k=1}^{r}((r+1-k)^s-(r-k)^s)=r^s.$$

Adding these two sums, we find that the total number of functions satisfying the given conditions is $(r+1)^{s+1}-r^{s+1}$.

Other commended solvers: **Jon GLIMMS, Michael HUI** and **Jeffrey HUI**, **Hei Chun LEUNG, Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 534. Prove that for any two positive integers m and n, there exists a positive integer k such that 2^k-m has at least n distinct prime divisors.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

We show by induction that there is $k\in\mathbb{N}$ such that 2^k-m has at least n odd prime divisors. If m is even, we can write $n=2^e s$ (with odd integer s) and take $k>e$ so we have $2^k-m=2^e(2^{k-e}-s)$. Then it is sufficient to show for $m=s$ (odd). Thus, we assume m is odd.

Taking $k\in\mathbb{N}$ such that $2^k-m>1$, we can take an odd prime divisor p of 2^k-m (which is odd). Assume we have $k\in\mathbb{N}$ such that 2^k-m has n odd prime divisors $p_1, p_2, \ldots, p_n$. For any i the pattern of $2^j \pmod{p_i}$ is periodic for j, which implies there are $e_i, f_i\in\mathbb{N}$ such that $2^j\equiv m \pmod{p_i}$ if and only if $j=e_it+f_i$ for some $t\in\mathbb{N}$. Since $p_i>2$, each e_i is greater than 1. Thus, we can take f_i' such that $f_i\not\equiv f_i' \pmod{e_i}$. By the Chinese remainder theorem, we can take f_i' such that $k'\equiv f_i' \pmod{e_i}$ and we have $p_i\nmid 2^{k'}-m$ for $1\le i\le n$. We can also select k' such that $2^{k'}-m>1$. Then we can take odd prime divisor p_{n+1} of $2^{k'}-m$, where p_{n+1} is different from any one of $p_1, p_2, \ldots, p_n$. Then we can choose j such that $2^j\equiv m \pmod{p_{n+1}}$, where $j=e_{n+1}t+f_{n+1}$ for some e_{n+1}, f_{n+1}. By the Chinese remainder theorem again, we can take K such that $K\equiv f_i'$ (mod e_i) and we have $p_i\nmid 2^K-m$ for $1\le i\le n+1$. Then 2^K-m has at least $n+1$ prime factors $p_1, p_2,\ldots,p_{n+1}$, completing the induction.

Problem 535. Determine all integers $n>4$ such that it is possible to color the vertices of a regular n-sided polygon using at most 6 colors such that any 5 consecutive vertices have distinct colors.

Solution. **CHUI Tsz Fung**, **Hei Chun LEUNG**, **LAU Chung Man** (Lee Kau Yan Memorial School), **LW Maths Solving Team** (SKH Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let the colors be *a, b, c, d, e, f.* Denote by S_1 the sequence *a, b, c, d, e* and by S_2 the sequence *a, b, c, d, e, f.* If $n>0$ is representable in the form $5x+6y$ for $x,y\ge 0$, then n satisfies the conditions of the problem: we may place x consecutive S_1 sequences, followed by y consecutive S_2 sequences, around the polygon. Setting y equal to 0, 1, 2, 3 or 4, we find that n may equal any number of the form $5x$, $5x+6$, $5x+12$, $5x+18$ or $5x+24$. The only numbers greater than 4 not of this form are 7, 8, 9, 13, 14 and 19. Below we will show that none of these numbers has the required property.

Assume for a contradiction that a coloring exists for n equal to one of 7, 8, 9, 13, 14 and 19. There exists a number k such that $6k<n<6(k+1)$. By the pigeonhole principle, at least $k+1$ vertices of the n-gon have the same color. Between any two of these vertices are at least 4 others, because any 5 consecutive vertices have different colors. Hence, there are at least $5k+5$ vertices, and $n\ge 5k+5$. However, this inequality fails for $n=7, 8, 9, 13, 14, 19$, a contradiction. Hence, a coloring is possible for all $n\ge 5$ except 7, 8, 9, 13, 14 and 19.

Notes on IMO 2019

(Continued from page 1)

Problem 3 is a graph algorithmic problem. The problem is not real hard, but the essential difficulty is hidden by the numbers, students also might find it hard because they do not have the language of graph theory. Namely the graph is connected, with at least three vertices and is not complete, and there is a vertex of odd degree. Then it is possible to find a vertex and apply the operation, and reduce the number of edges by 1, yet maintaining the essential initial conditions. There is no worry of the existence of a cycle, for instance, during the operations. Otherwise the cycle can only be shrunk to a triangle and get stuck. At least a solution is conceivable.

I do not know what to say about problem 2 and 6 (medium and hard geometry problem). Our team did not do too well. It suffices to say, problem 2 may be done by careful angle chasing, while problem 6 is more complicated, but there is a nice and not too complicated complex number solution.

In short, leaders generally agreed that those problems are do-able. If one understands what is going on, one should be able to do those problems, and there is no need of deep and/or obscure theorems. I recalled one of my teachers told us, there really is “no mystery”, if you get the point. Also it came to my mind Hilbert’s motto: *wir mussen wissen*, *wir werden wussen* (we must know, we will know). Indeed at the end, the cut-off scores were relatively high, 17 for bronze, 24 for silver, and 31 for gold, and in total 6 contestants obtained full mark.

After coordination and the final Jury meeting, we managed to get 1 silver medal (Harris) and 3 bronze (Wan, Daniel and Omega). Surely it was not too good, but not too bad either. Indeed they could do better. For instance, Bruce was only 1 point below bronze, and Sui Kei 3 points (he got a honorable mention by scoring full mark in a problem), should they not making several trivial mistakes (also made by members of several strong teams), they should get medals. Both Daniel and Wan solved three problems, and in my opinion potential silver medalists. On the whole, I notice they have been working hard during the last two months, so I don’t think I should blame them too much. One thing however I think our team members should watch out is, in case they will come back next time, they should know how much further effort they need to devote and know what they expect.

I have given my opinions and suggestions. Accordingly 2020 IMO will be held in Russia, 2021 in USA, 2022 in Norway, 2023 in Japan, 2024 in Shanghai China (probably) and 2025 in Australia. Some people have been working hard to make future IMOs possible. I hope Hong Kong will continue to join. However I cannot be too sure. For one thing, not sure if Hong Kong will be as relatively free/peaceful/prosperous to sustain events of this kind. Even so, I am not quite sure if our students may maintain their interest. Life is hard (as usual). Let’s hope for the best. Good Luck.